Thm: 2.1 |AN| The ∞-category $(\mathrm{CycSp}_p)_{\geq 0}$ forms the connective part of an accessible, left complete t-structure on CycSp_p, (which is compatible with the symmetric monoidal structure on CycSp_p)

So first goal is to try and actually understand what it states, hence we will first give a general introduction to t-structures on stable ∞-categories and then on CycSp_p

Notation: $\mathcal{C}[1] := \Sigma \mathcal{C}$ suspension
$\mathcal{C}[-1] := \Omega \mathcal{C}$ desuspension

Def. Let $\mathcal{C}$ be a stable ∞-category. A t-structure on $\mathcal{C}$ is a pair $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ of full subcategories of $\mathcal{C}$ s.t.

$\mathcal{C}_{\geq 0}$:= connective objects
$\mathcal{C}_{\leq 0}$:= coconnective objects

1) $\underbrace{\mathcal{C}_{\geq 0}[1] \subseteq \mathcal{C}_{\geq 0}}_{\text{stable under suspension}}$ and $\underbrace{\mathcal{C}_{\leq 0}[-1] \subseteq \mathcal{C}_{\leq 0}}_{\text{stable under desuspension}}$

2) $X \in \mathcal{C}_{\geq 0}$, $Y \in \mathcal{C}_{\leq 0} \Rightarrow \underbrace{\mathrm{Map}_{\mathcal{C}}}_{\text{mapping space}}(X, Y[-1])$ is contractible

3) $\forall X \in \mathcal{C}$ $\exists$ fiber sequence

$$\tau_{\geq 0} X \to X \longrightarrow \tau_{\leq -1} X$$

with $\tau_{\geq 0} X \in \mathcal{C}_{\geq 0}$, $\tau_{\leq -1} X[1] \in \mathcal{C}_{\leq 0}$

Notation: $\mathcal{C}_{\geq n} := \mathcal{C}_{\geq 0}[n]$
$\mathcal{C}_{\leq n} := \mathcal{C}_{\leq 0}[n]$

Ex: ① $\mathcal{C} = \mathrm{Sp}$

$\mathcal{C}_{\geq 0} = \{X \in \mathrm{Sp} \mid \pi_n X = 0 \ \forall n < 0\}$

$\mathcal{C}_{\leq 0} = \{X \in \mathrm{Sp} \mid \pi_n X = 0 \ \forall n > 0\}$

② $\mathcal{C} = \mathrm{Sp}^{BS^1}$

$\mathcal{C}_{\geq 0}$ = Those with underlying connective spectrum

③ $\mathcal{C} = \mathrm{Ch}(R\text{-Mod})$

$\mathcal{C}_{\geq 0} = \{X \mid H_* X = 0, * < 0\}$

$\mathcal{C}_{\leq 0} = \{X \mid H_* X = 0, * > 0\}$

$\mathcal{C}_{\leq 0}$: Those w/ underlying coconnective spectra

<u>Rem:</u> A t-structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ on $\mathcal{C}$ can be determined by either $\mathcal{C}_{\geq 0}$ or $\mathcal{C}_{\leq 0}$: when we know $\mathcal{C}_{\geq 0}$ is part of a t-structure

$\mathcal{C}_{\leq 0}$ is the full subcategory of objects $Y \in \mathcal{C}$ which satisfies that

$$\mathrm{Map}_{\mathcal{C}}(X, Y[1]) \simeq 0 \quad \forall X \in \mathcal{C}_{\geq 0}$$

$\mathcal{C}_{\leq 0}$ is right orthogonal to $\mathcal{C}_{\geq 0}$

Similarly: $\mathcal{C}_{\geq 0}$ is left orthogonal to $\mathcal{C}_{\leq -1}$.

<u>Def:</u> The inclusions of $\mathcal{C}_{\leq 0}$ and $\mathcal{C}_{\geq 0}$ admits adjoints:

$$\begin{tikzcd} \mathcal{C}_{\geq 0} \arrow[r, bend left, "\mathrm{incl}"] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, bend left, "\tau_{\geq 0}"] \arrow[r, bend left, "\tau_{\leq 0}"] \arrow[r, phantom, "\perp"] & \mathcal{C}_{\leq 0} \arrow[l, bend left] \end{tikzcd}$$

truncation

connective cover

Can be done for any n to obtain $\tau_{\leq n}, \tau_{\geq n}$

We have a commutative diagram of simplicial sets

$$\begin{tikzcd} \mathcal{C}_{\geq n} \arrow[r, hook] \arrow[d, "\tau_{\leq m}"'] \arrow[dr, phantom, "\circlearrowright"] & \mathcal{C} \arrow[d, "\tau_{\leq m}"] \arrow[l, bend right, "\tau_{\geq n}"'] \\ \mathcal{C}_{\geq n} \cap \mathcal{C}_{\leq m} \arrow[r, hook] & \mathcal{C}_{\leq m} \arrow[l, bend left, "\tau_{\geq n}"] \end{tikzcd}$$

which induces a natural transformation

$$\theta : \tau_{\leq m} \circ \tau_{\geq n} \longrightarrow \tau_{\geq n} \circ \tau_{\leq m}$$

of functors $\mathcal{C} \to \mathcal{C}_{\leq m} \cap \mathcal{C}_{\geq n}$, which can be shown to be an equivalence.

<u>Def:</u> The heart of a t-structure is the full subcategory

$$\mathcal{C}^{\heartsuit} = \mathcal{C}_{\geq 0} \cap \mathcal{C}_{\leq 0} \subseteq \mathcal{C}$$

- This is an abelian category

<u>Ex:</u>
- $\mathrm{Ch}(R\text{-Mod})^{\heartsuit} \simeq R\text{-Mod}$
- $\mathrm{Sp}^{\heartsuit} = \mathrm{Ab}$

Def: For each $n \in \mathbb{Z}$, we let $\pi_0 : \mathcal{C} \to \mathcal{C}^{\heartsuit}$ denote the functor

$$\tau_{\leq 0} \circ \tau_{\geq 0} \simeq \tau_{\geq 0} \circ \tau_{\leq 0} : \mathcal{C} \longrightarrow \mathcal{C}_{\geq 0} \cap \mathcal{C}_{\leq 0} = \mathcal{C}^{\heartsuit},$$

and

$$\begin{tikzcd} \pi_n : \mathcal{C} \arrow[rr] \arrow[dr, shift left, "\times{[n]}"] & & \mathcal{C}^{\heartsuit} \\ & \mathcal{C} \arrow[ul, shift left, "\Omega^n"] \arrow[ur, "\pi_0"'] & \end{tikzcd}$$

Now, what does it mean when we say a t-structure is ccessible and left complete?

Def: Let $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ be a t-structure on $\mathcal{C}$. We then define the left completion $\hat{\mathcal{C}}$ of $\mathcal{C}$ to be the homotopy limit of

$$\lim\left(\cdots \longrightarrow \mathcal{C}_{\leq 2} \xrightarrow{\tau_{\leq 1}} \mathcal{C}_{\leq 1} \xrightarrow{\tau_{\leq 0}} \mathcal{C}_{0} \xrightarrow{\tau_{\leq -1}} \cdots\right)$$

similar for right completion

and we say $\mathcal{C}$ is left complete if the canonical map

$$\mathcal{C} \longrightarrow \hat{\mathcal{C}}$$

is an equivalence

Def: We say that $\mathcal{C}$ is left seperated if the full subcategory of ∞-connective objects

$$\mathcal{C}_{\geq \infty} := \bigcap \mathcal{C}_{\geq n} \subseteq \mathcal{C}$$

is contractible.

Lem: Let $\mathcal{C}$ be a stable ∞-category w. a t-structure. Assume
AW:A.6
HA.1.2.1.19 that $\mathcal{C}$ admits countable products and that $\mathcal{C}_{\geq 0}$ is stable under countable products. Then $\mathcal{C}$ is

$$\text{left seperated} \iff \text{left complete.}$$

Def: An ∞-category $\mathcal{C}$ is presentable if it is "accessible" and admits all small colimits

Being "accessible" means that it can be generated (through so called Ind-completion) from a <u>small</u> ∞-category so the idea is that we can write any object in a presentable ∞-category as colimits of a small amount of objects (i.e. objects from a small ∞-cat).

Ex:
- ∞-category of spaces $\mathcal{S}$ is presentable
- Sp is presentable
- $CycSp_p$ is presentable – why? NS. theorem regarding LEq

we don't use this if we don't show the colocalizing

Rem: An important property of being presentable is that it implies existence of all colimits (??) – in particular final objects.

Def: Let $\mathcal{C}$ be a presentable ∞-category. We will say that a t-structure on $\mathcal{C}$ is accessible if $\mathcal{C}_{\geq 0} \subseteq \mathcal{C}$ is presentable

Rem: It can be shown that this is equivalent to $\mathcal{C}_{\geq 0}$ being accessible – which explains the name.

When working with stable presentable ∞-categories, there is a rather easy way, to obtain a t-structure namely we get that for any small collection of objects $\{X_\alpha\}$ of $\mathcal{C}$, there exists a t-structure 'generated' by the objects X_α.

More precisely, we have the following result:

Thm: |HA.1.4.4.11| Let $\mathcal{C}$ be a presentable stable ∞-category.

1) If $\mathcal{C}' \subseteq \mathcal{C}$ is a full subcategory which is presentable, closed under small colimits and closed under extension. Then there exists a t-structure on $\mathcal{C}$ s.t. $\mathcal{C}' = \mathcal{C}_{\geq 0}$.

2) Let $\{X_\alpha\}$ be a small collection of objects of $\mathcal{C}$, and let $\mathcal{C}'$ be the smallest full subcategory of $\mathcal{C}$ which contains each X_α and is closed under extensions and small colimits. Then $\mathcal{C}'$ is presentable.

We are only going to sketch the proof of (1).

Pf of HA.1.4.4.11 Note that this is a mix of proof of HA.1.2.1.16 & 1.4.4.11.

- $\mathcal{C}'$ is a colocalization of $\mathcal{C}$, i.e $\mathcal{C}' \subseteq \mathcal{C}$ admits a right adjoint
$$R : \mathcal{C} \longrightarrow \mathcal{C}'$$
w. essential image $\mathcal{C}'$.

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "R"] \arrow[r, phantom, "\perp"] & \mathcal{C}' \arrow[l, bend left, "\text{ind}"] \end{tikzcd}$$

We claim that
$$\mathcal{C}_{\leq -1} = \{A \in \mathcal{C} \mid RA = 0\} \qquad \text{So } \mathcal{C}_{\leq 0} = \Omega\, \mathcal{C}_{\leq -1}$$
$$\mathcal{C}_{\geq 0} = \{A \in \mathcal{C} \mid RA = A\} \simeq \mathcal{C}'$$
determines a t-structure on $\mathcal{C}$.

1) Let $X \in \mathcal{C}_{\geq 0}$, $Y \in \mathcal{C}_{\leq -1}$. Then
$$\pi_0 \mathrm{Map}_{\mathcal{C}}(X, Y) = \mathrm{Ext}^0_{\mathcal{C}}(X, Y)$$
$$\overset{\substack{\text{adjunction}\\ \text{incl} \dashv R,\\ \text{using } X \in \mathcal{C}'}}{\simeq} \mathrm{Ext}^0_{\mathcal{C}}(X, RY)$$
$$\simeq \mathrm{Ext}^0_{\mathcal{C}}(X, 0)$$
$$\simeq 0$$

2) - $\mathcal{C}_{\geq 0}[1] \subseteq \mathcal{C}_{\geq 0}$:

We have assumed that $\mathcal{C}' = \mathcal{C}_{\geq 0}$ is stable under small colimits, so in particular under suspensions (since this suspension is a pushout)

- $\mathcal{C}_{\leq 0}[-1] = \mathcal{C}_{\leq -1} \subseteq \mathcal{C}_{\leq 0}$: We have that

$$R : \mathcal{C} \longrightarrow \mathcal{C}_{\geq 0} = \mathcal{C}'$$

is a colocalization, so it in particular preserves limit. Now, consider $\{A_i\} \in \mathcal{C}$ s.t. $RA_i \simeq 0$. Then

$$\begin{aligned} R(\lim_i A_i) &\simeq \lim_i RA_i \\ &\simeq \lim_i 0 \\ &\simeq 0 \end{aligned}$$

So $\mathcal{C}_{\leq -1}$ is stable under limits - in particular loops, so

$$\mathcal{C}_{\leq -1}[-1] \subseteq \mathcal{C}_{\leq -1}.$$

3) Let $X \in \mathcal{C}$. We then form a cofiber sequence

$$RX \longrightarrow X \longrightarrow X',$$

which equivalently is a fiber sequence since $\mathcal{C}$ is assumed stable. We then claim that $X' \in \mathcal{C}_{\leq -1}$, i.e. $RX' = 0$:

$$RX \simeq RRX \xrightarrow{\sim} RX \longrightarrow RX'$$

$$\Rightarrow RX' = 0.$$

§2: Cyclotomic t-structure

Recall: p-typical cyclotomic spectra

$$CycSp_p := LEq\left(\mathrm{id}, (-)^{tC_p} : Sp^{BS^1} \rightrightarrows Sp^{BS^1}\right).$$

$$\begin{tikzcd} CycSp_p \arrow[r] \arrow[d] \arrow[dr, phantom, "\circledast \ \lrcorner \text{ def. in sSet}"] & (Sp^{BS^1})^{\Delta^1} \arrow[d, "(ev_0, ev_1)"] \\ Sp^{BS^1} \arrow[r, "(\mathrm{id}, (-)^{tC_p})"] & Sp^{BS^1} \times Sp^{BS^1} \end{tikzcd}$$

so a p-typical cyclotomic spectrum is a spectrum X with S^1-action and an S^1-equivariant map $\varphi: X \to X^{tC_p}$, where X^{tC_p} carries the residual $S^1 \cong S^1/C_p$-action

Def: Write $(CycSp_p)_{\geq 0} \subseteq CycSp_p$ for the full subcategory of p-typical cyclotomic spectra where the underlying spectrum is connective.
$\pi_n X = 0 \ \forall n < 0$

Note: Using that for $X \in (CycSp_p)_{\geq 0}$, $\tau_{\geq 0} X \cong X$ we get that the cyclotomic structure map $\varphi: X \to X^{tC_p}$ canonically factors through $\tau_{\geq 0}(X^{tC_p})$:

$$\begin{tikzcd} \tau_{\geq 0} X \arrow[r] \arrow[d, "\cong"] & \tau_{\geq 0}(X^{tC_p}) \arrow[d] \\ X \arrow[r] \arrow[ur] & X^{tC_p} \end{tikzcd}$$

Ex: ① If R is a connective associative and unital ring spectrum, then

$$THH(R) \in (CycSp_p)_{\geq 0}$$

② Any spectrum X with trivial S^1-action can be canonically turned into $X^{triv} \in CycSp_p$ (David mentioned this), and if X furthermore is connective, then $X^{triv} \in (CycSp_p)_{\geq 0}$.

To understand $(CycSp_p)_{\geq 0}$ better and actually prove that it indeed determines a t-structure on CycSp_p, we need the following lemma:

Lem: 2.10 The following diagram

$$\begin{tikzcd}
(\mathrm{Sp}^{BS^1})_{\geq 0} \arrow[r, shift left, "\mathrm{id}"] \arrow[r, shift right, "\tau_{\geq 0}(-)^{tC_p}"'] \arrow[d] & (\mathrm{Sp}^{BS^1})_{\geq 0} \arrow[d] \\
\mathrm{LEq}\Big(\mathrm{Sp}^{BS^1} \arrow[r, shift left, "\mathrm{id}"] \arrow[r, shift right, "(-)^{tC_p}"'] & \mathrm{Sp}^{BS^1}\Big) =: \mathrm{CycSp}_p
\end{tikzcd}$$

induces via the natural transformations of functors

$$\tau_{\geq 0}(-)^{tC_p} \Rightarrow (-)^{tC_p}$$

an equivalence

$$\mathrm{LEq}(\mathrm{id}, \tau_{\geq 0}(-)^{tC_p} : (\mathrm{Sp}^{BS^1})_{\geq 0} \rightrightarrows (\mathrm{Sp}^{BS^1})_{\geq 0}) \simeq (\mathrm{CycSp}_p)_{\geq 0}.$$

In particular we get that $(\mathrm{CycSp}_p)_{\geq 0}$ is presentable and the inclusion functor

$$(\mathrm{CycSp}_p)_{\geq 0} \hookrightarrow \mathrm{CycSp}_p$$

preserves colimits.

Pf:

- Equivalence:

$$\mathrm{LEq}(\mathrm{id}, \tau_{\geq 0}(-)^{tC_p} : (\mathrm{Sp}^{BS^1}) \rightrightarrows (\mathrm{Sp}^{BS^1}))$$
$$= \{ (X \in (\mathrm{Sp}^{BS^1})_{\geq 0}, X \to \tau_{\geq 0}(X^{tC_p})) \}$$

$$(\mathrm{CycSp}_p)_{\geq 0} = (X \in \underbrace{\mathrm{Sp}^{BS^1} \text{ connective}}_{\text{i.e. } \in (\mathrm{Sp}^{BS^1})_{\geq 0}},\quad \begin{tikzcd} \tau_{\geq 0}X \arrow[r] & \tau_{\geq 0}(X^{tC_p}) \arrow[d] \\ X \arrow[u, "\cong"] \arrow[ur] \arrow[r, "\varphi"'] & X^{tC_p} \end{tikzcd}$$

- Presentable: This comes from the following general result NS18.II.1.5(3): Let $\mathcal{C}$ be presentable, $\mathcal{D}$ accessible and assume we have two functors

$$F, G : \mathcal{C} \to \mathcal{D}$$

with F colimit preserving and G accessible preserves the kind of colimits that "generates" $\mathcal{C}$ and $\mathcal{D}$.

Then $LEq(F, G)$ is presentable.

In our case it is obviouse that id preserves colimits, and it can be shown that $\tau_{\geq 0}(-)^{tCp}$ is accessible.

- Colimit preserving: Follows by considering the following diagram:

$$\begin{tikzcd}
\text{colimit} \arrow[rrr, mapsto] \arrow[ddd, mapsto] & & & \text{cocone} \arrow[ddd, mapsto] \\
& (CycSp_p)_{\geq 0} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & CycSp_p \arrow[d] & \\
& (Sp^{BS^1})_{\geq 0} \arrow[r, hook] & Sp^{BS^1} & \\
\text{colimit} \arrow[rrr, mapsto] & & & \text{colimit}
\end{tikzcd}$$

Can be shown to preserve and detects colimits

closed under colimits

Since $CycSp_p \longrightarrow Sp^{BS^1}$ detects colimits, we get that the cocone in $CycSp_p$ needs to be a colimit.

Thm: 2.1 The ∞-category $(CycSp_p)_{\geq 0}$ forms the connective part of an accessible, left complete t-structure on $CycSp$.

Pf: To see that $(CycSp_p)_{\geq 0}$ indeed forms the connective part of a t-structure on $CycSp_p$, we prove that it satisfies the assumptions of HA. 1.4.4.11:

- Closed under extension: First we note that a fiber sequence in $CycSp_p$ also gives a fiber sequence on the underlying spectra. So let

$$X \longrightarrow Y \longrightarrow Z$$

be a fiber sequence in $CycSp_p$. Assume $X, Z \in (CycSp_p)_{\geq 0}$. Then

$$\underset{\overset{\shortparallel}{0}}{\pi_n X} \longrightarrow \pi_n Y \longrightarrow \underset{\overset{\shortparallel}{0}}{\pi_n Z} \qquad \forall n < 0,$$

hence $\pi_n Y = 0 \ \forall n<0 \rightsquigarrow Y \in (\mathrm{CycSp}_p)_{\geq 0}$

- presentable & $(\mathrm{CycSp}_p)_{\geq 0} \hookrightarrow \mathrm{CycSp}_p$ closed under small colimits: Lemma AN.2.10 ✓

$\rightsquigarrow$ HA.1.4.4.11 implies that we have an accessible t-structure on CycSp_p given by the earlier defined $(\mathrm{CycSp}_p)_{\geq 0}$ as connective part, and

$$(\mathrm{CycSp}_p)_{\leq -1} = \{X \in \mathrm{CycSp}_p \mid RX \simeq 0\}.$$

Next we wish to show that CycSp_p w. this t-structure is left complete. We do this by showing that it is left seperated and that $(\mathrm{CycSp}_p)_{\geq 0}$ is closed under countable products. since then the desired follows from HA.1.2.1.19

- CycSp_p left seperated: We need to show that

$$(\mathrm{CycSp}_p)_{\geq \infty} := \bigcap_{n \in \mathbb{Z}} (\mathrm{CycSp}_p)_{\geq n}$$

is contractible. We consider each term

$$(\mathrm{CycSp}_p)_{\geq 0} = \{X \in \mathrm{CycSp}_p \mid \pi_n X = 0 \ \forall n<0\}$$

$$\begin{aligned}(\mathrm{CycSp}_p)_{\geq 1} &= \Sigma (\mathrm{CycSp}_p)_{\geq 0} \\ &= \{X \in \mathrm{CycSp}_p \mid \Sigma \pi_n X = \pi_{n+1} X = 0 \ \forall n+1<0\} \\ &= \{X \in \mathrm{CycSp}_p \mid \pi_n X = 0 \ \forall n<1\end{aligned}$$

$\vdots$

$$(\mathrm{CycSp}_p)_{\geq m} = \{X \in \mathrm{CycSp}_p \mid \pi_n X = 0 \ \forall n<m\}$$

and similarly

$$(\mathrm{CycSp}_p)_{\geq -m} = \{X \in \mathrm{CycSp}_p \mid \pi_n X = 0 \ \forall n<-m.$$

So we get that an object in $(\mathrm{CycSp}_p)_{\geq \infty}$ will need to have trivial homotopy groups.

- $(\mathrm{CycSp}_p)_{\geq 0} \subseteq \mathrm{CycSp}_p$ closed under (countable) products:
  - AN.A.17: $(\mathrm{Sp}^{BS^1})_{\geq 0} \subseteq \mathrm{Sp}^{BS^1}$ is closed under products.

We claim it then is sufficient to show that $\mathrm{CycSp}_p \to \mathrm{Sp}^{BS^1}$ commutes with products:

If we assume this claim and we again consider the diagram

want this to be closed under products

$$\begin{tikzcd}
(\mathrm{CycSp}_p)_{\geq 0} \arrow[r, hook] \arrow[d] & \mathrm{CycSp}_p \arrow[d, "F"] \\
(\mathrm{Sp}^{BS^1})_{\geq 0} \arrow[r] & \mathrm{Sp}^{BS^1}
\end{tikzcd}$$

commutes w. products ← claim

closed under products

Let $X_i \in \mathrm{CycSp}_p$ be a countable family, all connective. Then

$$F(\prod_i X_i) \simeq \prod_i (FX_i),$$

where each FX_i again is connective. Since $(\mathrm{Sp}^{BS^1})_{\geq 0} \hookrightarrow \mathrm{Sp}^{BS^1}$ is closed under products we then get that $\prod_i F(X_i)$ is connective.

$$\rightsquigarrow \prod_i X_i \in (\mathrm{CycSp}_p)_{\geq 0}.$$

So need to show the claim.

- $\mathrm{CycSp}_p \to \mathrm{Sp}^{BS^1}$ commutes w. products of connective objects:

  NS18.II.1.5(v) gives us that the forgetful functor out of a lax equializer preserves a certain a certain limit, if the "right map" in our case this is $(-)^{tC_p}$ in LEq preserves that limit. In our case this boils down to that it is sufficient to show that for a product $\prod X_i$ w. X_i connective spectrum w. S^1-action, the natural map

$$(\prod_i X_i)^{tC_p} \to \prod_i (X_i^{tC_p})$$

is an equivalence. This follows by AN.2.11. $\square$

Now that we know $(CycSp_p)_{\geq 0}$ forms a t-structure, we recall that it generates $(CycSp_p)_{\leq 0}$ by saying

$$Y \in (CycSp_p)_{\leq 0} \iff Map_{CycSp_p}(X, Y[-1]) \simeq 0 \quad \forall X \in (CycSp_p)_{\geq 0}.$$

So using that

$$TC(X,p) := map_{CycSp_p}(\mathbb{S}^{triv}, X)$$

and $\mathbb{S}^{triv} \in (CycSp_p)_{\geq 0}$, we see that a necessary condition for $X \in (CycSp_p)_{\leq 0}$ is that

$$\pi_i TC(X) = 0 \quad \forall i > 0.$$

It can be shown that

$$\pi_i TC(\mathbb{Z}^{triv}) \simeq \mathbb{Z}_p \quad \text{for all } i > 0 \text{ odd},$$

so $\mathbb{Z}^{triv} \notin (CycSp_p)_{\leq 0}$ – so $(CycSp_p)_{\leq 0}$ is not just those with underlying coconnective spectrum.