§2.4: Lubin-Tate Theories

Def: If R is complex orientable, then the stratification of the moduli of formal groups by height provides a sequence of ideals

$$\mathfrak{m}_0 \subset \mathfrak{m}_1 \subset \cdots \subseteq \pi_* R$$

in $\pi_* R$ known as the Landweber ideals.

Write $\bar{v}_k \in (\pi_* R)/\mathfrak{m}_{k-1}$ for the k^{th} Hasse invariant of $\mathbb{G}^Q$

- $\bar{v}_k$ cuts out the locus where $\mathbb{G}^Q$ has height at least $k+1$.

- $\bar{v}_k$ is of degree $2(p^k-1)$.

Quillen's Work

R: Complex orientable homotopy commutative ring spectrum $\rightsquigarrow$ Formal group G^Q over the graded ring $\pi_* R$

Choice of complex orientation $\rightsquigarrow$ Choice of coordinate on G^Q

Quillen Formal Group

Eq: $\pi_2(R)$ projective of rank 1 as $\pi_0 R$-module
(& $\pi_2(R) \otimes_{\pi_0(R)} \pi_n(R) \xrightarrow{\sim} \pi_{n+2}(R)$)

R - complex periodic $\mathbb{E}_\infty$-ring

& weakly 2-periodic
& complex orientable

Construction: Exist essentially unique symmetric monoidal functor

w. smash product relative to R

$$\mathcal{S} \longrightarrow \mathrm{Mod}_R$$
$$X \longmapsto C_*(X;R)$$

which preserves small colimits

$\rightsquigarrow$ $C_*(X;R)$: spectrum w. homotopy groups = R-homology groups of X

Quillen Formal Group

Rem: M: free abelian group of rank $r<\infty$

The coalgebra $C_*(K(M,2);R)$ is smooth of dimension r

Construction:

Free ab. grps of finite rank

smooth coalgebra over R

$$\mathrm{Lat}^{op} \longrightarrow \mathrm{cCAlg}^{sm}_R$$

$$M \longmapsto C_*(K(M^\vee,2);R)$$

$\rightsquigarrow$ commutes w. finite products

$\rightsquigarrow$ Regard as abelian group object of cCAlg^{sm}_R

cospectrum

$\rightsquigarrow$ $M \mapsto \mathrm{cSpec}(C_*(K(M^\vee,2);R))$ Abelian group object of $\mathrm{Hyp}(R)$

hyperplanes over R

Formal group $\widehat{\mathbf{G}}^Q_R$ over R

Quillen Formal group over R

Quillen Formal Group – Classical

$\mathbb{G}_R^{Q_0} :=$ Image of $\mathbb{G}_R^{Q}$ under $\mathrm{FGrp}(R) \xrightarrow{\text{forgetful}} \mathrm{FGrp}(\pi_0 R)$

Classical Quillen formal group of R

Concretely:

$$\mathbb{G}_R^{Q_0} = \mathrm{Spf}(R^0(\mathbb{CP}^\infty)) \simeq c\mathrm{Spec}(R_0(\mathbb{CP}^\infty))$$

Note: Do not actually need R to be $\mathbb{E}_\infty$

Moduli of formal groups

$\mathcal{L} = \pi_* MU$ Lazard ring

G^+: Group scheme given by

$$G^+(R) := \{ g \in R[[x]] \mid g(t) = b_1 t + b_2 t^2 + \dots \text{ , w. } b_1 \in R^\times \}$$

$G^+ \circlearrowright \operatorname{Spec} \mathcal{L}$

$\rightsquigarrow$ Quotient stack $\mathcal{M}_{fg} = \operatorname{Spec}(L)/G^+$

$f \in FGL(R) \rightsquigarrow \mathcal{G}_f : Alg_R \longrightarrow Ab$

formal group associated to f

$A \longmapsto \{ a \in A : \exists n \text{ s.t. } a^n = 0 \} \subseteq A$, $(a,b) \mapsto f(a,b)$

group str.

makes sense, since a,b both nilpotent so only finitely many nonzero terms

$f \in FGL(R) \iff \mathcal{G}_f(A)$ group for every A

Formal group law over R = $\mathcal{G} : Alg_R \to Ab$ s.t. ...

$\mathcal{M}_{fg}(R)$ = Category of formal group laws of R

Stratification by height $\mathcal{M}_{fg} = \operatorname{Spec}(L)/G^+$

Construct closed substacks of $\mathcal{M}_{fg} \times \operatorname{Spec}\mathbb{Z}_{(p)}$

- $\mathcal{M}_{fg}^{\geq n} := \operatorname{Spec}(L_{(p)}/(v_0, \ldots, v_{n-1}))/G^+$ height at least n
- $\mathcal{M}_{fg}^{=n} := \mathcal{M}_{fg}^{\geq n} - \mathcal{M}_{fg}^{\geq n+1} = (\operatorname{Spec} L_{(p)}[v_n^{-1}]/(v_0, \ldots, v_{n-1}))/G^+$ height n $\operatorname{Spec}\overline{\mathbb{F}}_p/\mathbb{G}_n$
- $\mathcal{M}_{fg}^{\infty} := (\operatorname{Spec} L/(v_0, v_1, \ldots))/G^+$ infinite height

$\rightsquigarrow$ $\mathcal{M}_{fg}^{n}$ ~ open strata for a stratification of $\mathcal{M}_{fg}$

" $\bar{v}_k \in (\pi_* R)/\mathfrak{m}_{k-1}$ = k^{th} Hasse invariant of G^{Q} " ?

Brauer group: Isomorphism classes of central division algebras over a fixed field K

$Br(K)$

Group structure: Morita equivalence classes

$\mathrm{Inv}: Br(\mathbb{F}_p) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z} \cong Br(\mathbb{Q}_p)$ Can extend p-adic valuation to a central division algebra

$\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathrm{End}(H_n/\mathbb{F}_p) \mapsto \frac{1}{n}$

" $\mathrm{End}(H_n/\mathbb{F}_p) \cong$ Ring of integers in the central division $\mathbb{Q}_p$-algebra of Hasse invariant $1/n$ "

Hasse Invariant   Lurie: Elliptic Cohomology II

Def: $X := \Omega^\infty \circ \mathbb{G}$ ~ underlying formal hyperplane of $\mathbb{G} \in \mathcal{M}_{fg}^{\geq n}(R)$

$\eta \in X(\tau_{\geq 0} R)$ basepoint classified by an augmentation

The identity w.r.t. the group structure on $\mathbb{G}$

$\varepsilon : \mathcal{O}_X \to R$ over η

$\mathcal{O}_X(-\eta) :=$ fiber of ε considered as a module over $\mathcal{O}_X$

$\omega_{\mathbb{G}} := R \otimes_{\mathcal{O}_X} \mathcal{O}_X(-\eta)$ Dualizing line of $\mathbb{G}$

- Locally free R-module of rank 1
- $\omega_{\mathbb{G}}$ depends functorially on R and $\mathbb{G}$

$R \to R' \Rightarrow \omega_{\mathbb{G}_{R'}} \simeq R' \otimes_R \omega_{\mathbb{G}}$

$f : \mathbb{G} \to \mathbb{G}' \Rightarrow f^* : \omega_{\mathbb{G}'} \to \omega_{\mathbb{G}}$ eq. iff. f is an eq.

- Canonical fiber sequence of R-modules

$$\Sigma(\omega_{\mathbb{G}}) \to R \otimes_{\mathcal{O}_X} R \longrightarrow R$$

Ex: R commutative $\Rightarrow \omega_{\mathbb{G}} =$ R-linear dual of $\mathrm{Lie}(\mathbb{G})$

Hasse Invariant - Dualizing line of $\mathbb{G}^Q$

$\omega_{\mathbb{G}^Q_R}$ is characterized by the existence of the fiber sequence

Complex periodic $\mathbb{E}_\infty$-ring

$$\Sigma(\omega_{\mathbb{G}^Q_R}) \to R \otimes_{C^*(\mathbb{C}P^\infty; R)} R \xrightarrow{m} R$$

?? $\quad C^*(\mathbb{C}P^\infty; R) \simeq C^*(S^1; R)$

Evaluation at the base point of S^1

$$\rightsquigarrow \quad \Sigma(\omega_{\mathbb{G}^Q_R}) \simeq C^*_{red}(S^1; R) \simeq \Sigma^{-1} R$$

As R-modules

$$\rightsquigarrow \quad \omega_{\mathbb{G}^Q_R} \simeq \Sigma^{-2}(R)$$

Locally free of rank 1

Hasse Invariant

$\varphi_G : G \to G^{(p)}$ (relative) Frobenius map:

$G^{(p^n)}(x) := G(x^{1/p^n})$

$\varphi_x^n : x \to x^{1/p^n} \rightsquigarrow \varphi_G^n : G \to G^{(p^n)}$

<u>Prop:</u>

$f: G \to G'$ morphism of 1-dim. FG's over R.

Then TFAE:

- The pullback map $f^* : \omega_{G'} \to \omega_G$ vanishes
- f factors as a compositon $G \xrightarrow{\varphi_G} G^{(p)} \xrightarrow{T} G'$

If these holds, then T is moreover unique

$G \in \mathcal{M}_{fg}^{\geq n}(R)$

For $n > 0$: p vanishes in R, hence

$$\begin{tikzcd} G \arrow[rr, "{[p]}"] \arrow[dr, "\varphi_{G^n}"'] & \arrow[d, phantom, "\circlearrowleft"] & G \\ & G^{(p^n)} \arrow[ur, "T"', "\text{unique}"] & \end{tikzcd} \rightsquigarrow T^* : \omega_G \to \omega_{G^{(p^n)}} \simeq \omega_G^{\otimes p^n}$$

$$\rightsquigarrow v_n \in \omega_G^{\otimes (p^n - 1)}$$ **n^{th} Hasse Invariant**

- Can view v_n as an R-module homomorphism $f_{v_n} : \omega_G^{\otimes(1-p^n)} \to R$
- **v_n vanishes iff G has height $\geq n+1$**

For $n = 0$: Set $v_0 := p \in R \simeq \omega_G^{\otimes (p^0 - 1)}$

- Can be identified w. the endomorphisms of ω_G induced by $[p] : G \to G$

Landweber Ideal

$\mathbb{G} \in \mathcal{M}_{fg}(R)$, R commutative

Thm: There exists a finitely generated ideal $\mathfrak{m}_n \subseteq R$ $\forall n \geq 0$ s.t.

$R \to R'$ annihilates $\mathfrak{m}_n$ iff $\mathbb{G}$ has height $\geq n$

Proof: Descending induction on m — largest element $\{0, \ldots, n\}$ s.t. $\mathbb{G}$ has height $\geq m$

- If $m = n \Rightarrow \mathbb{G} \in \mathcal{M}_{fg}^{\geq n}(R) \rightsquigarrow$ Set $\mathfrak{m}_n = (0)$
- Otw: v_m m-th Hasse invariant

$$f_{v_m} : \omega_{\mathbb{G}}^{\otimes(1-p^m)} \longrightarrow R$$

$\rightsquigarrow \operatorname{Im}(f_{v_m}) \subseteq R$ finitely generated ideal

$\rightsquigarrow \mathbb{G}_{R/\operatorname{Im}(f_{v_m})}$ height $\geq m+1$. since vanishing m^{th} Hasse invariant

$\rightsquigarrow$ Exists fin. gen. ideal $I \subseteq R/\operatorname{Im}(f_{v_m})$ s.t.

$R/\operatorname{Im}(f_{v_m}) \to R'$ annihilates I iff

$\mathbb{G}_{R'}$ has height $\geq n$

$\rightsquigarrow$ Let $\mathfrak{m}_n := \operatorname{Im}^{-1}(R \longrightarrow R/\operatorname{Im}(f_{v_m}))$ m^{th} Landweber ideal

Landweber Ideal

$\mathfrak{m}_n^{\mathbb{G}} = \mathrm{Im}^{-1}(R \to R/\mathrm{Im}(f_{v_m})) \subseteq R$

$\rightsquigarrow$ $\mathbb{G}$ determines formal group $\mathbb{G}_0$ over $\pi_0(R)$

variant

R $\mathbb{E}_\infty$-ring Set $\mathfrak{m}_n^{\mathbb{G}} := \mathfrak{m}_n^{\mathbb{G}_0} \subseteq \pi_0(R)$ $\mathbb{G}$ has height n if $\mathfrak{m}_n^{\mathbb{G}} = \pi_0 R$

If R commutative, $\mathbb{G} \in M_{\mathbb{G}}^{fg}(R)$, $\omega_{\mathbb{G}}$ trivial $\sim\ \cong R$ as R-modules

$\mathfrak{m}_\eta$ ~~$\mathfrak{m}_n$~~ is generated by n elements

Intuition: Generated by the Hasse invariants $v_0 = p, v_1, -, v_{n-1}$, modulo that each v_m is only well-defined modulo the ideal $(v_0, -, v_{m-1})$

The construction of v_m requires $\mathbb{G}$ has height $\geq m$

Landweber Ideal - For $\mathbb{G}^{\mathcal{Q}}$

- R complex periodic $\rightsquigarrow$ $\mathfrak{m}_n^R := \mathfrak{m}_n^{\mathbb{G}_R^{\mathcal{Q}}}$ n^{th} Landweber ideal of R
- Extending scalars along $\pi_0 R \to \pi_0 R / \mathfrak{m}_n^R$
  $\rightsquigarrow$ New formal group $\mathbb{G}_R^{\mathcal{Q}_n}$
  - ↳ height $\geq n$
  - ↳ dualizing line is given by $\pi_2(R)/\mathfrak{m}_n^R\, \pi_2(R)$
- v_n n^{th} Hasse invariant of $\mathbb{G}_R^{\mathcal{Q}_n}$
  $\rightsquigarrow$ Regard as an element of $\pi_{2p^n-2}(R)/\mathfrak{m}_n^R\, \pi_{2p^n-2}(R)$
  $\rightsquigarrow$ Write $\bar{v}_n \in \pi_{2p^n-2}(R)$ for any lift of v_n

$\mathfrak{m}_{n+1}^R$ is generated by $\mathfrak{m}_n^R + \bar{v}_n\, \pi_{2-2p^n}(R)$

stratification of the moduli of formal groups by height provides a sequence of ideals

$$\mathfrak{m}_0 \subset \mathfrak{m}_1 \subset \cdots \subseteq \pi_* R$$

in $\pi_* R$ known as the Landweber ideals.

$$\mathfrak{m}_0 = (v_0)$$

$$\mathfrak{m}_1 = (v_0, v_1)$$

$$\vdots$$

Back to the paper!

Def: If R is complex orientable, then the stratification of the moduli of formal groups by height provides a sequence of ideals

$$\mathfrak{m}_0 \subset \mathfrak{m}_1 \subset \cdots \subseteq \pi_* R$$

in $\pi_* R$ known as the Landweber ideals.

Write $\bar{v}_k \in (\pi_* R)/\mathfrak{m}_{k-1}$ for the k^{th} Hasse invariant of $\mathbb{G}^Q$

- $\bar{v}_k$ cuts cut the locus where $\mathbb{G}^Q$ has height at least $k+1$.

- $\bar{v}_k$ is of degree $2(p^k-1)$.

Conventions

- R complex orientable $\rightsquigarrow$ Fix lifts $v_0, v_1, \ldots \in \pi_* R$ of Hasse invariants
- R complex periodic $\rightsquigarrow$ Reformulate $\mathbb{G}^Q$ as formal group on $\pi_0 R$
  w. $\omega_{\mathbb{G}^Q} \cong \pi_2(R)$

  \+ Even $\rightsquigarrow$ $\pi_*(R) \cong \begin{cases} \omega_{\mathbb{G}^Q}^{\otimes k} & * = 2k \\ 0 & \text{otw} \end{cases}$
- If $\pi_*(R)$ has a unit in degree 2: Write u for such a choice
  $\rightsquigarrow$ Trivialization of $\omega_{\mathbb{G}^Q}$ $\rightsquigarrow$ Push v_i into degree zero
  write u_i instead of v_i

  $$\boxed{v_i = u_i u^{p^i - 1}}$$
- Focus on height n $\rightsquigarrow$ $m := m_{n-1}$ Landweber ideal which cuts out the locus of height $\geq n$

Lubin-Tate Theories

Goerss-Hopkins-Miller: k perfect field, $\mathrm{char}(k) = p > 0$, $\mathbb{G} \in \mathcal{M}^n_{fg}(k)$:

Exists complex periodic $\mathbb{E}_\infty$-ring $E(k, \mathbb{G}) = E$

In $FG = \left\{ \begin{array}{l} (R, \mathbb{G}),\ R \text{ com. ring} \\ \qquad\qquad \mathbb{G} \text{ FG} \\ + \text{ morphisms} \end{array} \right.$

$+ \quad \alpha : (k, \mathbb{G}) \xrightarrow{\sim} (\pi_0(E)/\mathfrak{m}^E_{\mathrm{LT}}, \mathbb{G}_E^{Q_n})$

s.t.

- $E(k, \mathbb{G})$ is even periodic, $K(n)$-local
- $(\pi_0(E), \mathbb{G}_E^{Q_0}) \to (\pi_0(E)/\mathfrak{m}^E_{\mathrm{LT}}, \mathbb{G}_E^{Q_n}) \xrightarrow{\alpha^{-1}} (k, \mathbb{G})$

  exhibits $\mathbb{G}_E^{Q_0}$ as the universal deformation of $\mathbb{G}$

  ↳ In particular: $\pi_0(E)$ = Lubin-Tate ring of $\mathbb{G}$.

⋮

Lubin-Tate Theories

$E(k, \mathbb{G})$ Lurie generalizes $\rightsquigarrow$ $E(-;-) : \int_{perf} \mathcal{M}^n_{fg} \longrightarrow \mathrm{CAlg}(\mathrm{Sp}_{T(n)})$

$(A, \mathbb{H}) \longmapsto E(A; \mathbb{H}_0)$

perfect $\mathbb{F}_p$-algebra

Formal group of height exactly n

Thm: 2.27

1) $E(-;-)$ fully faithful
2) $E(A, \mathbb{H})$ is $K(n)$-local & even periodic
3) Any sequence $v_0, \ldots, v_{n-1} \in \pi_* E(A; \mathbb{H})$ lifting the Hasse invariants is regular

In particular the Landweber ideals

4) $\mathrm{Im}\, E(-;-) := \{ R \in \mathrm{CAlg}$ satisfying (2), (3) & $\pi_0(R)/m$ perfect$\}$
5) Natural identification

$$(\pi_0 E(A;\mathbb{H})/m \, , \; \mathbb{G}^Q_{\pi_0 E(R;\mathbb{H})/m}) \sim (A; \mathbb{H})$$

which, when A is a perfect field, exhibits $\mathbb{G}^Q$ as a universal deformation of $\mathbb{H}_0$.

Derived Quotient

Construction:

R even commutative ring, $x \in \pi_* R$

$\rightsquigarrow$ $\mathbb{E}_1$-R-algebra R/x

For us: R complex orientable, $v_0, v_1, \ldots \in \pi_* R$ lifts of Hasse invariant

$\rightsquigarrow$ $\mathbb{E}_1$-R-algebra $R/\!/\mathfrak{m}_j$

Underlying R-module

$$(R/v_0) \otimes_R \cdots \otimes_R (R/v_j)$$

S commutative R-algebra

In the paper it says '$\mathbb{E}_1$-A-algebra'?

$\rightsquigarrow$ $\mathbb{E}_1$-R-algebra $S/\!/\mathfrak{m}_j := R/\!/\mathfrak{m}_j \otimes_R S$ obtained through base-change

Note: S $T(n)$-local $\Rightarrow$ $S/\!/\mathfrak{m}_j$ $T(n)$-local

Homotopy of Morava K-theory

A : perfect $\mathbb{F}_p$-algebra , $\mathbb{H}_0 \in \mathcal{M}_{fg}^{=n}(A)$

$\rightsquigarrow E(A;\mathbb{H}_0) \in \mathrm{CAlg}(\mathrm{Sp}_{T(n)})$ Lubin-Tate theory

$\rightsquigarrow E(A;\mathbb{H}_0)/\!/\mathfrak{m}$ Morava K-theory

$$\pi_*(E(A;\mathbb{H}_0)/\!/\underset{\overset{\shortparallel}{\mathfrak{m}_{n-1}}}{\mathfrak{m}}) \cong \begin{cases} \omega_{\mathbb{H}_0}^{\otimes k} & * = 2k \\ 0 & \text{otw} \end{cases}$$

Since the Landweber ideal in $\pi_*(E(A;\mathbb{H}_0))$ is generated by a regular sequence

Lemma 2.29: Assume $R \in CAlg(Sp_{T(n)})$, $V_n \otimes R$ even for some type n generalized Moore spectrum V_n. Then

1) $R \otimes V$ is even for all generalized Moore spectrum V
2) R is $K(n)$-local, even & complex orientable
3) $u_0, \dots, u_{n-1}$ is regular in $\pi_* R$ & $\pi_*(R/\!\!/\mathfrak{m}) \cong \pi_*(R)/\mathfrak{m}$

Helps us recognise if $R \simeq E(A, \mathbb{H}_0)$ for some $A, \mathbb{H}_0$.

Recall

- X type n spectrum: $K(n)_*(X) \neq 0$, $K(m)_*(X) \simeq 0$ for $m < n$.
- Moore spectrum for $G \in Ab$: The spectrum MG characterized by
  - $\pi_{<0} MG = 0$
  - $\pi_0 MG = G$
  - $H_{>0}(MG, \mathbb{Z}) = \pi_{>0}(MG \wedge H\mathbb{Z}) = 0$

Type n generalized Moore Spectrum

- Brown-Peterson spectrum BP — Characterized by $\pi_* BP = BP_* \cong \mathbb{Z}_{(p)}[v_1, v_2, \_\_]$, $|v_k| = 2(p^k - 1)$

  proper ideal, $p^{e_0} \neq 0$, $v_i^{e_i}$ not a zero-divisor on $BP_*/(p^{e_0}, v_1^{e_1}, \_\_ v_{i-1}^{e_{i-1}})$ for $i>0$

- Invariant ideal $\mathcal{J}_n = (p^{e_0}, v_1^{e_1}, \_\_, v_{n-1}^{e_{n-1}}) \subset BP_*$ — e_i positive integers, $n \geq 1$

  $e_0 - 1 \leq e_1$, $s_i \leq p^{e_{i+1}-e_i-e_0+1}$, $1 \leq i \leq n$ $*$??

- Type n generalized Moore spectrum: Spectrum $M\mathcal{J}$

$$BP_*(M\mathcal{J}) = BP_*/\mathcal{J}_n$$

Theorem: For each invariant ideal $\mathcal{J}_n$ there exists a type n generalized Moore spectrum $M\mathcal{J}'_n$ w. $\mathcal{J}'_n \subset \mathcal{J}_n$ invariant ideal of the form $*$

For us:

$$M\mathcal{J}_n = V_n = \mathbb{S}/(p^{i_0}, v_1^{i_1}, \_\_, v_{n-1}^{i_{n-1}})$$

Lemma 2.29 - Proof

$R \in \mathrm{CAlg}(\mathrm{Sp}_{T(n)})$, $V_n \otimes R$ even

Claim: R is even

$V_n = \mathbb{S}/(p^{i_0}, v_1^{i_1}, \ldots, v_{n-1}^{i_{n-1}}) \rightsquigarrow$ Tower of V_m's s.t. $V_m = V_{m-1}/v_{m-1}^{i_{m-1}}$

Tactic: $V_m \otimes R$ even by downward induction

$m = n$: Assumption on R

Induction step: Assume $V_{m+1} \otimes R$ even $\rightsquigarrow$ $v_m^{i_m}$-Bockstein tower

$$\begin{tikzcd}
 & \Sigma^{2|v_m^{i_m}|} V_m/v_m^{i_m} \otimes R \arrow[d] & \Sigma^{|v_m^{i_m}|} V_m/v_m^{i_m} \otimes R \arrow[d] & \\
\cdots \arrow[r] & V_m/v_m^{3i_m} \otimes R \arrow[r] & V_m/v_m^{2i_m} \otimes R \arrow[r] & V_m/v_m^{i_m} \otimes R
\end{tikzcd}$$

- $T(n)$-local inverse limit ( ) $\simeq V_m \otimes R$
- $V_{m+1} = V_m/v_m^{i_m}$ even $\rightsquigarrow$ Every term is even (Extension of even objects is even)

$\rightsquigarrow$ Maps are surjective on homotopy groups $\rightsquigarrow$ Inverse limit is even

Case $V_0 = \mathbb{S}$ $\rightsquigarrow$ R even

Lemma 2.29 - Proof $R \in \mathrm{CAlg}(\mathrm{Sp}_{T(n)})$, $v_n \otimes R$ even, R even

Claim: R is $K(n)$-local & complex orientable

Even $\rightsquigarrow$ Complex orientable

Complex orientable $\rightsquigarrow$ [$T(n)$-local $\Rightarrow$ $K(n)$-local]
Bousfield localization functors & Hopkins' zeta conjecture, 1.10, Hovey

X finite type n spectrum: $A_n X = \mathrm{fib}(\mathrm{Tel}(X) = L_n^f(X) \to L_n(X) = L_{K(n)} X)$

Bousfield class of $A_n X$ does not depend on X: Write $A(n)$

$E \sim F \iff E \wedge X = 0$ iff $F \wedge X$ $\forall X$

$A(n) \wedge K(m) = 0$ $\forall m$, since $L_n^f X \to L_n X$ iso on $K(m)$-homology $\forall m$

Prop: $\langle BP \wedge \mathrm{Tel}(n) \rangle = \langle K(n) \rangle$,

$\rightsquigarrow$ $\langle BP \wedge A(n) \rangle \overset{?}{=} \langle BP \wedge \mathrm{Tel}(n) \wedge A(n) \rangle = \langle K(n) \wedge A(n) \rangle = 0$

$\rightsquigarrow$ $L_n^f X \to L_n X$ BP-equivalence

Lemma 2.29 - Proof

$R \in \mathrm{CAlg}(\mathrm{Sp}_{T(n)})$, $V_n \otimes R$ even, R even, $K(n)$-local, complex orientable

Claim: $v_0, \ldots, v_{n-1}$ is regular in $\pi_* R$ & $\pi_*(R/\!/\mathfrak{m}) \cong \pi_*(R)/\mathfrak{m}$

R complex orientable, so can pick classes v_i in $\pi_* R$

Cofiber sequence

$$\Sigma^{|v_m^{i_m}|} V_m \otimes R \xrightarrow{v_m^{i_m}} V_m \otimes R \longrightarrow V_{m+1} \otimes R$$

LES on homotopy + all even terms $\rightsquigarrow$ $v_m^{i_m}$ acts injectively on $\pi_*(V_m \otimes R)$

Inducting on m $\rightsquigarrow$ $\begin{cases} \pi_*(V_m \otimes R) \cong (\pi_* R)/(p^{i_0}, v_1^{i_1}, \ldots, v_{m-1}^{i_{m-1}}) \\ p^{i_0}, v_1^{i_1}, \ldots, v_{m-1}^{i_{m-1}} \text{ regular in } \pi_* R \end{cases}$

$\rightsquigarrow$ $p, v_1, \ldots, v_{m-1}$ regular in $\pi_* R$

Claim: $R \otimes V$ is even for all generalized Moore spectra V

Follows by LES on homotopy $\square$

Corollary 2.30

Assume $R \in \mathrm{CAlg}(\mathrm{Sp}_{T(n)})$ s.t. $R \otimes V$ even for some type n generalized Moore spectrum V. Then

Lemma 2.29

R is even and we have the $\mathbb{E}_1$-R-algebra $R/\!/\mathfrak{m}$.

If furthermore $\pi_*(R/\!/\mathfrak{m})$ is an even periodic algebraically closed field, then $R \cong E(\pi_0(R/\!/\mathfrak{m}))$.

Theorem 2.27

$(\pi_0 E(A;\mathbb{H})/\mathfrak{m}, \; \mathbb{G}^{Q}_{\pi_0 E(R;\mathbb{H})/\mathfrak{m}}) \sim (A;\mathbb{H})$, $\mathbb{G}^{Q}$ universal deformation

When over an algebraically closed field, all formal groups of height n are isomorphic & we therefore drop the formal group from our notation

Relative Lubin-Tate

Convention for rest of paper

Fix
- perfect field k
- Formal group G of height n over k

$\rightsquigarrow E(k) = E(k; G)$

Fix a choice of unit $u \in \pi_2 E(k)$

$\rightsquigarrow$ Determines $u_0 = p, u_1, \ldots, u_{n-1} \in \pi_0 E(k)$

& $\pi_0 E(k) \cong W(k)[[u_1, \ldots, u_{n-1}]][u^{\pm 1}]$

Definition Relative version of Lubin-Tate

$$E(-) : \mathrm{Perf}_k \longrightarrow \mathrm{CAlg}^{\wedge}_{E(k)}$$

$$(k \to A) \longmapsto E(A) = E(A; G_A)$$

5.4.2, Elliptic cohomology II $\rightsquigarrow$ $\pi_* E(A) \overset{\text{can}}{\cong} W(A)[[u_1, \ldots, u_{n-1}]][u^{\pm 1}]$ w. our conventions

↑ canonical

Corollary 2.33

$R \in \mathrm{CAlg}^{\wedge}_{E(k)}$ is in the essential image of

$$E(-) : \mathrm{Perf}_k \longrightarrow \mathrm{CAlg}^{\wedge}_{E(k)}$$

if & only if $R/\mathfrak{m}$ is even & $\pi_0(R)/\mathfrak{m}$ is perfect

$R \simeq E(A) \Rightarrow$ $R/\mathfrak{m}$ even, $\pi_0(R)/\mathfrak{m}$ perfect

Proof:

Only if: $R \simeq E(A) \rightsquigarrow E(A)/\mathfrak{m}$ even

& $\pi_0(E(A))/\mathfrak{m} \cong W(A)[[v_0, \dots, v_{n-1}]][u^{\pm 1}]/\mathfrak{m}$ perfect

If: $R/\mathfrak{m}$ even, $\pi_0(R)/\mathfrak{m}$ perfect

$\rightsquigarrow R$ even, $v_0, \dots, v_{n-1}$ is regular

$\overset{2.27(4)}{\rightsquigarrow}$ $\boxed{R \simeq E(A)}$