Adapted homology theories & deformations

Reference: "Adams spectral sequence & Franke's algebraicity conjecture" by Patchkoria & Pstrągowski

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Assume throughout that

- $\mathcal{C}$ is a stable ∞-category
- $\mathcal{A}$ is an abelian category with enough injectives.

1) Adapted homology theory

Def A functor $H : \mathcal{C} \to \mathcal{A}$ is homological if it is additive, and for all cofiber sequences

$$c \to d \to e$$

in $\mathcal{C}$,

$$H(c) \to H(d) \to H(e)$$

is exact in $\mathcal{A}$.

Def A local grading on some ∞-category $\mathcal{D}$ is an equivalence

$$[1] : \mathcal{D} \xrightarrow{\sim} \mathcal{D},$$

and we write $[n] := \underbrace{[1] \circ \cdots \circ [1]}_{n}$

Ex:
- $\mathcal{C}$ stable, $[1] := \Sigma : \mathcal{C} \to \mathcal{C}$
- R_* a graded ring, then Mod_{R_*} has a local grading

$$(M[1])_* := M_{*-1}.$$

Def: A homology theory on $\mathcal{C}$ is a homological functor with a isomorphism $H \circ \Sigma \cong [1] \circ H$

So a functor of locally graded ∞-categories with underlying homological functor

Ex: $H_*(-,A): Sp \longrightarrow Vect(A)$

$$X \longmapsto \pi_*(X \wedge HA)$$

Def: A homology theory $H: \mathcal{C} \to \mathcal{A}$ is adapted if

- For any injective $i \in \mathcal{A}$ there exists an object $i_{\mathcal{C}} \in \mathcal{C}$ — injective lift associated to i —

  together with a map $H(i_{\mathcal{C}}) \to i$ (structure map of the lift) s.t. for any $c \in \mathcal{C}$, the induced composite

  $$[c, i_{\mathcal{C}}] \to Hom_{\mathcal{A}}(H(c), H(i_{\mathcal{C}})) \longrightarrow Hom_{\mathcal{A}}(H(c), i)$$

  is an isomorphism of abelian groups — i.e.: $i_{\mathcal{C}}$ represents the the functor $Hom_{\mathcal{A}}(H(-), i)$, $\mathcal{C}^{op} \to \mathcal{A}b$ in $h\mathcal{C}$.

  Every injective lifts

- The structure map $H(i_{\mathcal{C}}) \to i$ is an isomorphism

Thm: | Classification of adapted homology theories | If $\mathcal{C}$ is idempotent complete stable ∞-category, then an adapted homology theory $H: \mathcal{C} \to \mathcal{A}$ is uniquely determined by the epimorphism class of $\mathcal{A}$ consisting of the H-epimorphisms $f \in \mathcal{C}(c,d)$ H-epi $\Leftrightarrow$ $H(f)$ epi in $\mathcal{A}$

Thm: | Adams SS | Let $H: \mathcal{C} \to \mathcal{A}$ be an adapted homology theory Then $\forall c, d \in \mathcal{C}$ there exists a spectral sequence

$$E_2^{s,t} = Ext^s_{\mathcal{A}}(H(c), H(d)[-t])$$

which 'sometimes' converges to $[c, d]$

↳ E.g. if H is conservative & $\mathcal{A}$ has finite cohomological dimension.

Pf: Let $d \in \mathcal{C}$ be some object and write $d^0 := d$. Since $\mathcal{A}$ has enough injective we have a monomorphism

$$H(d^0) \hookrightarrow i^0$$

into some injective $i^0 \in \mathcal{A}$. Using that H is adapted this lifts to a morphism

$$j^0 : d^0 \longrightarrow i^0_{\mathcal{C}}$$

injective lift of i^0

since $[d^0, i^0_{\mathcal{C}}] \xrightarrow[H]{\cong} [H(d^0), i^0]$. Now set

$$d^1 := \mathrm{cofib}(j^0 : d^0 \longrightarrow i^0_{\mathcal{C}}),$$

and repeat the construction so we obtain the diagram

$$\begin{tikzcd}
d^0 \arrow[r, "j^0"'] & i^0_{\mathcal{C}} \arrow[r] \arrow[d] & i^1_{\mathcal{C}} \arrow[r] \arrow[d] & i^2_{\mathcal{C}} \arrow[r] & \cdots \\
& d^1 \arrow[ul] \arrow[ur, "j^1"'] & d^2 \arrow[l] \arrow[ur, "j^2"'] & \arrow[l] &
\end{tikzcd}$$

$\mathrm{cofib}(j^0)$ $\quad$ $\mathrm{cofib}(j^1)$

Adams resolution

which satisfies that if we apply $H(-)$ to the top row we obtain an injective resolution of $H(d^0)$. Applying $[c,-]_*$ to this diagram for some $c \in \mathcal{C}$ we obtain the diagram

$\mathrm{Hom}(H(c), i^0)$

$\cong$ by adaptness

$$\begin{tikzcd}
{[c,d^0]} \arrow[r] & {[c,i^0_{\mathcal{C}}]_*} \arrow[r] \arrow[d, "k_0"] & {[c,i^1_{\mathcal{C}}]_*} \arrow[r] \arrow[d, "k^1"] & \cdots \\
& {[c,d^1]_*} \arrow[ul] \arrow[ur, "j^1"'] & {[c,d^2]_*} \arrow[l] \arrow[ur, "j^2"'] & \arrow[l]
\end{tikzcd}$$

By this we obtain an exact couple

$$D^{s,t} = [c, \Sigma^{s-t} d^s]$$

$$E^{s,t} = \mathrm{Hom}_{\mathcal{A}}(H(c), i^s[s-t])$$

This bigrading is the Adams grading

$\deg(\partial) = (-1,-1)$, $\deg(j) = (0,0)$, $\deg(k) = (1,0)$

$\rightsquigarrow$ There is a spectral sequence with

H-Adams spectral sequence

$$E_2^{s,t} = H_*(E, j \circ k) = \mathrm{Ext}^s(H(c), H(d)[-t])$$

2) The Freyd envelope

An adapted homology theory can further be classified by a certain factorization through a certain abelian category:

Recall that we write $\mathrm{Fun}_\Sigma(\mathcal{C}^{op}, \mathrm{Ab}) \subseteq \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Ab})$ for the full subcat of spherica presheaves, and that we have

$$\begin{aligned} y : \mathcal{C} &\longrightarrow \mathrm{Fun}_\Sigma(\mathcal{C}^{op}, \mathrm{Ab}) \\ c &\longmapsto y(c) : \mathcal{C}^{op} \longrightarrow \mathrm{Ab} \\ & \qquad\qquad d \longmapsto y(c)(d) := [d,c] \end{aligned}$$

$\pi_0 \mathrm{Map}_{\mathcal{C}}(d,c)$!!

associated representable

<u>Def:</u> The **Freyd envelope** $A(\mathcal{C}) \subseteq \mathrm{Fun}_\Sigma(\mathcal{C}^{op}, \mathrm{Ab})$ spanned by those spherical presheaves which are cokernels of representables, i.e. those $X : \mathcal{C}^{op} \to \mathrm{Ab}$ for which there exists a cokernel of spherical presheaves

$$y(c) \to y(d) \to X \to 0$$

– such spherical presheaves are called finitely presented

<u>Rem:</u> We get a **discrete Yoneda functor**

$$\begin{aligned} y : \mathcal{C} &\longrightarrow A(\mathcal{C}) \\ c &\longmapsto y(c). \end{aligned}$$

Since Ab forms an ordinary category any $X \in \mathrm{Fun}_\Sigma(\mathcal{C}^{op}, \mathrm{Ab})$ will factor uniquely through $h\mathcal{C}^{op}$

$$\begin{tikzcd} \mathcal{C}^{op} \arrow[rr, "X"] \arrow[dr] & & \mathrm{Ab} \\ & h\mathcal{C}^{op} \arrow[ur] & \end{tikzcd}$$

which gives an equivalence $A(\mathcal{C}) \simeq A(h\mathcal{C})$.

We can also promote $y : \mathcal{C} \to A(\mathcal{C})$ to a functor of locally graded categories. To see this we first equip $A(\mathcal{C})$ with a local grading induced by the grading on $\mathcal{C}$: For $X \in A(\mathcal{C})$ we define

$$(X[1])(c) := X(\underbrace{\Sigma^{-1}c}_{c[-1]}),$$ (so using the local grading in $\mathcal{C}$

and we see that y preserves exactly this grading:

$$\begin{aligned} y(\underbrace{\Sigma c}_{c[1]_{\mathcal{C}}})(d) &= [d, \Sigma c] \\ &= [\Sigma^{-1}d, c] \\ &= y(c)(\Sigma^{-1}d) = (y(c)[1]_{\mathcal{A}(\mathcal{C})})(d). \end{aligned}$$

Thm: 2.51 | Freyd | $y: \mathcal{C} \to \mathcal{A}(\mathcal{C})$ is homological and the universal such functor in the sense that for any homological functor $H: \mathcal{C} \to \mathcal{A}$ into some $\mathcal{A}$ abelian, there is an essentially unique exact functor $L: \mathcal{A}(\mathcal{C}) \to \mathcal{A}$ of abelian categories s.t.

$$\begin{tikzcd} \mathcal{C} \arrow[r, "H"] \arrow[dr, "y"'] & \mathcal{A} \\ & \mathcal{A}(\mathcal{C}) \arrow[u, "\exists! L"'] \end{tikzcd}$$

Rem: Using that the Yoneda embedding $y: \mathcal{C} \to \mathcal{A}(\mathcal{C})$ is a functor of locally graded ∞-categories, so it follows that y is a homology theory. If $H: \mathcal{C} \to \mathcal{A}$ is a homology theory, it can be shown that the unique functor $L: \mathcal{A}(\mathcal{C}) \to \mathcal{A}$ can be promoted to a functor of locally graded categories, so $y: \mathcal{C} \to \mathcal{A}(\mathcal{C})$ is also the universal homology theory.

Thm: 2.56 | Characterization of adapted homology theories | Let $\mathcal{C}$ be an idempotent complete stable ∞-category and $H: \mathcal{C} \to \mathcal{A}$ a homology theory s.t. $\mathcal{A}$ has enough injectives. Then the following is equivalent:

1. H is adapted

② The induced functor $L: A(\mathcal{C}) \to \mathcal{A}$ admits a fully faithful right adjoint. Always exists, but needs adapted for fully faithful

③ The kernel of the induced functor $L: A(\mathcal{C}) \to \mathcal{A}$ is a localizing subcategory of the Freyd envelope and L identifies $\mathcal{A}$ with the Gabriel quotient; i.e. $\mathcal{A} \simeq A(\mathcal{C})/\ker(L)$ That is, any exact functor $\underline{G}$ $G: A(\mathcal{C}) \to \mathcal{D}$ w $\mathcal{D}$ some abelian out of $A(\mathcal{C})$ annihilating $\ker(L)$ factors uniquely through $\mathcal{A}$.

Pf: Will only prove $(1) \Rightarrow (2) \Rightarrow (3)$

$(1) \Rightarrow (2)$ If L admits a right adjoint R it'll satisfy

$$\mathrm{Hom}_{A(\mathcal{C})}(x, Ra) \simeq \mathrm{Hom}_{\mathcal{A}}(Lx, a)$$

for all $x \in A(\mathcal{C})$, $a \in \mathcal{A}$. Since $A(\mathcal{C})$ is generated by the image of $\mathcal{C}$ under finite colimits and L is right exact, it is enough to have

$$(Ra)(c) \simeq \mathrm{Hom}_{A(\mathcal{C})}(y(c), Ra) \simeq \mathrm{Hom}_{\mathcal{A}}(\overset{L\circ y(a)}{\overset{\wr}{H(c)}}, a)$$

which forces the definition on us: $Ra \simeq \mathrm{Hom}_{\mathcal{A}}(H(-), a)$. Hence we need to show $Ra \in A(\mathcal{C})$.

Case 1: $i \in \mathcal{A}$ is injective and let $i_{\mathcal{C}}$ denote the injective lift.

Then

$$\overset{\mathrm{Hom}_{P\mathcal{C}}(c, i_c)}{\overset{\|}{[c, i_{\mathcal{C}}]}} \simeq \mathrm{Hom}_{A(\mathcal{C})}(y(c), y(i_{\mathcal{C}}))$$

adapted $\to$ $\wr$ $\leftarrow$ Fully faithful

$$\mathrm{Hom}_{\mathcal{A}}(H(c), i)$$

which implies $R(i) \simeq y(i_{\mathcal{C}})$

Case 2: A general $a \in \mathcal{A}$ can be written as a kernel

$$0 \to a \to i \to j$$

of a map between injectives, since $\mathcal{A}$ is assumed to have enough injectives, so

$$Ra \cong \ker(y(i_e) \to y(j_e)) \in A(\mathcal{C}).$$

To se fully faithfulness we first note that b the adjunction

$$\mathrm{Hom}_{A(\mathcal{C})}(Ra, Rb) \cong \mathrm{Hom}_{\mathcal{A}}(LRa, b) \xleftarrow{\circ \text{ Counit}} \mathrm{Hom}_{\mathcal{A}}(a, b)$$

$$(LRa \xrightarrow{\text{counit}} a \to b) \longleftarrow\!\shortmid \; a \to b$$

So need to show the counit is an isomorphism. In the case $i \in \mathcal{A}$ injective we use the above identification $Ri \cong y(i_e)$ to see that the counit map corresponds to the structure map

$$LRi \cong L(y(i_e)) \cong H(i_e) \longrightarrow i,$$

which is an isomorphism exactly when H is adapted. Since LR is left exact we deduce that this is always an isomorphism - not just on injectives.

(2) ⇒ (3) As just shown we have $LR \cong \mathrm{id}_{\mathcal{A}}$, so we get that for any $a \in \mathcal{A}$, the unit map

$$a \longrightarrow RLa$$

$G : A(\mathcal{C}) \to \mathcal{B}$ exact

has kernel and cokernel contained in $\ker(L) \subseteq \ker(G)$ since

$$(*) \quad 0 \to \ker(u) \to a \xrightarrow{u} RLa \to \operatorname{coker}(u) \to 0 \quad \text{exact}$$

$\Big\downarrow$ L exact

since $LR \cong \mathrm{id}_{\mathcal{A}}$

$$0 \to L(\ker(u)) \to La \xrightarrow{\sim} LRLa \to L(\operatorname{coker}(u)) \to 0 \quad \text{exact}$$

So $L(\ker(u)) = L(\operatorname{coker}(u)) = 0$. Hence

$$G(a) \xrightarrow{G(u)} G(RLa)$$

is an isomorphism since applying G to $(*)$ gives exact

$$0 \to \underbrace{G(\ker(u))}_{=0} \to Ga \to GRLa \to \underbrace{G(\operatorname{coker}(u))}_{=0} \to 0$$

So we get a factorization
$$\begin{tikzcd} & \mathcal{A} \arrow[dr, "GR"] & \\ A(\mathcal{C}) \arrow[ur, "L"] \arrow[rr, "G"'] & & \mathcal{B} \end{tikzcd}.$$

As an application of this we get a factorization of homology theories through adapted homology theories.

Thm 3.27 | Adapted factorization | Let $H : \mathcal{C} \to \mathcal{A}$ be a homology theory such that $\mathcal{A}$ has enough injectives which lift to injectives of $\mathcal{C}$. Then there is a unique factorization

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "H"] \arrow[dr, "H^*"'] & & \mathcal{A} \\ & \mathcal{A}^* \arrow[ur, "U"'] & \end{tikzcd}$$

where H^* is an adapted homology theory and U is a comonadic exact left adjoint. In particular, $\mathcal{A}^*$ is the category of comodules over a certain left exact comonad C over $\mathcal{A}$. $\mathcal{A}^* = \mathrm{Comod}_C(\mathcal{A})$, $C = LR$

Pf: By 2.56, to give a factorization as above is equivalent to giving a factorization

$$\begin{tikzcd} \mathcal{A}(\mathcal{C}) \arrow[rr, "L"] \arrow[dr, "L^*"'] & & \mathcal{A} \\ & \mathcal{A}^* \arrow[ur, "U"'] & \end{tikzcd}$$

where L^* is a quotient by localizing subcategory, U is a comonadic exact left adjoint and $\mathcal{A}^*$ has enough injectives.

$$\begin{tikzcd} \mathcal{C} \arrow[dr, "y"] \arrow[drr, bend left, "H"] \arrow[ddr, bend right, "H^*"'] & & \\ & \mathcal{A}(\mathcal{C}) \arrow[r, "L"] \arrow[d, "L^*"] & \mathcal{A} \\ & \mathcal{A}^* \arrow[ur, "U"'] & \end{tikzcd}$$

Since $\mathcal{A}^*$ having enough injectives lets us use 2.56 on the diagram

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "H^*"] \arrow[dr, "y"'] & & \mathcal{A}^* \\ & \mathcal{A}(\mathcal{C}) \arrow[ur, "L^*"'] & \end{tikzcd},$$

so if L^* is a quotient by a localizing subcategory we get that H^* is adapted – The part w. U is just part of the theorem we wish to prove. By A.1 the category of comodules over the comonad $C := L\underline{R}$ has the first two properties – and it has enough injectives by A.6, since $\mathcal{A}$ has

the right adjoint to L

So $\mathcal{A}^* \simeq \mathrm{Comod}_{LR}(\mathcal{A})$

In the case where we have an Adams type ring spectrum, this recovers Comod_{E_*E}.

Ex: 3.28 Let E be an Adams type ring spectrum, then we have a homology theory

$$E_*: Sp \to \mathrm{Mod}_{E_*}$$

where Mod_{E_*} has enough injectives - it is in fact 'Groethendiech' and these all lifts to injectives by Brown representability. Hence we get by 3.27

$$\begin{tikzcd} Sp \arrow[rr, "E_*"] \arrow[dr, "(E_*)^*"'] & & \mathrm{Mod}_{E_*} \\ & (\mathrm{Mod}_{E_*})^* \arrow[ur] & \end{tikzcd}$$

Note: We have this even if E is not Adam's type

The question is - what is this?

Assuming that E is Adams-type it follows by a classical result by Devinatz that E_* factors through Comod_{E_*E}, s.t

$$\begin{tikzcd} Sp \arrow[rr, "E_*"] \arrow[dr, "E_*"'] & & \mathrm{Mod}_{E_*} \\ & \mathrm{Comod}_{E_*E} \arrow[ur, "U"'] & \end{tikzcd}$$

with E_* adapted and U simply forgets the comodule structure so it is in particular conservative - i.e. preserves epimorphisms. Using that E is Adams we in particular get that E_*E is flat, so U is exact. This means that L and $(E_*)^*$ determines the same epimorphism classes, hence due to the uniqueness of adapted homology theories by the epimorphism classes they determine, we get that the two adapted homology theories agrees, so we get that the universal abelian category is exactly Comod_{E_*E}.

2) Deformations

Now, given an adapted homology theory $H: \mathcal{C} \to \mathcal{A}$ we can form the full subcategory $\mathcal{A}^{\omega}(\mathcal{C}) \subseteq \mathrm{Fun}_{\Sigma}(\mathcal{C}^{op}, \mathcal{S}) =: \mathcal{P}_{\Sigma}(\mathcal{C})$ consisting of the 'perfect presheaves' - those which is a finite colimit of representables, hence $\mathcal{A}^{\omega}(\mathcal{C})$ is the smallest subcategory of $\mathcal{P}_{\Sigma}(\mathcal{C})$ closed under finite colimits. It can be shown that $\underbrace{\mathcal{A}^{\omega}(\mathcal{C})}_{\text{prestable}}{}^{\heartsuit} \simeq \mathcal{A}(\mathcal{C})$. Recall that prestable implies that it is equivalent to the connective part of the t-structure on some stable ∞-category, which gives us this notion of "heart"

Next, we recall that an adapted homology theory $H: \mathcal{C} \to \mathcal{A}$ is completely determined by the H-epimorphisms, which determines a Groethendieck pretopolgy on $\mathcal{P}_{\Sigma}(\mathcal{C})$

$X: \mathcal{C}^{op} \to \mathcal{S}$ s.t. $\forall\ a \to \underset{H\text{-epi}}{\underline{b \to c}}$ fib sequence

$\rightsquigarrow$ Sheafification $L: \mathcal{P}_{\Sigma}(\mathcal{C}) \to \mathcal{P}_{\Sigma}(\mathcal{C})$ $\Rightarrow X(c) \to X(b) \to X(a)$ fib sequence in $\mathcal{S}$

which can be shown to preserve perfect presheaves, so we define the perfect derived ∞-category of $\mathcal{C}$ w.r.t. H by

$$\mathcal{D}^{\omega}(\mathcal{C}) := \mathcal{A}^{\omega, sh}(\mathcal{C}) = Sh^{H\text{-epi}}_{\omega}(\mathcal{C})$$

Since $\mathcal{A}^{\omega}(\mathcal{C})$ is prestable and $\mathcal{D}^{\omega}(\mathcal{C})$ is a localization of this we get $\mathcal{D}^{\omega}(\mathcal{C})$ is again prestable, and we get a canonical equivalence

$$\begin{aligned}\mathcal{D}^{\omega}(\mathcal{C})^{\heartsuit} &\simeq (\mathcal{A}^{\omega}(\mathcal{C})^{\heartsuit})^{sh} \\ &\simeq \mathcal{A}(\mathcal{C})^{sh} \\ &\simeq \mathcal{A}.\end{aligned}$$

This $\mathcal{D}^{\omega}(\mathcal{C})$ is the 'deformation' we are interested in. To understand how this describes such ∞-categorical deformations, we wish to construct a synthetic analouge and a τ-map as in the synthetic setting.

Synthetic analouge

The discrete Yoneda $y: \mathcal{C} \to \mathcal{A}(\mathcal{C})$ induces a canonical factorization through $\mathcal{A}^{\omega}(\mathcal{C})$ by taking the mapping space so we get

$$\begin{tikzcd} \mathcal{C} \arrow[r, "y"] \arrow[dr, "\tilde{y}"'] & \mathcal{A}(\mathcal{C}) \\ & \mathcal{A}^{\omega}(\mathcal{C}) \arrow[u, "\pi_0"'] \end{tikzcd}$$

- left adjoint to the inclusion of the heart

Using that $\tilde{y}(c) := \mathrm{Map}_{\mathcal{C}}(-,c)$ preserves limits and $\mathcal{C}$ is stable we get a fully faithful embedding $\nu: \mathcal{C} \to \mathcal{D}^{\omega}(\mathcal{C})$ called the **synthetic analouge** fitting in to the diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "H"] \arrow[dr, "\nu"'] & \mathcal{A} \\ & \mathcal{D}^{\omega}(\mathcal{C}) \arrow[u, "\pi_0"'] \end{tikzcd},$$

τ-Map

Local grading on $\mathcal{D}^{\omega}(\mathcal{C})$:

$$[1]: \mathcal{D}^{\omega}(\mathcal{C}) \to \mathcal{D}^{\omega}(\mathcal{C})$$
$$X \longmapsto X[1](c) = X(c[-1])$$
$$\mathcal{C}^{op} \xrightarrow{\Sigma^{-1}} \mathcal{C}^{op} \xrightarrow{X} \mathcal{S}$$

$\rightsquigarrow$ $\tau: \Sigma X \to X[1]$ natural map adjoint to

$X \to \Omega X[1]$ level wise given by

$$X(c) \simeq X(\Sigma\Omega c) \to \Omega X(\Omega c)$$
$$X[1](\Omega c) = X(\Omega c[-1])$$

which measures the extent to which X takes suspension to loops.

$\rightsquigarrow$ $C\tau \otimes X := \mathrm{cofib}(\tau: \Sigma X \to X[1])$

Just notation! Not necessarily monoidal

Thm 5.45: | Special fibre | The endofunctor

$$C\tau \otimes - : \mathcal{D}^\omega(\mathcal{C}) \longrightarrow \mathcal{D}^\omega(\mathcal{C})$$

admits a canonical structure of a monad. Moreover, the inclusion of the heart induces a canonical equivalence

$$\underbrace{\mathrm{Mod}_{C\tau}(\mathcal{D}^\omega(\mathcal{C}))}_{\text{modules over the monad}} \simeq \mathcal{D}^b(\mathcal{A})$$

connective bounded derived ∞-category = sheaves w.r.t. 4-epi top on almost perfect presheaves

$X \simeq \operatorname{colim}_\Delta X_\bullet$ w. X_n represent.

Thm: | The generic fiber | Write

$$\begin{tikzcd} \mathcal{C} \arrow[r, "id"] \arrow[d, "\nu"'] & \mathcal{C} \\ \mathcal{D}^\omega(\mathcal{C}) \arrow[ur, "\tau^{-1} :: \mathrm{Lan}"'] & \end{tikzcd}$$

left Kan extension of the synthetic analouge

Then

1) τ^{-1} is both left and right exact

2) sends all the maps $\tau : \Sigma X[-1] \to X$ to equivalences

and τ^{-1} is universal w.r.t. these two properties.

These two results together tells us that we can think of $\mathcal{D}^\omega(\mathcal{C})$ as an ∞-categorical deformation in the same way as we have seen with synthetic spectra and comodules:

$$\begin{tikzcd}[column sep=small] & \mathcal{D}^\omega(\mathcal{C}) \arrow[dl, squiggly, "\text{kill } \tau \; 5.45"'] \arrow[dr, squiggly, "\text{invert } \tau \; 5.47"] & & & \mathrm{Syn}_E \arrow[dl, squiggly, "\text{Kill } \tau \; \text{Talk 6}"'] \arrow[dr, squiggly, "\text{Invert } \tau \; \text{Talk}"] & \\ \mathcal{D}^b(\mathcal{A}) \simeq \mathrm{Mod}_{C\tau} & & \mathcal{C} & \mathrm{Mod}_{C\tau} \simeq \mathrm{Stable}_{E_*E} & & \mathrm{Sp} \end{tikzcd}$$

<u>Syn_E vs $\mathcal{D}^\omega(\mathrm{Sp})$</u>: $\mathcal{D}^\omega(\mathrm{Sp})$ can be thought of as somehow dual to Syn_E.

- Syn_E = sheaves on Sp_E^{fin} – finite spectra w. projective E_*-homology

- $\mathcal{D}^\omega(Sp)$ = sheaves defined on all of Sp, but need something in "exchange" for the finite condition – therefor the "perfect" condition

Syn_E talks about projectives while $\mathcal{D}^\omega(Sp)$ relies on injectives through the H-epimorphism topology.

Why $\mathcal{D}^\omega(Sp)$ instead of Syn_E?

- Adams SS collapses for injectives ⇝ Gives better bounds in e.g. Franke's conjecture.
- On the other hand: Injective lifts in Sp are hard to write down

Another connection between $\mathcal{D}^\omega(Sp)$ and synthetic spectra follows from the universal property of $\mathcal{D}^\omega(\mathcal{C})$:

Thm | Universal property of $\mathcal{D}^\omega(\mathcal{C})$ | $\nu : \mathcal{C} \to \mathcal{D}^\omega(\mathcal{C})$ is the universal 'prestable enhancement' of $H : \mathcal{C} \to \mathcal{A}$, i.e.

left exact + additive + $\pi_0 \circ \nu \simeq H$

left Kan extension along ν induces an equivalence between

1) Exact functors $G : \mathcal{D}^\omega(\mathcal{C}) \to \mathcal{D}$, w. $\mathcal{D}$ prestable
2) Exact functors $G_0 : A \to \mathcal{D}^\heartsuit$ together with a prestable enhancement of $G_0 H : \mathcal{C} \to \mathcal{D}^\heartsuit$.

So in the case of E being an Adams type ring spectrum we recall that we have

$$\underset{\mathcal{D}}{Syn_E^\heartsuit} \simeq \underset{\mathcal{A}}{Comod_{E_*E}}$$

and we have a small coproduct-preserving prestable enhancement

$$\nu_E : \underset{\mathcal{C}}{Sp} \to \underset{\mathcal{D}}{Syn_E}$$

of the E_*-homology functor

$$\begin{tikzcd}
\mathrm{Sp} \arrow[rr, "E_*"] \arrow[dr, "E_*"'] & & \mathrm{Mod}_{E_*} \\
& \mathrm{Comod}_{E_*E} \arrow[ur, "u"'] &
\end{tikzcd}$$

Adapted factorization

Hence we get a canonical exact functor

$$\mathcal{D}^{\omega}(\mathrm{Sp}) \longrightarrow \mathrm{Syn}_E$$

with image generated under finite colimits by the synthetic analouge of ordinary spectra.

Final notes about the paper

Thm: |Franke's algebraicity conjecture| Let $H: \mathcal{C} \to \mathcal{A}$ be an adapted homology theory and assume that

1) $\mathcal{A}$ admits a splitting of order $q+1$
2) $\mathcal{A}$ is of finite cohomological dimension d
3) $d \leq q$.

Then, we have an equivalence of homotopy $(q+1-d)$- categories

$$h_{q+1-d}(\mathcal{C}) \simeq h_{q+1-d}(\mathcal{D})^{Per}(\mathcal{A}))$$

What does these term means?

- Splitting of order $q+1$ w. $q \geq 0$: Let $\mathcal{A}$ be an abelian category with a local grading $[1]: \mathcal{A} \to \mathcal{A}$. A splitting of order $q+1$ of $\mathcal{A}$ is a collection of Serre subcategories

$$\mathcal{A}_\phi \subseteq \mathcal{A}$$

indexed by $\phi \in \mathbb{Z}/q+1$ s.t.

1) $[k](\mathcal{A}_\phi) \subseteq \mathcal{A}_{\phi+k} \quad \forall k \in \mathbb{Z}$
2) $\prod_{\phi \in \mathbb{Z}/(q+1)} \mathcal{A}_\phi \xrightarrow{\sim} \mathcal{A}$

$$(a_0, \dots, a_q) \longmapsto \bigoplus_\phi a_\phi$$

abelian w. local grading $[1]$

- $\mathcal{D}^{Per}(\mathcal{A})$: A differential object in $\mathcal{A}$ is a pair (a, ∂) with $a \in \mathcal{A}$ and $\partial: a \to a[1]$ s.t. $\partial \circ \partial[-1] = 0$

$\sim$ So can think of this as some sort of (periodic) chain complex

$$\dots \to a[-1] \xrightarrow{\partial[-1]} a \xrightarrow{\partial} a[1] \xrightarrow{\partial[1]} a[2] \to \dots$$

Define the homology of such (a, ∂):

$$H(a,\partial) := \operatorname{Ker}\partial / \operatorname{Im}\partial[-1] \in \mathcal{A}.$$

We say that a morphism

$$f:(a,\partial)\to(b,\delta)$$

is a quasi-isomorphism if $H(a,\partial)\xrightarrow[\cong]{Hf}H(b,\delta)$. We write

$$\underset{\text{stable } \infty\text{-cat}}{\mathcal{D}^{per}(\mathcal{A})} := \text{Differential objects}[\text{quasi-iso's}^{-1}] \quad \leftarrow \infty\text{-cat. localization}$$

an call it the periodic derived category of $\mathcal{A}$

$\rightsquigarrow$ Adapted conservative homology theory

$$H:\mathcal{D}^{per}(\mathcal{A})\to\mathcal{A}.$$

Note: Idea of the assumption of the order and dimension is that when $\dim\mathcal{A}<q+1$, $H(c)\in\mathcal{A}_i$, $H(d)\in\mathcal{A}_j$

$\Rightarrow$ Adams spectral sequence collapses.

Examples of applications:

1) R an associative ring spectrum, $\mathcal{C}=\mathrm{Mod}_R$. Then

$$\begin{aligned}\pi_*:\mathrm{Mod}_R &\longrightarrow R_*\text{-gr.Mod}=:\mathcal{A} \\ M &\longmapsto \pi_*M\end{aligned}$$

(π_*R)

is a conservative adapted homology theory. Assume R_* is $(q+1)$-sparse, (Piotr's thesis) i.e. $R_i=0$ unless $i\equiv 0 \bmod (q+1)$. So

$$M\in\mathcal{B}\subseteq\mathcal{A}\iff M_i=0 \quad \text{unless } i\equiv 0 \bmod (q+1)$$

to obtain a splitting of order $q+1$. Further assume that $\dim R_*=:d<q+1$. Then the theorem gives us

$$h_{q+1-d}(\mathrm{Mod}_R)\simeq h_{q+1-d}(\text{Diff. graded modules over } R_*).$$

Ex of such R:

1) $R=BP\langle n\rangle$ truncated Brown-Peterson spectrum, so in particular

$$BP\langle n\rangle_*=\mathbb{Z}_{(p)}[v_1,\dots,v_n],$$

where $|v_i|=2(p^i-1)$. This is a ring of graded global dimension $n+1$, concentrated in degrees divisible by $2p-2$.

$\rightsquigarrow$ If $2p-2 > n+1$ then

$$h_* \mathrm{Mod}_{BP\langle n\rangle} \simeq h_* \mathrm{Mod}_{H\mathbb{Z}_{(p)}[v_1, \dots, v_n]}.$$

2) $R = E(n)$ - Johnson-Wilson spectrum.

$$E(n)_* \cong \mathbb{Z}_p[v_1, \dots, v_{n-1}, v_n^{\pm 1}], \quad |v_i| = 2(p^i - 1).$$

is a ring of graded global dimension n and are concentrated in degrees divisible by $2p-2$. So if $2p-2 > n$

$$\rightsquigarrow \quad h_* \mathrm{Mod}_{E(n)} \simeq h_* \mathrm{Mod}_{H\mathbb{Z}_{(p)}[v_1, \dots, v_{n-1}, v_n^{\pm 1}]}.$$

Comment on the proof:

We have Goers-Hopkins obstruction theory

$$\dots \to M_2(\mathcal{C}) \to M_1(\mathcal{C}) \to M_0(\mathcal{C}) \simeq A \subseteq \mathcal{D}^\omega(\mathcal{C})$$

where $M_n(\mathcal{C}) \subseteq \mathcal{D}^\omega(\mathcal{C})$ full subcategory s.t.

$$X \in M_n(\mathcal{C}) \iff \begin{array}{l} X(c) \text{ is } n\text{-truncated} \\ \& \ \pi_i \tau \text{ isomorphism } \forall\, 0 \le i \le n-1 \end{array}$$

$\underset{\text{Adams SS}}{\Longrightarrow}$ $\mathcal{C} \simeq \varprojlim M_q(\mathcal{C}) = M_\infty(\mathcal{C})$

For all $q \ge d$ the canonical truncation map

$$\mathcal{C} \to M_q \mathcal{C}$$

induces an equivalence

$$h_{q-d+1} \mathcal{C} \simeq h_{q-d+1} M_q(\mathcal{C}) \quad (1)$$

of homotopy $(q-d+1)$-categories.

monadicity argument

$$h_{q+1} \mathcal{C} \overset{(1)}{\simeq} h_{q+1-d} M_q(\mathcal{C}) \overset{(2)}{\simeq} h_q M_q(\mathcal{D}^{per}(A))$$

$$\overset{(1)}{\simeq} h_{q-d+1}(\mathcal{D}^{per}(A)).$$