Section 2.D: Difference Equations Basics

# Definition

A dynamical system helps us model a variable that is changing over time. We use the following idea to model change:

Change = Future – Present.

If the change is continuous, we use a *differential equation*. Differential equations have been studied extensively for centuries, and many resources are available. We will discuss a few examples in a later section.

If the change is discrete, we use a *difference equation*. If represents a quantity after units of time, then a **difference equation** is the discrete dynamical system of the form:

where represents the future value of the variable and represents the present value of the variable.

# Example #1:

*I put $5 into a shoebox every day. How much will I have in the box after days? After 5 years?*

## Answer:

Let represent the amount in the shoebox after days. We will assume Then our model for how the amount in the box is changing is We can also write Solving for we have This recursive equation can be iterated quickly with a spreadsheet or calculator. Its simple form also allows us to write it in explicit form, So, we find that after five years,

# Example #2:

*I put $5000 into a Certificate of Deposit that earns 5% annual interest, compounded quarterly. How much will I have after quarters? After 5 years?*

## Answer:

Let represent the amount in the CD after quarters with Notice the interest we earn is *proportional* to the amount in the CD. So, we can model the amount in the CD with where is the interest rate for the quarter. Solving for as before, we have This equation can also be written in explicit form,

In five years (i.e., 20 quarters), the CD is worth

# Example #3:

*I put $1000 into a savings account that earns 3% annual interest, compounded monthly. I deposit $100 into it each month (after the interest compounds). How much will I have after months? After 5 years?*

## Answer:

Let represent the amount in the savings account after months with In this case, Our difference equation is which yields have The explicit form for this equation is more difficult to find (see Problem #1). It is

In five years (i.e., 60 months), the amount in the savings account is

# Question #1:

What is the difference between a differential equation and a difference equation?

# Question #2:

Verify the explicit equation for given in Example #2. Hint: write in terms of and then and so forth. (Or use mathematical induction to write a formal proof.)

# Exercise #1:

Assume Find the next five terms for the following difference equations:

# Exercise #2:

Write down an investment scenario that corresponds to each of the following difference equations:

1. compounding quarterly.
2. compounding monthly.
3. compounding quarterly.
4. compounding monthly.

# Exercise #3:

Use a dynamical system to find how much each of the following investment strategies will yield after 40 years.

1. Invest $54,000 initially in an account that pays 2%, compounded quarterly.
2. Invest $120 at the end of each month in an account that pays 3.5%, compounded monthly.

# Problem #1:

Verify the explicit equation for given in Example #3. Hint: Proceed as in Question #2 and then use a geometric series.

# Problem #2:

Suppose you would like to retire at age 65 with an annuity that will pay you $2000 monthly for 25 years. Assume this annuity will have an annual interest rate of 3%, compounded monthly. At age 25, you decide to start depositing a constant amount into a different annuity that pays 7% interest, compounded monthly. What is the minimum monthly deposit you will need to make to have enough when you retire?

# Problem #3:

How does your answer to Problem #2 change if you wish to receive $2000 monthly indefinitely?

# Problem #4:

Suppose upon graduation from college a relative gives you $20,000 to buy a new car. You go to the dealership and find a car you like for that price. The dealership gives you a choice between $1000 cash back, or $500 cash back plus 0% financing for 36 months. You currently are keeping the $20,000 in an account that earns 3% interest, compounded monthly. Which is the better offer? Explain.

# Problem #5:

Given you are years from retirement, how much do you need to save a month to have $100,000 when you retire? Assume a 10% annual return on your savings. Express your answer as a table where

# Problem #6:

Suppose a 15-year mortgage charges 4% annual interest. The borrower can afford a maximum monthly payment of $900. What is the maximum amount of money that can be borrowed? Your answer should be accurate to within $500.

# Problem #7:

A friend owes $1375 on a credit card that charges 1% interest each month. What is the minimum monthly payment needed to reduce the credit card balance? (Assume no more charges are made.)

# Project #1:

Create a spreadsheet that will solve situations like the one in Problem #2. Allow the user to input all the relevant information. Choose a scenario that fits you personally and then analyze it by tweaking the various parameters. Summarize your findings.

# Project #2:

An article from the Kansas City Star published in 1999 claimed that if someone working at minimum wage could save 10% of their after-tax pay, they could retire a millionaire, even if they never received a raise. Under what conditions would this be true? What would a similar statement look like today?