Section 2.D: Difference Equations Basics

# Definition

A dynamical system helps us model a variable that is changing over time. We use the following idea to model change:

Change = Future – Present.

If the change is continuous, we use a *differential equation*. Differential equations have been studied extensively for centuries, and many resources are available. We will discuss a few examples in a later section.

If the change is discrete, we use a *difference equation*. If $a\_{n}$ represents a quantity after $n$ units of time, then a **difference equation** is the discrete dynamical system of the form:

$$∆a\_{n}=a\_{n+1}-a\_{n}=f\left(a\_{n}\right), a\_{0}=a constant,$$

where $a\_{n+1}$ represents the future value of the variable and $a\_{n}$ represents the present value of the variable.

# Example #1:

*I put $5 into a shoebox every day. How much will I have in the box after* $n$ *days? After 5 years?*

## Answer:

Let $a\_{n}$ represent the amount in the shoebox after $n$ days. We will assume $a\_{0}=0.$ Then our model for how the amount in the box is changing is $∆a\_{n}=5.$ We can also write $a\_{n+1}-a\_{n}=5.$ Solving for $a\_{n+1},$ we have $a\_{n+1}=a\_{n}+5.$ This recursive equation can be iterated quickly with a spreadsheet or calculator. Its simple form also allows us to write it in explicit form, $a\_{n}=5n.$ So, we find that after five years, $a\_{1836}=\$9130.$

# Example #2:

*I put $5000 into a Certificate of Deposit that earns 5% annual interest, compounded quarterly. How much will I have after* $n $*quarters? After 5 years?*

## Answer:

Let $a\_{n}$ represent the amount in the CD after $n$ quarters with $a\_{0}=5000.$ Notice the interest we earn is *proportional* to the amount in the CD. So, we can model the amount in the CD with $∆a\_{n}=ra\_{n},$ where $r=\frac{0.05}{4}=0.0125$ is the interest rate for the quarter. Solving for $a\_{n+1}$ as before, we have $a\_{n+1}=1.0125a\_{n}.$ This equation can also be written in explicit form,

$$a\_{n}=1.0125^{n}a\_{0}.$$

In five years (i.e., 20 quarters), the CD is worth $a\_{20}≈\$6410.$

# Example #3:

*I put $1000 into a savings account that earns 3% annual interest, compounded monthly. I deposit $100 into it each month (after the interest compounds). How much will I have after* $n $*months? After 5 years?*

## Answer:

Let $a\_{n}$ represent the amount in the savings account after $n$ months with $a\_{0}=1000.$ In this case, $r=\frac{0.03}{12}=0.0025. $ Our difference equation is $∆a\_{n}=ra\_{n}+100 $which yields have $a\_{n+1}=1.0025a\_{n}+100.$ The explicit form for this equation is more difficult to find (see Problem #1). It is

$$a\_{n}=1.0025^{n}\left(a\_{0}+\frac{100}{0.0025}\right)-\frac{100}{0.0025}.$$

In five years (i.e., 60 months), the amount in the savings account is $a\_{60}≈\$7626.$

# Question #1:

What is the difference between a differential equation and a difference equation?

# Question #2:

Verify the explicit equation for $a\_{n}$ given in Example #2. Hint: write $a\_{n+1}$ in terms of $a\_{n-1},$ and then $a\_{n-2},$ and so forth. (Or use mathematical induction to write a formal proof.)

# Exercise #1:

Assume $a\_{0}=0.6.$ Find the next five terms for the following difference equations:

1. $a\_{n+1}=a\_{n}-4.$
2. $a\_{n+1}=3a\_{n}+n.$
3. $∆a\_{n}=0.4a\_{n}+11.$
4. $∆a\_{n}=1.5a\_{n}-7.$
5. $∆a\_{n}=2a\_{n}(1-a\_{n}).$

# Exercise #2:

Write down an investment scenario that corresponds to each of the following difference equations:

1. $∆a\_{n}=0.01a\_{n}, a\_{0}=1000,$ compounding quarterly.
2. $∆a\_{n}=0.005a\_{n}, a\_{0}=500,$ compounding monthly.
3. $∆a\_{n}=0.02a\_{n}-100, a\_{0}=1,000,000,$ compounding quarterly.
4. $∆a\_{n}=0.0025a\_{n}+200, a\_{0}=5000,$ compounding monthly.

# Exercise #3:

Use a dynamical system to find how much each of the following investment strategies will yield after 40 years.

1. Invest $54,000 initially in an account that pays 2%, compounded quarterly.
2. Invest $120 at the end of each month in an account that pays 3.5%, compounded monthly.

# Problem #1:

Verify the explicit equation for $a\_{n}$ given in Example #3. Hint: Proceed as in Question #2 and then use a geometric series.

# Problem #2:

Suppose you would like to retire at age 65 with an annuity that will pay you $2000 monthly for 25 years. Assume this annuity will have an annual interest rate of 3%, compounded monthly. At age 25, you decide to start depositing a constant amount into a different annuity that pays 7% interest, compounded monthly. What is the minimum monthly deposit you will need to make to have enough when you retire?

# Problem #3:

How does your answer to Problem #2 change if you wish to receive $2000 monthly indefinitely?

# Problem #4:

Suppose upon graduation from college a relative gives you $20,000 to buy a new car. You go to the dealership and find a car you like for that price. The dealership gives you a choice between $1000 cash back, or $500 cash back plus 0% financing for 36 months. You currently are keeping the $20,000 in an account that earns 3% interest, compounded monthly. Which is the better offer? Explain.

# Problem #5:

Given you are $n$ years from retirement, how much do you need to save a month to have $100,000 when you retire? Assume a 10% annual return on your savings. Express your answer as a table where $15\leq n\leq 47.$

# Problem #6:

Suppose a 15-year mortgage charges 4% annual interest. The borrower can afford a maximum monthly payment of $900. What is the maximum amount of money that can be borrowed? Your answer should be accurate to within $500.

# Problem #7:

A friend owes $1375 on a credit card that charges 1% interest each month. What is the minimum monthly payment needed to reduce the credit card balance? (Assume no more charges are made.)

# Project #1:

Create a spreadsheet that will solve situations like the one in Problem #2. Allow the user to input all the relevant information. Choose a scenario that fits you personally and then analyze it by tweaking the various parameters. Summarize your findings.

# Project #2:

An article from the Kansas City Star published in 1999 claimed that if someone working at minimum wage could save 10% of their after-tax pay, they could retire a millionaire, even if they never received a raise. Under what conditions would this be true? What would a similar statement look like today?