Section 2.F: Systems of Difference Equations

# Definition

A **system of difference equations** is used to model the behavior of two or more interacting variables. We will limit our examples to two or three variables. For more complex systems, matrices and the tools of linear algebra are useful for calculation and analysis. We first examine a well-defined system.

# Example #1:

*Suppose a car rental company has offices in Kansas City, Missouri, and Emporia, Kansas. Most customers return their car to their original rental office. But occasionally, customers will one want a one-way rental to/from the airport in KC. Find the long-term distribution of 100 cars between the two offices using the following assumptions for daily usage:*

* *75% of cars rented in Emporia are returned to Emporia.*
* *25% of cars rented in Emporia are returned to KC.*
* *85% of cars rented in KC are returned to KC.*
* *15% of cars rented in KC are returned to Emporia.*

## Answer:

Let represent the number of cars in Emporia after days. Let represent the number of cars in KC after days. The given assumptions yield the following system:

We can iterate this system using a spreadsheet for varying values of and Table 1 shows two such examples. Values are rounded to the nearest integer.

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| **Day** | **Emporia** | **KC** |  | **Day** | **Emporia** | **KC** |
| 0 | 50 | 50 |  | 0 | 100 | 0 |
| 1 | 45 | 55 |  | 1 | 75 | 25 |
| 2 | 42 | 58 |  | 2 | 60 | 40 |
| 3 | 40 | 60 |  | 3 | 51 | 49 |
| 4 | 39 | 61 |  | 4 | 46 | 54 |
| 5 | 38 | 62 |  | 5 | 42 | 58 |
| 6 | 38 | 62 |  | 6 | 40 | 60 |
| 7 | 38 | 62 |  | 7 | 39 | 61 |
| 8 | 38 | 62 |  | 8 | 39 | 61 |
| 9 | 38 | 62 |  | 9 | 38 | 62 |
| 10 | 38 | 62 |  | 10 | 38 | 62 |

Table 1: Distribution of cars in Emporia and KC for two different initial conditions.

Notice the number of days to achieve equilibrium will depend on the initial conditions, but the equilibria values remain the same. We can solve for these values by taking the limit of each side of the equation as approaches infinity. If we let and then our system becomes

Simplifying, each equation becomes

Which is equivalent to We have a second equation, Substituting for and solving for yields as expected.

If we made this calculation in advance, we could initially distribute our fleet of 100 cars to match the equilibria and the number of cars at each office would remain constant. Keep in mind that we are modeling an idealized situation. In reality, these trends are probabilistic, and we would expect variation in the number of cars rented each day, the duration of the rental, and the proportion that are returned to each location. We can edit the parameters in our model to account for some of this variation, but we may choose to create a more sophisticated model using simulation.

# Variable interaction

Sometimes there is a more complex level of interaction between two variables. A famous example is the predator-prey relationship, first analyzed by Lotka and Volterra independently in the 1920s. Volterra analyzed the proportion of prey fish and their predators (sharks and skates) caught by Italian fisherman in the 1910s [M].

The model assumes the prey will grow exponentially in the absence of predators. Conversely, the predators will decline exponentially if no prey exist. Furthermore, the predator will benefit from more “interaction” between the two species, while the prey will suffer. We need an interaction term that will be large when the population of both species is large, but small when the populations are small. Similar to the logistic model, Lotka and Volterra assumed the change in population due to interaction would be proportional to the product of the two variables.

To aid our discussion, let represent the number of predators after time periods and represent the number of prey. We can summarize the above discussion with the following system of difference equations:

where Depending on the choices for these parameters and the initial populations, the model will predict a variety of outcomes. We can solve for the equilibria as before, replacing all population terms with and , as appropriate:

We immediately see one possible equilibrium, by dividing through by and The remaining equilibrium can be found by solving for and to arrive at

# Example #2:

*Letting , use a spreadsheet to iterate the populations for 60 time periods. How does the long-term behavior depend on the initial populations?*

## Answer:

Solving for the equilibrium, we find This is unstable as even a small change in these values will result in increasingly larger variations in the populations over time. Figure 1 shows the predicted populations when and The populations oscillate for but it is unclear if that trend will continue indefinitely. If we alter the initial populations to and then extinction for both species occurs, as Figure 2 illustrates.

Figure 1: Lotka-Volterra predator-prey model prediction when and

Figure 2: Lotka-Volterra predator-prey model prediction when and

# Model Analysis

Notice that the model predicts the prey is more likely to survive as the proportion of predators increases. This counterintuitive result could be interpreted as a flaw in our model. However, Volterra observed a similar phenomenon in fishing data collected in Fiume, Italy. The proportion of prey fish actually decreased during World War I, when there were fewer fishermen.

This illustrates one potential use of mathematical modeling. We do not always need to fit a model to data. Observing trends and making conclusions can be valuable on its own. In this case, we observed:

* Oscillation in populations can be expected in a predator-prey relationship.
* A decrease in predators may not benefit the long-term survival of prey.
* A small change in population can have a dramatic effect on their long-term survival.

These effects have been observed in nature, as can be seen in some of the problems below.

# Question #1:

Is the equilibrium in Example #1 stable or unstable?

# Question #2:

What is the “interaction” term for the Lotka-Volterra model? Justify the signs (+/-) for each term in that model.

# Exercise #1:

Iterate the Lotka-Volterra model of Example #2 for new values of and What do you observe?

# Exercise #2:

Tweak the Lotka-Volterra model for slightly different values of and iterate. Does the model behave as you would expect? Why or why not?

# Problem #1:

One predator-prey relationship that is commonly analyzed is the lynx and snowshoe hare of Canada. During the late 1800s and early 1900s, the Hudson Bay Company tracked the number of pelts they bought for each of these species. It is assumed that the number of pelts bought is proportional to the population of each animal. A graph of the data [K] is shown in Figure 3.

Figure 3: Number of pelts bought by the Hudson Bay Company from 1847 to 1903

Without finding a specific model, do you think this data could be fit with a Lotka-Volterra model? Give at least one reason for and against this possibility.

# Problem #2:

Systems of difference equations can also be used to analyze competing species relationships. Consider an environment where great white sharks () and bottlenose dolphins () are competing for the same prey [NC]. Using the following system, deduce the sign of each coefficient and explain your answer.

# Problem #3:

Consider the competing species scenario from Problem #2. Assume , , , and . Deduce the correct sign for each coefficient and then solve for the equilibrium value(s) for this system. Are they stable or unstable?

# Problem #4:

Iterate the system of Problem #3 for the following initial values and summarize the short-term and long-term predictions for each.

1. and .
2. 0 and 0.
3. and
4. and

# Project #1:

Open the file, [EpidemicData.xlsx](https://emporia-my.sharepoint.com/personal/bhollenb_emporia_edu/Documents/Desktop/Modeling/Tools%20of%20MM/EpidemicData.xlsx) and answer the following questions.

1. Select the H1N1 (25 wks) tab. This data reflects the ``Swine Flu'' outbreak of 2009. The third column shows the **cumulative** number of H1N1 cases reported to the CDC by collaborating laboratories for Region 7. Region 7 consists of Kansas, Missouri, Iowa, and Nebraska. Fit a reasonable unconstrained growth model to this data. How well does it fit? Do you notice anything strange about the data? Use your model to predict the total number of cases as of February 13, 2010 (Week 40).
2. Next select the H1N1 (40 wks) tab. Using this additional information, fit a constrained growth model to the data. What value did you use for the ``carrying capacity'' and why? How well does this model fit?
3. Now click on the SIR tab. Notice the SIR model does not analyze cumulative infections, but rather ``current'' infections. These have been calculated in Column D. Apply the SIR model to this data. Unfortunately, the CDC numbers do not track every infection so the infected numbers will be greatly underestimated. So you will need to adjust your initial susceptible population accordingly. One possible way to do this is to use the info in the article linked at the bottom of the sheet. Assuming 0% are ``recovered'' initially, you should have all three initial values and you may now construct your predictions using the SIR model:
4. Notice that you only need to estimate and If you assume that no one is contagious for a week or more, then you can deduce directly. What is it? Thus is your only unknown parameter. Use trial-and-error to find the best value for How well does your model fit the actual data?
5. Summarize your findings. What are the strengths and weaknesses of each model? Can one be considered the ``best''? If so, which one, and why?

# Project #2:

We can also use the SIR model to help us understand epidemics in a general sense. Click on the SIR 2 tab of [EpidemicData.xlsx](https://emporia-my.sharepoint.com/personal/bhollenb_emporia_edu/Documents/Desktop/Modeling/Tools%20of%20MM/EpidemicData.xlsx). This represents a generic model where 95% of the population is susceptible and 5% is already infected. Recall that indicates a recovery (or removal) rate of 3 days. is a measure of how easily the infection spreads. By adjusting the appropriate values on this sheet, answer the following questions, justifying all answers. (It is up to you to figure out which values to change and how to change them.)

1. Using the default values, notice that nearly everyone who was susceptible is eventually infected. Is this true regardless of the value of ?
2. Now let and assume the model represents a serious disease that results in death after 3 days (). You should notice that eventually 65% of the population will eventually die from this disease, according to the model. Which disease would result in a lower overall death rate: a disease that kills more quickly, or less quickly, than 3 days?
3. For this problem, assume and . If 5% initially are infected, the model shows that eventually 83% will become infected. To avoid a “spike” (i.e. increase) in infections, immunizations are often used. Obviously, if all 95% of the susceptible population were immunized on Day 0, then a spike would be avoided. However, can a spike still be avoided if a smaller proportion of the population is immunized? In other words, what is the smallest proportion of the susceptible population that would need to be immunized to avoid a spike in infections?

# Project #3:

Select the SIRS tab from [EpidemicData.xlsx](https://emporia-my.sharepoint.com/personal/bhollenb_emporia_edu/Documents/Desktop/Modeling/Tools%20of%20MM/EpidemicData.xlsx). Notice the SIR model has already been built into this sheet:

We would like to tweak the model to allow for the fact that recovered individuals eventually lose their immunity and will become susceptible again. Assume that change is proportional to the number of recovered individuals.

1. Rewrite the system above taking this into account, using to represent the new parameter.
2. Edit the SIRS sheet to reflect this change.
3. Suppose we want to see if the SIRS system above can model the general trend of the number of Omicron infections of COVID-19 in the U.S. As you can see from the chart (https://www.nytimes.com/interactive/2021/us/covid-cases.html), cases were low in November of 2021, then spiked dramatically within two months. Cases dropped significantly in the spring but rose again in the summer of 2022. Assume and only 1 person out of 100,000 was infected with Omicron on "Day 0" of our spreadsheet. Assume 80% of the population is susceptible, and the rest are recovered from a prior infection. Assume on average, a person is contagious for about 5 days, and immune after infection for about 200 days. Find
4. If done correctly, you should see a larger spike on about Day 46 (corresponding to mid-January) and a smaller spike on Day 216 (early July). What day will infections start to rise again, according to the model?
5. Notice the three variables (susceptible, infected, recovered) appear to be tending towards an equilibrium. Estimate the equilibrium for the proportion of infected.
6. Suppose recovered individuals lose their immunity after 50 days instead of 200. How does the ``infected'' equilibrium value change?
7. Continue to assume recovered individuals lose their immunity after 50 days instead of 200. And now suppose individuals are only contagious for two days (or perhaps they choose to isolate more quickly due to widespread testing.) How does the ``infected'' equilibrium value change?
8. Tweak one or more of the parameters until you find something interesting. Describe what you changed and the result you observed. Include a graph of your situation on your Excel file.

# Project #4:

From [M]: *“Humberto D'Ancona was an Italian biologist who, in 1924, completed a statistical study of fish populations in the Adriadic Sea. He observed that, during the time of reduced fishing in World War I, his populations showed an increasing percentage of predator fish, especially sharks and skates (or a decrease of prey fish). This increase in population of the predator fish declined after the war. Vito Volterra, his father-in-law, was an Italian mathematician who had recently retired, and D'Ancona asked Volterra if there was a mathematical model which could explain this observed relative change in the populations of fish species. Within a couple of months, Volterra produced a series of mathematical models for the interaction of two or more species. Below is a table showing his data with a graph. Why should World War I affect the relative frequency of fish in Italian ports?”*

|  |  |
| --- | --- |
| Year | % |
| 1914 | 12 |
| 1915 | 21 |
| 1916 | 22 |
| 1917 | 21 |
| 1918 | 36 |
| 1919 | 27 |
| 1920 | 16 |
| 1921 | 16 |
| 1922 | 15 |
| 1923 | 11 |

Table 2: Percentages of predators in the Fiume fish catch

Create a predator-prey model that includes an extra predator, human fishermen. Does your model explain the jump in the data during World War I?

# Project #5:

Collect some data from another epidemic and illustrate the modeling process by finding a model of its infection level. Organize your discussion using the stages of modeling. Explain if you need to work cycle through the stages a second time or more. Make sure that your work shows that you understand the modeling process. (However, you may skip the implementation phase.) Your discussion should include your choice for the ``best'' model, and your opinion of how well it fits this new set of data.

# The SIR Game

**Rules**:

* On “Day 0”, everyone starts with a coin and either a red or a gray sheet. Those students will be known as RED and GRAY below.
* To begin, write a small “0” on your sheet, preferably in the corner.
* On “Day 1” everyone should *meet* exactly one person.
  + If RED *meets* RED, do nothing except write a “1” on your sheet (next to the “0”).
  + If GRAY *meets* GRAY, do nothing except write a “1” on your sheet.
  + If RED *meets* GRAY, flip a coin. If it is **tails**, do nothing, except write a “1” on your sheet. If it is **heads**, then GRAY puts their sheet away and RED tears off half of their sheet and gives it to GRAY. Keep the half with your numbers on it. And GRAY has now become a RED. They each write a “1” on their (red) sheet.
* On “Day 2” repeat the instructions for Day 1, except write a “2” instead of “1.” Make sure you meet a **new** person.
* On “Day 3” repeat the instructions for Day 2, except write a “3” instead of “2.” Also, if you have been RED for all 3 days, put your sheet away after your meeting. You are no longer RED or GRAY and may sit down until the game is done.
* Continue the above for each Day until we finish. Don’t forget that you are only allowed to be RED for 3 full days (i.e. you will only write 4 numbers on your red sheet before you are done.)
* When we are done, we will track how many in the class were RED, GRAY, or neither each day.

**Variations**:

* Start with more or fewer RED.
* You are allowed to meet more than one person per day.
* You may meet the same person more than once.
* After you are done as a RED, you may continue to meet others on the remaining Days.
* REDs remain RED for more days or fewer days.

# Bibliography

[JD…] <https://besjournals.onlinelibrary.wiley.com/doi/10.1111/j.1365-2656.2005.00977.x>

[K] <http://katalog.ub.uni-heidelberg.de/titel/66489211>

[M] <https://jmahaffy.sdsu.edu/courses/f09/math636/lectures/lotka/qualde2.html>

[NC] <https://ncstate.pressbooks.pub/appliedecology/chapter/chapter-6-content-competition/>

[W] <http://www.washingtonpost.com/wp-dyn/content/article/2010/02/12/AR2010021202204.html>