The Role of Topology in the Interpretation of Quantum Event Structures

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Duality

Algebra $\leftrightarrows$ Geometry

- Algebraic Structures are dual to Geometric Spaces
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Boolean Logic $\leftrightarrows$ Set Theory

- Logic of Classical Physics $\rightarrow$ Boolean Logic
- Logic of Quantum Physics $\rightarrow$ Non-Boolean Logic

Embodiment of Duality

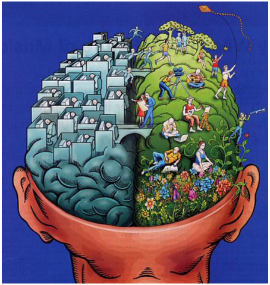

Figure : Left and Right Hemispheres

• Algebras are really Geometric Spaces seen from the other side of your brain up to Isomorphism.

Formalization of Duality

$$\bigwedge(-,-) : \mathcal{C}^{op} \times \mathcal{D} \to \mathcal{S}$$

- The bifunctor $\bigwedge(-,-)$ is bi-representable, that is representable within both $\mathcal{C}^{op}$, $\mathcal{D}$ if and only if there exist two functors: $\mathbf{R} : \mathcal{D} \to \mathcal{C}^{op}$ (encoding) and $\mathbf{L} : \mathcal{C}^{op} \to \mathcal{D}$ (decoding) such that

$$\mathbf{Hom}_{\mathcal{D}}(\mathbf{L}(C), D) \cong \bigwedge(C, D) \cong \mathbf{Hom}_{\mathcal{C}^{op}}(C, \mathbf{R}(D))$$

- The functors **L**, **R** form a categorical adjunction.
- **L**: left adjoint, **R**: right adjoint.
- Unit morphism at $\mathcal{C}$: $\eta_C : C \to \mathbf{R}(\mathbf{L}(C))$
- Co-unit morphism at $\mathcal{D}$: $\varepsilon_D : \mathbf{L}(\mathbf{R}(D)) \to D$
- Duality: Adjunction such that η, ε isomorphisms.

Isomorphism

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- The idea of isomorphism demarcates the self-evident, insurmountable boundary of cognition.
- Through the disclosure of isomorphic relations it is possible to transfer any insights gained in one field to the isomorphic field.

Hermann Weyl

"Philosophy of Mathematics and Natural Science".

Duality Pairing: Events

- Evaluation of an Observable at a State $\rightarrow$ Event.
- Evaluation of a Proposition at a Locus $\rightarrow$ Event.

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- Ultimate Facts of Nature are Events.
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- Extension is the most General Scheme of Real Potentiality.
- Extensive Event Continuum: Part/Whole and Local/Global Relations.
- System $\leftrightarrows$ Relational Complex: Internal Relations Binding the Events into a Nexus.

Mathematical Model: Sheaf

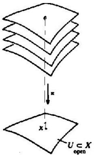
Sheaf of Observable Germs

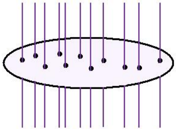
• **Idea**: Bidirectional Compatibility of Observable Information under the Inverse Operations of:

I. Extension or Induction from the Local to the Global.

II. Restriction or Reduction from the Global to the Local.

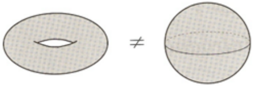
• **Principle**: A Global Information Structure can be Synthesized by means of Compatible Interlocking Families of Local or Partial Information Carriers.





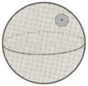
Topological Ideas

1. Geometric Theory of Elastic Shape.
2. Invariant Properties under Continuous Deformation.
3. Allowed Deformation Operations: Stretching, Shrinking, Bending, and Twisting.
4. Not-Allowed Deformation Operations: Cutting and Gluing.
5. Same Elastic Shape $\rightarrow$ Homeomorphism.



Localization

- Locally the Torus and the Sphere look like the 2-d Euclidean plane.

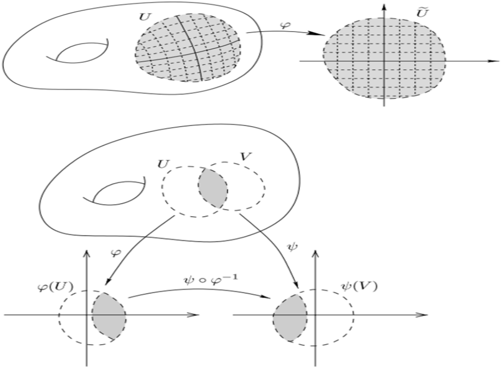
Topological Space

- **Idea**: Topology $\rightarrow$ Collection of Local Probes or Local Covers of a space closed under arbitrary unions and finite intersections.
- Continuous Function $\rightarrow$ No Breaks or Separations.
- Continuity is a Local Property.
- **Principle**: Local Trivialization or Local Homeomorphism

Manifolds

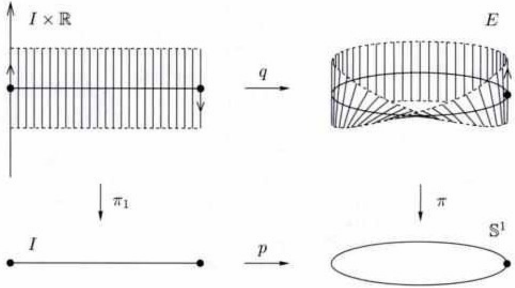
Topological Manifold

- **Idea**: Manifold $\rightarrow$ Topological Space that looks locally like some Model Space.



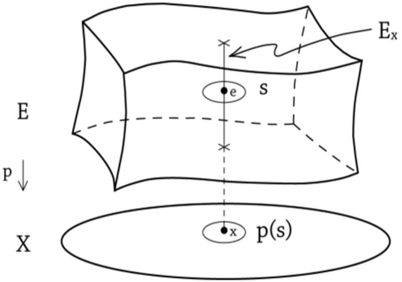
Fiber Spaces

- **Fiber Bundles**: Local Triviality $\rightarrow$ Locally looks like a Product with a fixed Model Fiber. Globally it can be Non-Trivial.



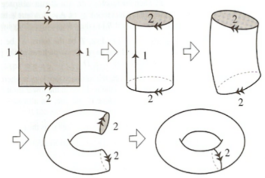
Display Spaces

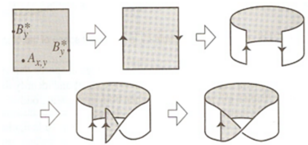
- **Display Bundles**: Local Homeomorphism $\rightarrow$ Locally looks like having the same Elastic Shape but Not Globally.

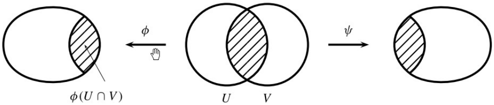


Gluing in Practice

- Torus: Glue together opposite sides of a square.
- Moebius Band: Glue the left to the right side of a square after making a half twist.



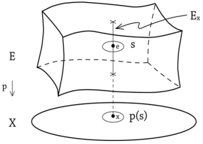


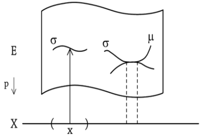


Sheaves

Unifying Idea: The Notion of a Sheaf

- A Sheaf is its Sections $\rightarrow$ Observables.
- Covering Families of a Global Object by Local Probes.
- Local Identity: Two Sections are Identical Globally if they are Identical Locally $\rightarrow$ Identical over every Local Cover of a Covering Family.
- Gluing: Local Sections can be Glued together Uniquely $\rightarrow$ Compatibility over Overlapping Local Covers





Formalization: Presheaf Frame

Presheaf Frame: Global $\rightarrow$ Local Compatibility

A presheaf frame **F** of sets of observables on a topological space X, consists of the following data:

1. For every open set U of X, a set of local observables $\mathbf{F}(U)$, and
2. For every inclusion $V \hookrightarrow U$ of open sets of X, a restriction morphism of sets in the opposite direction:

$$r(U|V) : \mathbf{F}(U) \to \mathbf{F}(V)$$

- $r(U|U) =$ identity at $\mathbf{F}(U)$ for all open sets U of X and
- $r(V|W) \circ r(U|V) = r(U|W)$ for all open sets $W \hookrightarrow V \hookrightarrow U$.
- Elements of $\mathbf{F}(U)$ are called local sections s.
- Notationally we denote: $r(U|V)(s) := s|_V$

Formalization: Sheaf Frame

Sheaf Frame: Global $\leftrightarrow$ Local Compatibility

A presheaf frame **F** of sets of observables on a topological space X, is defined to be a sheaf frame if it satisfies the following two conditions, for every family V_a, $a \in I$, of local open covers of V, where V open locus in X, such that $V = \cup_a V_a$:

- Local Identity Axiom of Sheaf Frame:
  Given $s, t \in \mathbf{F}(V)$ with $s|_{V_a} = t|_{V_a}$ for all $a \in I$, then $s = t$.
- Gluing Axiom of Sheaf Frame:
  Given $s_a \in \mathbf{F}(V_a)$, $s_b \in \mathbf{F}(V_b)$, $a, b \in I$, such that: $s_a|_{V_a \cap V_b} = s_b|_{V_a \cap V_b}$ for all $a, b \in I$, then there exists a unique section $s \in \mathbf{F}(V)$, such that: $s|_{V_a} = s_a \in F(V_a)$ and $s|_{V_b} = s_b \in F(V_b)$.

Generalized Covers

- Generalization: Abstract the notion of Local Cover from the case of Open Locus. Consider general Local Frames.
- Every Locally Determined Physical Property gives rise to a Sheaf.

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- Generalization: Abstract the notion of Local Cover from the case of Open Locus. Consider general Local Frames.
- Every Locally Determined Physical Property gives rise to a Sheaf.
- Grothendieck Topology: Abstracts the constitutive properties of a Localization Process in relational algebraic terms forming a topological theory of Congruence.
- Sieves: Internally structured families of arrows targeting some object in a category. A B-sieve $\mathbf{S}$ on some object B in a category $\mathcal{B}$ is a family of arrows targeting B such that: If $C \to B \in \mathbf{S}$ and $D \to C$ is any $\mathcal{B}$-arrow, then $D \to C \to B \in \mathbf{S}$.

Covering Sieves-Sites

- B-sieve $\mathbf{S} \cong$ Subobject of $\mathbf{Hom}_{\mathcal{B}}(-, B)$
- The set of all B-sieves forms a Heyting algebra.
- Localization congruence: Covering Sieves
  1. Maximal Sieves are covering.
  2. Covering sieves are stable under restriction (pullback).
  3. Covering sieves are transitive: if the restriction of a sieve to each frame of a covering sieve is covering, then it is also a covering sieve.
- A function which assigns to each object of the category of frames $\mathcal{B}$ a collection of covering sieves defines a Grothendieck topology $\mathbf{J}$.
- Site $(\mathcal{B}, \mathbf{J})$: A category of frames equipped with a theory of topological localization congruence.

Application in Quantum Theory

Bridge with Quantum Theory

- Local Cover $\leftrightarrow$ Boolean Algebra $\leftrightarrow$ Boolean Frame.
- Local Determination $\leftrightarrow$ Measurement of Observable $\leftrightarrow$
$\leftrightarrow$ Observed Events w.r.t. Boolean Frame.
- Gluing $\leftrightarrow$ Compatibility of Observables $\leftrightarrow$ Transition Function $\leftrightarrow$
$\leftrightarrow$ Nexus of Events.

1. Complete Boolean algebras act as local Boolean logical frames for probing the quantum domain by measurement.
2. Contextual coordinatization of quantum events by local Boolean frames.
3. Sheaf-theoretic localization of a quantum event algebra w.r.t. local Boolean frames covering it.
4. Quantum Logic is a patchwork of non-trivially interlocking local Boolean frames.

Basic Tenets

1. Events
    - Presuppose Boolean frame $\leftrightarrow$ Context of Measurement of Observable.
    - Probabilistic Inference prior to Measurement w.r.t. Boolean Frame.
    - Affirmation follows Registration by Measuring Device.

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   - Non-Simultaneous Observability.
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3. Entanglement
   - Einstein-Podolsky-Rosen correlations $\leftrightarrow$ Non-Separability.
   - Non-Individuation of the Parts Independently of the Whole.
   - Whole is more than the Sum of the Parts.

Localization of the Quantum

Resolution

Sheaf-theoretic Localization of Quantum Events w.r.t. Boolean Logical Frames.

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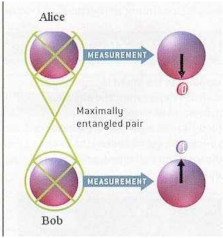
Resolution

Sheaf-theoretic Localization of Quantum Events w.r.t. Boolean Logical Frames.

1. Individuation only w.r.t. Boolean Frames.
2. Quantum Event Continuum is a Sheaf w.r.t. Covering Families of Boolean Frames.
3. Quantum Logic is a Global Evolving Network of Topologically Interlocking Local Boolean Frames.
4. Equivalence classes of Sheaf-Compatible Boolean frames $\leftrightarrow$ Boolean Germs.
5. Boolean Germs are the Seeds of Extensive Connection in the Quantum Event Continuum.

Non-Separability

- Composite Quantum System Consisting of Two Subsystems.
- Subsystems Do Not have Individual, Separable State Independently of the State of the Composite System.
- Form of Correlation Not Dynamical and Not Dependant on Spatial Distance.
- Registered Only by Correlated Pairs of Events.
- Conserved Physical Property of the Total System.



Topological Explanation

1. Fundamental Significance of Events $\leftrightarrow$ Measurement Outcomes.
2. Notion of Subsystem Meaningful Only w.r.t. Local Boolean Frame.
3. The Two Subsystems correspond to Localizations or Restrictions of a Sheaf to Two Boolean Frames.
4. Compatible Boolean frames $\leftrightarrow$ Compatible Observables of Subsystems.
5. Physical Compatibility Condition $\leftrightarrow$ Conservation Principle.
6. Sheaf Compatibility Condition $\leftrightarrow$ Topological Gluing of Boolean frames.
7. Entangled Pair Forms a Boolean Germ in the Quantum Event Continuum.