MR4505842 81P05 03A10 18Axx 18F99 81P13 81P15

Karakostas, Vassilios; Zafiris, Elias (D-PCCFS)

On the structure and function of scientific perspectivism in categorical quantum mechanics. (English. English summary)

British J. Philos. Sci. **73** (2022), *no.* 3, 811–848.

Summary: "Contemporary scientific perspectivism is primarily viewed as a methodological framework of how we obtain and form scientific knowledge of nature, through a broadly perspectivist process, especially, with reference to quantum mechanics. In the present study, this is implemented by representing categorically the global structure of a quantum algebra of events in terms of structured interconnected families of local Boolean probing frames, realized as suitable perspectives or contexts for measuring physical quantities. The essential philosophical meaning of the proposed approach implies that the quantum world can be consistently approached and comprehended through a multilevel structure of locally variable perspectives, which interlock, in a category-theoretical environment, to form a coherent picture of the whole in a nontrivial way."

MR4404639 81S10 81Q70 81S30

Zafiris, Elias (D-PCCFS)

The equiareal Archimedean synchronization method of the quantum symplectic phase space: I. Spinorial amplitudes, transition probability, and areal measure of time. (English. English summary)

Found. Phys. **52** (2022), *no.* 2, *Paper No.* 44, 31 *pp.*

Summary: "The quantum transition probability bears the symmetry of a process 'evolving' around a symplectic area-bounding loop in the projective Hilbert space that essentially underlies the notion of a global geometric phase. The basic idea is that this symmetry can be associated with a synchronization procedure in the quantum phase space, which is only implicit due to the insistence of interpreting the temporal variable as a classical one, overlooking the subtle interrelation of the complex with the symplectic structure. This leads to the qualification of quantum amplitudes in terms of symplectic spinorial objects which doubly cover the corresponding complex vectors obtained through squaring. Their interrelation can be simply formulated in terms of null vectors of a 3-d Minkowski space, which proves to be instrumental for the Hermitian and unitary representations of the symplectic group. In this light, we show that the quantum transition probability assignment constitutes an equiareal transformation from the annulus of spinors to the disk of complex vectors, which makes it equivalent to the equiareal measure-preserving transformation of Archimedes. This realization ignited a process of re-evaluating the original works of Archimedes in light of their conceptual significance. It turns out that the symplectic method of equiareal projection on a disk pertains to a precise synchronization procedure that can be applied to the quantum phase space. From this viewpoint, we examine the pertinent notions of time, entangled symplectic area, objective indistinguishability, and qualify physically the non-squeezing theorem of symplectic geometry."

MR4404638 81S10 81Q70 81S30

Zafiris, Elias (D-PCCFS)

The equiareal Archimedean synchronization method of the quantum symplectic phase space: II. Circle-valued moment map, integrality, and symplectic Abelian shadows. (English. English summary)

Found. Phys. **52** (2022), *no.* 2, *Paper No.* 43, 32 *pp.*

Summary: "The quantum transition probability assignment is an equiareal transformation from the annulus of symplectic spinorial amplitudes to the disk of complex state vectors, which makes it equivalent to the equiareal projection of Archimedes. The latter corresponds to a symplectic synchronization method, which applies to the quantum phase space in view of Weyl's quantization approach involving an Abelian group of unitary ray rotations. We show that Archimedes' method of synchronization, in terms of a measure-preserving transformation to an equiareal disk, imposes the integrality of the quantum of action, and requires the extension of the classical moment map from the real line to the circle. Additionally, the same synchronization method is encoded in the structure of the Heisenberg group, viewed as a principal bundle with a connection, whose curvature and anholonomy is expressed in terms of area bounding loops in relation to the underlying Abelian shadow on a symplectic plane. In this manner, we show that the geometric phase pertains to the minimal synchronized area $\pi\hbar$ of the 2-d symplectic Abelian shadow of the symplectic ball, modulo $\mathbb{Z}$. The integrality condition naturally leads to the consideration of modular commutative observables pertaining to the role of the discrete Heisenberg group. We prove that the structural transition from non-commutativity to modular commutativity in accordance to Weyl's group-theoretic commutation relations takes place via universal factorization through the discrete Heisenberg group. In this way, we derive a homology-theoretic formulation of the synchronization method in terms of the area-bounding cells of the modular lattice $\frac{\mathbb{R}^2}{\mathbb{Z}^2}$ in relation to any Abelian symplectic shadow. Thus, we finally obtain the physical interpretation of the analytic representation of quantum states as theta functions corresponding to the sections of a complex line bundle with an integral symplectic structure."

MR4368642 22E70 46N50 81S10

Zafiris, Elias (D-PCCFS); **von Müller, Albrecht** (D-PCCFS)

On the discrete Heisenberg group and commutative modular variables in quantum mechanics: II. Synchronization of unitary actions and homological abelianization. (English. English summary)

Quantum Stud. Math. Found. **9** (2022), *no.* 1, 71–92.

Summary: "Since, the quantum modular variables are encoded in terms of one-parameter unitary groups, we are led to a careful re-evaluation of the Heisenberg group, from where the commutation relations emerge from. In particular, the re-evaluation pertains to the discrete Heisenberg group from the perspective of its genuine descent from a more fundamental layer of structure targeting the origin of the non-commutativity of quantum observables. Due to the fact that the modular variables commute, they give rise to an integrality condition inherent to the structure of the discrete Heisenberg group. In this manner, this group should mediate in the structural transition from non-commutativity to its integral Abelian shadow, which is qualified symplectically via the non-squeezing theorem. In particular, the integrality of symplectic area pertains to the global topology of the torus, meaning that the phase space of the 2-d Abelian shadow is topologically toroidal and isomorphic to the modular lattice $\frac{\mathbb{R}^2}{\mathbb{Z}^2}$, such that it is universally covered by $\mathbb{R}^2$, where $\mathbb{Z}^2$ is the free Abelian group in two generators acting by integer translations. The main conclusion of this work is that the structural

transition from the non-Abelian-free fundamental group of based loops Θ_2, expressing the synchronization condition of the joint action of modular variables, to the Abelian-free homology group of cycles $\mathbb{Z}^2$, where the entangled symplectic area pertains, factorizes though the discrete Heisenberg group. This elucidates the role and status of modular variables in quantum mechanics and constitutes a viable theoretical explanation of the nature and appearance of quantum interference phenomena underlying the significance of the notion of a geometric phase."

MR4320565 22E70 46N50 81S10

Zafiris, Elias (D-PCCFS); **von Müller, Albrecht** (D-PCCFS)

On the discrete Heisenberg group and commutative modular variables in quantum mechanics: I. The Abelian symplectic shadow and integrality of area. (English. English summary)

Quantum Stud. Math. Found. **8** (2021), *no.* 4, 391–410.

Summary: "Global interference phenomena start to manifest in quantum transitions involving at least two projection filters which cannot be realized simultaneously, like in the case of the double-slit experiment. In terms of the 3-vertex invariants of the projective space of rays, thought of as a symplectic phase space, we encounter products involving the complex-valued $\mathbf{I}_3(\Psi_{in}, \Psi_b, \Psi_{fin})$ with the complex conjugate $\mathbf{I}_3^*(\Psi_{in}, \Psi_b', \Psi_{fin})$, where the projection operators $|\psi_b\rangle\langle\psi_b|$ and $|\psi_b'\rangle\langle\psi_b'|$ are not simultaneously realizable. In this way, interference can be expressed in terms of products of this form defined over squares in the space of rays, which can be triangulated. Triangles in this space encode the invariant information of the geometric phase factor. From this perspective, we qualify the proposal of Aharonov and collaborators pertaining to the consideration of commutative modular variables evaluated in $\mathbb{R}/\mathbb{Z}$ in deciphering the quantum interference pattern of the double-slit experiment. The quantum modular variables pertaining to conjugate observables are encoded in terms of one-parameter unitary groups acting jointly on the phase space, thus modeled through the continuous group action of $\mathbb{R}^2$. We show that $\frac{h}{2}$ expresses the minimal indistinguishable invariant area of the 2D symplectic Abelian shadow of the symplectic ball of radius $R = \sqrt{\hbar}$ in the $2n$-phase space of the conjugate position and momenta. The above conclusion leads to a re-estimation of Weyl's view of the quantum kinematical space in terms of an Abelian group of unitary ray rotations, and in particular the role that the discrete Heisenberg group plays in this conundrum."

MR4465975 53-03 53-02

Zafiris, Elias (GR-UATH)

★**Contributions of abstract differential geometry (ADG) in physics.**

Topological algebras and their applications, 15–17, *Proc. Pure Appl. Math.*, *Academic Services*, *Wilmington*, *DE*, [2020],

MR4036925 81P10

Zafiris, Elias (D-PCCFS); **Karakostas, Vassilios** (GR-UATHS-QH)

Category-theoretic interpretative framework of the complementarity principle in quantum mechanics. (English. English summary)

Internat. J. Theoret. Phys. **58** (2019), *no.* 12, 4208–4234.

Summary: "This study aims to provide an analysis of the complementarity principle in quantum theory through the establishment of partial structural congruence relations between the quantum and Boolean kinds of event structure. Specifically, on the basis of the existence of a categorical adjunction between the category of quantum event algebras and the category of presheaves of variable Boolean event algebras, we establish a twofold complementarity scheme consisting of a generalized/global and a restricted/local conceptual dimension, where the latter conception is subordinate to and constrained by the former. In this respect, complementarity is not only understood as a relation between mutually exclusive experimental arrangements or contexts of comeasurable observables, as envisaged by the original conception, but it is primarily comprehended as a reciprocal relation concerning information transfer between two hierarchically different structural kinds of event structure that can be brought into partial congruence by means of the established adjunction. It is further argued that the proposed category-theoretic framework of complementarity naturally advances a contextual realist conceptual stance towards our deeper understanding of the microphysical nature of reality."

MR3821544 83-01 81-01 81R40 81T20

von Müller, Albrecht; Zafiris, Elias

★**Concept and formalization of constellatory self-unfolding.**

A novel perspective on the relation between quantum and relativistic physics.

On Thinking.

Springer, *Cham*, 2018. *xxi*+217 *pp.* *ISBN* 978-3-319-89775-2; 978-3-319-89776-9

Publisher's description: "This volume develops a fundamentally different categorical framework for conceptualizing time and reality. The actual taking place of reality is conceived as a 'constellatory self-unfolding' characterized by strong self-referentiality and occurring in the primordial form of time, the not yet sequentially structured 'time-space of the present.' Concomitantly, both the sequentially ordered aspect of time and the factual aspect of reality appear as emergent phenomena that come into being only after reality has actually taken place. In this new framework, time functions as an ontophainetic platform, i.e., as the stage on which reality can first occur. Events are merely the 'tracks' that the actual taking place of reality leaves behind on the co-emergent 'canvas" of local spacetime.

"The view of time proposed here is particularly relevant to the recent debate over the 'ER=EPR' conjecture targeting the relation between quantum physics and general relativity theory. The novelty of this radically different framework is that it allows quantum reduction and singularities to be addressed as inverse transitions into and out of the factual layer of reality: In quantum physical state reduction, reality 'gains' the chrono-ontological format of facticity, and the sequential aspect of time becomes applicable. In singularities, by contrast, the opposite happens: Reality loses its local spacetime formation and reverts back to its primordial, pre-local shape—making the use of causality relations, Boolean logic and the dichotomization of subject and object obsolete in the process.

"For our understanding of the relation between quantum and relativistic physics, this new view opens up fundamentally new perspectives: Both are legitimate views of time and reality; they simply address very different chrono-ontological portraits, and thus should not lead us to erroneously prefer one view over the other.

"The task of the book is to provide a formal framework in which this radically different view of time and reality can be suitably addressed. The mathematical approach is based on the logical and topological features of the Borromean Rings, and draws upon concepts and methods from algebraic and geometric topology—especially the theory of sheaves and links, group theory, logic and information theory in relation to the standard constructions employed in quantum mechanics and general relativity, shedding new light on the problems of their compatibility. The intended audience includes physicists, mathematicians and philosophers with an interest in the conceptual and mathematical foundations of modern physics."

MR3620240 18B25 81R10

Karakostas, Vassilios (GR-UATHS-QH); **Zafiris, Elias** (H-EOTVO-L)

Contextual semantics in quantum mechanics from a categorical point of view. (English. English summary)

Synthese **194** (2017), *no.* 3, 847–886.

Let $\mathcal{H}$ be a complex separable Hilbert space, possibly infinite-dimensional. The paper under review explores the relation between the quantum-mechanical lattice of projections $\mathbb{L}(\mathcal{H})$ and the local classical frames given by the Boolean (sub)algebras of the mutually commuting projections. These Boolean algebras (Bas) can be assigned to every set of commuting self-adjoint operators (observables) [cf. G. Takeuti, *Two applications of logic to mathematics*, Iwanami Shoten, Publishers, Tokyo, 1978; MR0505474].

The lattice can be also analyzed from the point of view of operator algebras such as von Neumann algebras of types I, II and III or C^*-algebras.

Another approach to $\mathbb{L}$ in the context of quantum mechanics (QM) is via category and model theories. Here we have two main schools—the Isham and Landsman schools of topos theory. All these approaches lead to better understanding of QM. Many interesting questions remain to be answered.

The paper is devoted to some mathematical aspects of QM seen via the relation of $\mathbb{L}$ and Bas, also from a broad philosophical perspective. This aim is achieved by applying certain category-theoretic constructions. Even though detailed formal reasoning is not prevalent in the paper, which rather focuses on descriptions of various results obtained elsewhere, one mathematical relation seems especially important. This is the adjointness of the categories $\mathrm{Set}^{\mathcal{B}^{op}}$ and $\mathcal{L}$. The first one is the category of presheaves on $\mathcal{B}$—here, $\mathcal{B}$ is the category of Boolean event algebras, whose objects are σ-Bas of events and whose arrows are the corresponding Boolean algebraic homomorphisms; $\mathcal{B}^{op}$ is the opposite category of $\mathcal{B}$; the objects of $\mathrm{Set}^{\mathcal{B}^{op}}$ are the presheaves on $\mathcal{B}$, that is, contravariant set-valued functors $P\colon \mathcal{B}^{op} \to \mathrm{Set}$, and the morphisms are the natural transformations between such functors. The category $\mathcal{L}$ is the category of quantum event algebras, whose objects are orthomodular σ-orthoposets, that is, partially ordered sets of quantum events with a maximum and an operation of orthocomplementation, and whose morphisms are the quantum algebraic homomorphisms.

These structures are not generically Boolean, though they determine families of Boolean subalgebras. The point is that these families, denoted $\{B's\}_L$ for $L \in \mathcal{L}$, give rise to presheaves $F_L \in \mathrm{Set}^{\mathcal{B}^{op}}$, $L \in \mathcal{L}$. The entire correspondence is based on the existence of the pair of adjoint functors

$$\mathrm{Left}\colon \mathrm{Set}^{\mathcal{B}^{op}} \leftrightarrows \mathcal{L}{:}\mathrm{Right},$$
$$F_L \mapsto L.$$

In fact $\mathrm{Left}(F)$ is the colimit of algebras from $\{B's\}_L$ giving rise to L, whereas $\mathrm{Right}(L)$ is the family $\{B's\}_L$ forming the presheaf in $\mathrm{Set}^{\mathcal{B}^{op}}$.

This construction allows also for considering the intersections between different local

Boolean frames (algebras). Precisely, this is enabled by the existence of pullbacks in the topos $\mathrm{Set}^{\mathcal{B}^{op}}$ plus the Yoneda embedding

$$\mathcal{B} \hookrightarrow \mathrm{Set}^{\mathcal{B}^{op}}.$$

These mathematical constructions I consider as especially important results of the paper with many possible further applications. *Jerzy Król*

MR3558384 81T30

Zafiris, Elias (D-PCCFS)

Loops, projective invariants, and the realization of the Borromean topological link in quantum mechanics. (English. English summary)

Quantum Stud. Math. Found. **3** (2016), *no.* 4, 337–359.

Summary: "All the typical global quantum mechanical observables are complex relative phases obtained by interference phenomena. They are described by means of some global geometric phase factor, which is thought of as the 'memory' of a quantum system undergoing a 'cyclic evolution' after coming back to its original physical state. The origin of a geometric phase factor can be traced to the local phase invariance of the transition probability assignment in quantum mechanics. Beyond this invariance, transition probabilities also remain invariant under the operation of complex conjugation. Most important, geometric phase factors distinguish between unitary and antiunitary transformations in terms of complex conjugation. These two types of invariance functions as an anchor point to investigate the role of loops and based loops in the state space of a quantum system as well as their links and interrelations. We show that arbitrary transition probabilities can be calculated using projective invariants of loops in the space of rays. The case of the double slit experiment serves as a model for this purpose. We also represent the action of one-parameter unitary groups in terms of oppositely oriented-based loops at a fixed ray. In this context, we explain the relation among observables, local Boolean frames of projectors, and one-parameter unitary groups. Next, we exploit the non-commutative group structure of oriented-based loops in 3-d space and demonstrate that it carries the topological semantics of a Borromean link. Finally, we prove that there exists a representation of this group structure in terms of one-parameter unitary groups that realizes the topological linking properties of the Borromean link."

MR3444903 58A03 18F20 53C05 81S40 81T13 81T20 81T60 81V19

Mallios, Anastasios (GR-UATH-NDM); **Zafiris, Elias** (GR-UATH-NDM)

★**Differential sheaves and connections.**

A natural approach to physical geometry.

Series on Concrete and Applicable Mathematics, 18.
World Scientific Publishing Co. Pte. Ltd., *Hackensack*, *NJ*, 2016. *xv*+285 *pp.*
ISBN 978-981-4719-46-9

We read in the preface of this book that "the present work constitutes a confluence of the ideas and methods emanating from two sources: The first source is the theory of Abstract (alias Modern) Differential Geometry [...], developed by [one of the authors, the late Anastasios] Mallios, using the notions of vector sheaves, connections, and sheaf cohomology [...] applied in the formulation and interpretation of gauge theories and general relativity [...]. The second source is the theory of topoi, developed by Grothendieck, as generalized localization environments [...] in algebraic geometry and homological algebra going beyond [...] topological spaces". The goal of the work is to address what the authors refer to as "[t]he basic question [...] [of whether] there exists a natural description of physical geometry", with this latter phrase meaning in essence that one seeks a mathematical description of physical objects and relations, i.e., the stuff that physics is made of, related to or even based on "the pertinent physical relations themselves together with their empirical realization in terms of observed events". Thus, Mallios and Zafiris are going for a more intrinsic rendering of fundamental physics, with the mathematical framework suggested directly (or at least more directly) by what the physical data, via measurements, experiments, and the like, might provide. To make mathematical sense of this approach, i.e., to get an idea of the nuts and bolts of the constructs Mallios and Zafiris present as "natural", it is perhaps reasonable to focus briefly on the most novel notion in the above list of mathematical tools, and the main order of business of Mallios' ADG (or abstract differential geometry), viz. vector sheaves. To wit, in their Prolegomena the authors state that "from a physical viewpoint, the construction of a sheaf constitutes the natural outcome of a complete localization process", e.g., the situation in quantum mechanics where localization is taken to refer to "the Hamiltonian eigenstates of an electron [. . .] constituting a Hilbert space basis of vectors with respect to some base space of control parameters". This already reveals part of their particular way of looking at, here, and as something of a paradigm, quantum physics. The other part to this is captured by the following characterization: "[t]he notion of a vector sheaf of states generalizes the notion of a vector space of states when there exists for instance a parametric dependence of a system from a topological space of control variables". The authors then go on as follows, explicitly highlighting the philosophy of sheaves: "[. . .] locally every section of a finite rank vector sheaf can be written as a finite linear combination of a basis of sections with coefficients from the local observable algebra [. . .] [f]or example, if X is a smooth algebra and [the observable algebra] is the [complex] algebra or [real] algebra sheaf of germs of smooth functions of X, then every section can be locally written as a finite linear combination or superposition of a basis of sections with coefficients being real-valued smooth function". And then: "the physical notion of time dependence may be implicitly introduced via time-parameterized paths on X". After this it's on to (vector) sheaf cohomology, and of course the other themes the authors mention above: the point is that they proceed to develop a formalism involving Mallios' abstract differential geometry, sheaf cohomology tailored to the needs of physics as just illustrated, and the theory of topoi to address the question of how to deal properly with locality (again, in the sense of the physicists), all to become able to deal with physics in what the authors believe to be the right geometrical way. In order to prosecute this ambitious program, Mallios and Zafiris carry out by way of preliminaries, i.e., as part of their eighty-page-long prolegomena, a careful delineation of relevant themes in quantum theory and differential and algebraic topology, with gauge theory featured, of course, and with a particular emphasis placed on what they call "the functorial imperative". The latter phrase relates to a particular contribution on the part of the second author, Zafiris, to the effect that "the notion of adjoint functors, and in particular the Hom-Tensor adjunction, should play the role of a connecting bridge between ADG and topos theory". His description of Mallios' reaction to his ideas is quite vivid and amusing, and also poignant given Mallios' recent passing; see p. viii. It is fair to say that this philosophy of adjoint functors is at the very heart of it all. In any event, once all this scaffolding is done, the authors proceed to the business of physical geometry proper, with the "General Theory" taking up the first long chapter and the second long chapter being devoted to "Applications: Fundamental Adjunctions". Under the first heading such physics topics as the Bohr Correspondence, a characterization of elementary particles in ADG terms, and even aspects of a unified field theory are addressed. In the second chapter they go on to hit, e.g., the Feynman Calculus (yes, Feynman diagrams are discussed) and the Stone–von Neumann Theorem: the former is interpreted in light of ADG, and the latter is approached via the vaunted adjunction manoeuvres. The same philosophy is applied to other deep and even hallowed physics themes—e.g., we get such things as "Schrödinger-Hamilton adjunction", "de Broglie–Einstein–Feynman adjunction", and even the exceptionally tantalizing "Einstein's equation: the 'Standard Model'". The book is manifestly dense, idiosyncratic, and ambitious in its aims. It is an interesting approach indeed to a very deep set of questions at the intersection of physics and mathematics, employing novel and even avant-garde techniques and perspectives.

Michael C. Berg

MR3337389 81Q30 78A35

Zafiris, Elias (GR-UATH)

The global symmetry group of quantum spectral beams and geometric phase factors. (English. English summary)

Adv. Math. Phys. **2015,** *Art. ID* 124393, 16 *pp.*

It has been well established that the physically significant factor in either the optical or the quantum mechanical phase is the phase difference between two "beams". The main focus of this paper is to connect a cohomological model of quantum state spaces with the formulation of a global geometric phase factor. It is also intended to pave the way for understanding the global symmetry group of a quantum spectral beam.

A. Mallios [*Modern differential geometry in gauge theories. Vol. I*, Birkhäuser Boston, Boston, MA, 2006; MR2189128] has utilized the methods of sheaf theory and sheaf cohomology to represent the essential aspects of gauge theories such as electrodynamics and Yang-Mills theories. in the paper under review, the author employs similar sheaf-cohomological concepts to focus on the crucial gauge theoretical aspect of quantum mechanics, that is, the local phase invariance of a quantum state vector.

After a brief summary of the cohomological description of quantum state spaces, the author introduces the notion of Hermitian differential line sheaves, or unitary rays, and classifies their gauge equivalence classes in terms of a global differential invariant given by the de Rham cohomology class of the curvature. Using the Chern-Weil integrality theorem, the author establishes the curvature recognition integrality theorem for unitary rays. Utilizing this recognition theorem, he then defines the notion of a quantum spectral beam and shows that it has an affine space structure with structure group given by the characters of the fundamental group.

Kotik K. Lee

MR3101153 81P10 06E99 18C50

Zafiris, Elias (GR-UATH); **Karakostas, Vassilios** (GR-UATH-QHS)

A categorial semantic representation of quantum event structures. (English. English summary)

Found. Phys. **43** (2013), *no.* 9, 1090–1123.

Summary: "The overwhelming majority of the attempts in exploring the problems related to quantum logical structures and their interpretation have been based on an underlying set-theoretic syntactic language. We propose a transition in the involved syntactic language to tackle these problems from the set-theoretic to the category-theoretic mode, together with a study of the consequent semantic transition in the logical interpretation of quantum event structures. In the present work, this is realized by representing categorically the global structure of a quantum algebra of events (or propositions) in terms of sheaves of local Boolean frames forming Boolean localization functors. The category of sheaves is a topos providing the possibility of applying the powerful logical classification methodology of topos theory with reference to the quantum world. In particular, we show that the topos-theoretic representation scheme of quantum event algebras by means of Boolean localization functors incorporates an object of truth values, which constitutes the appropriate tool for the definition of quantum truth-value assignments to propositions describing the behavior of quantum systems. Effectively, this scheme induces a revised realist account of truth in the quantum domain of discourse. We also include an Appendix, where we compare our topos-theoretic representation scheme of quantum event algebras with other categorial and topos-theoretic approaches."

MR2926916 03G30 68T30

Zafiris, Elias (GR-UATH)

Rosen's modelling relations via categorical adjunctions. (English. English summary)

Int. J. Gen. Syst. **41** (2012), *no.* 5, 439–474.

Summary: "Rosen's modelling relations constitute a conceptual schema for the understanding of the bidirectional process of correspondence between natural systems and formal symbolic systems. The notion of formal systems used in this study refers to information structures constructed as algebraic rings of observable attributes of natural systems, in which the notion of observable signifies a physical attribute that, in principle, can be measured. Due to the fact that modelling relations are bidirectional by construction, they admit a precise categorical formulation in terms of the category-theoretic syntactic language of adjoint functors, representing the inverse processes of information encoding/decoding via adjunctions. As an application, we construct a topological modelling schema of complex systems. The crucial distinguishing requirement between simple and complex systems in this schema is reflected with respect to their rings of observables by the property of global commutativity. The global information structure representing the behaviour of a complex system is modelled functorially in terms of its spectrum functor. An exact modelling relation is obtained by means of a complex encoding/decoding adjunction restricted to an equivalence between the category of complex information structures and the category of sheaves over a base category of partial or local information carriers equipped with an appropriate topology."

MR2818955 (2012j:81117) 81S10 58A12

Mallios, Anastasios (GR-UATH); **Zafiris, Elias** (GR-UATH)

The homological Kähler-de Rham differential mechanism: II. Sheaf-theoretic localization of quantum dynamics. (English. English summary)

Adv. Math. Phys. **2011,** *Art. ID* 189801, 13 *pp.*

Summary: "The homological Kähler-de Rham differential mechanism models the dynamical behavior of physical fields by purely algebraic means and independently of any background manifold substratum. This is of particular importance for the formulation of dynamics in the quantum regime, where the adherence to such a fixed substratum is problematic. In this context, we show that the functorial formulation of the Kähler-de Rham differential mechanism in categories of sheaves of commutative algebras, instantiating generalized localization environments of physical observables, induces a consistent functorial framework of dynamics in the quantum regime."

{For Part I see [A. Mallios and E. Zafiris, Adv. Math. Phys. **2011**, Art. ID 191083; MR2801347].}

MR2801347 (2012f:58002) 58A05 83C99

Mallios, Anastasios (GR-UATH); **Zafiris, Elias** (GR-UATH)

The homological Kähler-de Rham differential mechanism part I: Application in general theory of relativity. (English. English summary)

Adv. Math. Phys. **2011,** *Art. ID* 191083, 14 *pp.*

The authors apply the construction of Kähler differentials (common in algebraic geometry) to the particular case of the algebra of smooth functions on a manifold $\mathcal{A} = C^\infty(M)$, obtaining the well-known algebraic de Rham complex (e.g., see §16.6 in [A. Grothendieck, Inst. Hautes Études Sci. Publ. Math. No. 32 (1967), 361 pp.; MR0238860]) and other geometric structures with applications to general relativity. The problem is that the $\mathcal{A}$-module of 1-forms is assumed to be isomorphic to the $\mathcal{A}$-module of Kähler differentials, which is not true, as they forget to complete $\mathcal{A} \otimes_{\mathcal{R}} \mathcal{A}$ with respect to the projective topology in the sense of Definition 2 in Chapter I, §1, no. 1 in [A. Grothendieck, Mem. Amer. Math. Soc. **1955** (1955), no. 16, 140 pp.; MR0075539], thus unfortunately invalidating the subsequent constructions. *Jaime Muñoz Masqué*

MR2762573 (2012h:81015) 81P10 03G30 06E99 18F20

Zafiris, Elias (GR-UATH-IM)

Boolean information sieves: a local-to-global approach to quantum information. (English. English summary)

Int. J. Gen. Syst. **39** (2010), *no.* 8, 873–895.

The principal purpose of this article is to provide an alternative framework to that of the quantum logic introduced by Birkhoff and von Neumann. Quantum logic, in focusing on the non-distributive lattice of quantum propositions (events), to some extent neglects the overlapping structure of the Boolean subalgebras of commuting observables or, to use the terminology of Zafiris, the "measurement contexts". Zafiris proposes a more global perspective where it is the "gluing" of these contexts that is regarded as fundamental. In formulating such a shift Zafiris appeals to language and tools of category theory, a growing trend in approaches to quantum foundations and information.

An important consequence of the Kochen-Specker theorem is that one can only consistently define a valuation function separately for each of the commuting subalgebras of the non-commuting set of all bounded operators on the Hilbert space. In this sense, the commuting subalgebras represent "classical span-shots" for which quantum theory "looks" realistic. By the same token a given measurement context cannot capture the complete structure of the algebra of observables. Zafiris, motivated in part by the observation that a set of simultaneous measurements is performed in such a measurement context, introduces a global representation of the algebra of observables using the system of interlocking commuting subalgebras. Zafiris's claim is that "the real significance of a quantum structure proves to be, not at the level of events, but at the level of gluing together overlapping Boolean localisation contexts".

The basic building blocks of Zafiris's construction are: (i) The quantum event category, whose objects are quantum event algebras (think orthocomplemented lattices of orthogonal projections on Hilbert spaces). Its arrows are the appropriately defined homomorphisms. Correspondingly, the classical event category consists of Boolean event algebras. (ii) The quantum observable category $\mathcal{O}_Q$, whose objects are quantum observables, i.e. appropriate homomorphisms from the reals to quantum event algebras. (For example, when the event algebra is an orthocomplemented lattice of orthogonal projections the observables are in injective correspondence with the hypermaximal Hermitian operators on the Hilbert space.) Its arrows are commutative triangles. Correspondingly, the Boolean observable category $\mathcal{O}_B$ consists of homomorphisms from the reals to classical event algebras. (iii) The functor category of Boolean observable presheaves $\mathbf{Sets}^{\mathcal{O}_B^{\mathrm{op}}}$, i.e. the category of functors from $\mathcal{O}_B^{\mathrm{op}}$ to **Sets** and their natural transformations. It is well known that this forms a topos. Finally, (iv) the Boolean coordinatisation functor, which assigns to Boolean observables in $\mathcal{O}_B$ the underlying quantum observables from $\mathcal{O}_Q$ and similarly for the homomorphisms.

Zafiris proceeds, using the above ingredients, by constructing a pair of adjoint functors, $\mathbf{L}\colon \mathbf{Sets}^{\mathcal{O}_B^{\mathrm{op}}} \to \mathcal{O}_Q$ and $\mathbf{R}\colon \mathcal{O}_Q \to \mathbf{Sets}^{\mathcal{O}_B^{\mathrm{op}}}$, inducing an adjunctive correspondence between $\mathbf{Sets}^{\mathcal{O}_B^{\mathrm{op}}}$ and $\mathcal{O}_Q$. Then the quantum observables may be "coordinatised" by what Zafiris calls the Boolean information sieve of prelocalisations. More precisely, for a given observable $\widehat{A} \in \mathcal{O}_Q$ the Boolean information sieve of prelocalisations is a sub-functor $\mathbf{S}$ of the right adjoint functor $\mathbf{R}(\widehat{A})$, where $\mathbf{S}(\widehat{a}) \subseteq [\mathbf{R}(\widehat{A})](\widehat{a})$ for all $\widehat{a} \in \mathcal{O}_B$. For this "coordinatisation" to be consistent it must satisfy a set of compatibility conditions, in which case it is called a Boolean information sieve of localisations. These conditions capture the consistent gluing relations between the overlaps of the Boolean covers and may be understood in terms of a pullback in the category $\mathcal{Q}_Q$.

In summary, Zafiris proposes a representation of quantum observables in terms of overlapping Boolean subalgebras, pasted together by sheaf-theoretic means. This construction rests on a categorical adjunction between the topos of Boolean presheaves and category of quantum observables. Moreover, Zafiris demonstrates that this representation is consistent if and only if it forms a Boolean information sieve of localisations, such that the counit of the adjunction is an isomorphism.

A different, but closely related, set of ideas using topos theory in quantum foundations has been developed in the work of Isham and Döring; see, e.g., [A. Döring and C. J. Isham, in *New structures for physics*, 753–937, Lecture Notes in Phys., 813, Springer, Heidelberg, 2011; MR2767057] for a recent review. *Leron Borsten*

MR2488328 (2010b:81043) 81P99 18B25

Zafiris, Elias (GR-UATH)

A sheaf-theoretic topos model of the physical 'continuum' and its cohomological observable dynamics. (English. English summary)

Int. J. Gen. Syst. **38** (2009), *no.* 1, 1–27.

In recent years physicists have been turning more and more to such abstract theories as sheaf theory and topos theory to adequately describe quantum phenomena. The author of this paper has been working for quite a while on this program, in particular on the version called ADG (Abstract Differential Geometry), in which the notions of general relativity are re-formulated in a more abstract setting. This program has had some success, although it still has not managed to obtain the non-vacuum Einstein equations. Nevertheless, it is a very interesting program. What the present paper purports to do is to eliminate the normal formulation of a continuum (usually the real numbers, $\mathbb{R}$), by instead replacing it with an abstract, partially ordered set of events (a bit like a causal set, but more abstract). The usual localisation scheme is replaced with a more general one involving a topological covering. *Frank Antonsen*

MR2292776 (2008b:81017) 81P05 58A03
Zafiris, Elias (GR-UATH-IM)
Quantum observables algebras and abstract differential geometry: the topos-theoretic dynamics of diagrams of commutative algebraic localizations. (English. English summary)
Internat. J. Theoret. Phys. **46** (2007), *no.* 2, 319–382.
Zafiris is a very productive author, and he is generally very interesting to follow, despite the typical length and verbosity of his papers. The present paper summarizes his recent research based on a categorical description of physics. In his formalism, classical physics is described by a small category $\mathcal{A}_C$ consisting of commutative, unital $\mathbb{R}$-algebras as objects and with unit-preserving $\mathbb{R}$-algebra morphisms as its arrows. In his approach, quantum systems are similarly described using commutative structures, which is one of the main novelties. To start with, quantum systems are described by another small category, $\mathcal{A}_Q$, whose objects are unital $\mathbb{R}$-algebras (the observables) and with, once more, unit-preserving $\mathbb{R}$-algebra morphisms as arrows. The idea is now to reformulate this inherently noncommutative structure in terms of locally commutative ones. This is achieved by assuming the existence of a coordinatization functor $\mathbf{M}\colon \mathcal{A}_C \to \mathcal{A}_Q$. The covariant Hom functor (the Yoneda embedding) defines a functor $\mathcal{A}_C \to \mathbf{Sets}^{\mathcal{A}_C^{\rm op}}$, and a similar functor with $\mathcal{A}_Q$ as domain can be defined as

$$\mathbf{R}\colon \mathcal{A}_Q \to \mathbf{Sets}^{\mathcal{A}_C^{\rm op}}, \quad \mathbf{R}(A_Q)\colon A_C \mapsto \mathrm{Hom}_{\mathcal{A}_Q}(\mathbf{M}(A_C), A_Q).$$

The author also constructs a left adjoint functor $\mathbf{L}\colon \mathbf{Sets}^{\mathcal{A}_C^{\rm op}} \to \mathcal{A}_Q$. All of this can also be found in his earlier work. The functor $\mathbf{R}$ turns out to send quantum observable algebras A_Q in $\mathcal{A}_Q$ not just into presheaves, but, with respect to a certain Grothendieck topology, into sheaves of commutative observable algebras; i.e. a quantum observable algebra is equivalent to a sheaf of classical commutative algebras. In other words, the non-commutativity can be "removed" locally. The construction is carried out in fairly great detail in the paper. The next step is to invoke the general ideas in Mallios' Abstract Differential Geometry (ADG), in which the usual C^∞-notions of forms and de Rham cohomology are given an abstract algebraic formulation. This allows the author to introduce a covariant derivative, and hence a curvature 2-form. The vanishing of this form is then tantamount to Einstein's vacuum equations for gravity. All in all, this is a nice summary of the author's recent findings and of the ADG-programme.

Frank Antonsen

MR2289343 (2008c:18009) 18F20 18B25 18D30 81P10
Zafiris, Elias (GR-UATH-IM)
Topos-theoretic classification of quantum events structures in terms of Boolean reference frames. (English. English summary)
Int. J. Geom. Methods Mod. Phys. **3** (2006), *no.* 8, 1501–1527.
This paper is a sequel to an earlier paper by the same author [J. Geom. Phys. **50** (2004), no. 1-4, 99–114; MR2078221] in which quantum events were formalized in terms of families of Boolean algebras. The paper is part of the recent trend, pioneered by C. Isham, to apply topos theory to the foundations of quantum physics—an approach which keeps yielding important insights. By definition, a quantum events structure is a small cocomplete category, $\mathcal{L}$, whose objects are orthomodular σ-orthoposets. Similarly, a classical events structure is a small category, $\mathcal{B}$, with σ-Boolean algebras as objects. The author then studies functors from $\mathcal{L}, B$ to $\mathbf{Sets}^{\mathcal{B}^{\rm op}}$—the functor category of presheaves on Boolean event algebras. In the aforementioned earlier paper, a pair of

adjoint functors $\mathbf{R}, \mathbf{L}$ is defined:

$$\mathbf{L}\colon \mathbf{Sets}^{\mathcal{B}^{\rm op}} \leftrightarrows \mathcal{L}\colon \mathbf{R}.$$

Combining these with the Yoneda embedding, $\mathbf{y}\colon \mathcal{B} \to \mathbf{Sets}^{\mathcal{B}^{\rm op}}$ given by $\mathbf{y}[B] = \mathrm{Hom}_{\mathcal{B}}(-, B)$, he then constructs a Boolean coefficients functor $\mathbf{M}$ as $\mathbf{M}(B) = \mathbf{L}\mathbf{y}[B](B)$. All of this can be found in his earlier paper as well. The next step is to consider sub-functors of the Hom-functor $\mathbf{R}(L)$ for some object L in $\mathcal{L}$. These are called systems of Boolean pre-localizations; they are equivalent to right ideals in $\mathbf{R}(L)$. He then constructs pullbacks $\psi_B\colon \mathbf{M}(B) \to L$, and arrows $\psi_{B,B'}\colon \mathbf{M}(B) \times_L \mathbf{M}(B') \to \mathbf{M}(B)$, and finds that the pasting isomorphisms $\Omega_{B,B'} = \psi_{B,B'} \circ \psi^{-1}_{B',B}$ satisfy the cocycle conditions. This allows him to construct Boolean localizations. The paper finishes with a thorough study of the subobject classifier, which leads to a definition of quantum sets quite analogous to the usual definition of (classical) sets. With this a concept of truth associated to a quantum event structure can be defined, and the author finishes with a short discussion of how this relates to the problem of quantum measurements. *Frank Antonsen*

MR2262674 (2007h:81017) 81P10 03G12

Zafiris, Elias (GR-UATH-IM)

Sheaf-theoretic representation of quantum measure algebras. (English. English summary)

J. Math. Phys. **47** (2006), *no.* 9, 092103, 22 *pp.*

The von Neumann-Birkhoff formulation of quantum theory in terms of quantum logics (i.e., orthomodular σ-orthoposets) is beginning to receive renewed attention. The paper under review considers a small category, denoted by $\mathcal{Q}$, whose objects are pairs (M, p), where M is a quantum logic and $p\colon M \to [0, 1]$ is a probability measure on M. The arrows, $(M, p) \to (L, q)$, of this category are quantum logic morphisms $l\colon M \to L$ with $p = q \circ l$. A small category, $\mathcal{C}$, of classical logics (i.e., Boolean algebras) is defined quite analogously. What is important here, is that the topos $\mathrm{Sets}^{\mathcal{C}^{\rm op}}$—the functor category of presheaves on classical (Boolean) measure algebras—becomes related to the quantum category $\mathcal{Q}$ through a pair of adjoint functors. In a sense, then, any quantum logic can be considered locally as a classical, Boolean, logic, which is a very interesting way to look at these non-Boolean logics. This concept of localization is then expanded upon in the remainder of the paper. First of all, to define the adjective "local", one must first endow $\mathcal{C}$ with a topology. Here the Grothendieck topology is used, defined by associating a set of sieves to each object (B, p) in $\mathcal{C}$, satisfying three conditions (maximality, stability and transitivity). With this topology, $\mathcal{C}$ becomes a generating model category for $\mathcal{Q}$, making precise the intuitive notion of "quantum logics as being locally Boolean". This new notion of "categorical locality", quite devoid of the normal spatial context, will probably turn out to be a very fruitful concept, with many important generalizations and results ahead. It also sheds some interesting new light on just how quantum physics differs from its classical counterpart, and it ties in very nicely with recent views (proposed by Isham) that quantization should be considered in topos language. *Frank Antonsen*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

MR2212740 (2007a:81014) 81P10 03G12 18B99

Zafiris, Elias (GR-UATH)

Category-theoretic analysis of the notion of complementarity for quantum systems. (English. English summary)

Int. J. Gen. Syst. **35** (2006), *no.* 1, 69–89.

This paper presents a complementarity scheme in quantum systems. The scheme is based on a category-theoretic analysis of quantum systems. The paper also explains the philosophical attitude behind the work. The attitude is claimed to be different from the classical one.

The categorical notion of adjunction is proved to exist between the category of quantum event algebras, and the topos of presheaves of Boolean event algebras. The complementarity scheme is an instance of such an adjunction. The scheme consists of a vertical complementarity, and a horizontal complementarity. In horizontal complementarity, a covering system is made out of Boolean reference frames, observing the quantum system. In vertical complementarity, a pair of adjoint functors, connecting quantum event structures, and Boolean event structures, is made.

In classical philosophical attitude, an object has a representation, and is a substance possessing properties. The attitude presented here, claims to generalize such an attitude. Here, the formulation of contextual theoretical structures grounding on the object is done. The structures allow different (maybe overlapping) phenomenal descriptions.

The paper is organized into 8 sections. The category-theoretic analysis is presented in sections 3, 4, and 5. This includes preliminaries of category theory for readers new to category theory. The complementarity scheme is described in the sixth section. The seventh section presents the philosophical discussion. The remaining sections are for prologue and epilogue. *Maulik A. Dave*

MR2202817 (2006k:81030) 81P10 81P15

Zafiris, Elias (GR-UATH)

Generalized topological covering systems on quantum events' structures. (English. English summary)

J. Phys. A **39** (2006), *no.* 6, 1485–1505.

Localization systems of the structure of quantum events are studied by means of Grothendieck topologies on the base category of Boolean events algebras. It is shown that the quantum events algebra can be represented by using a Grothendieck sheaf-theoretic structure over the base category of local Boolean frames. *Jan Hamhalter*

MR2092313 (2005g:81023) 81P10 18B25

Zafiris, Elias (BG-SOFIM)

Quantum event structures from the perspective of Grothendieck topoi. (English. English summary)

Found. Phys. **34** (2004), *no.* 7, 1063–1090.

The fascinating theory of topoi is becoming ever more important in theoretical physics. This fits very well with modern physics increasingly becoming about "local structures". Einstein's general relativity made geometry (and gravity) local by using differential geometry. Quantum theory too benefits from such a move. The present paper shows one interesting way of using topoi in quantum theory. It has been known for a long time that quantum logics differ from ordinary, classical or Boolean logic. This paper sheds some interesting light on this difference. The author shows how quantum logics can be formulated in terms of local Boolean logics. As a starting point, we have quantum event algebras, which are posets endowed with a maximum element and an orthocomplentation relation. The first step is the construction of an adjunctive correspondence between Boolean presheaves of (classical) event algebras and the aforementioned quantum event algebras. By endowing the base category of Boolean event algebras with a Grothendieck topology, this leads to an equivalence between the category of quantum event algebras, called $\mathcal{L}$ by the author, and a Grothendieck topos $\mathbf{Sh}(\mathcal{B}, J)$. Here, $\mathcal{B}$ is the author's symbol for the category of Boolean event algebras, and J is the epimorphic family defining the Grothendieck topology. I find the theorem very interesting, and the paper overall quite well written, although the author doesn't seem to have made up his mind as to the readership. He spends a page or so introducing categories, but then jumps straight in and talks about small categories and colimits without defining these; readers who know the latter will not need the former, and, conversely, readers needing the introduction to categories will not understand the rest. Other than that, it is quite well written. *Frank Antonsen*

MR2088285 (2005e:81015) 81P05 18C99

Zafiris, Elias (BG-SOFIM)

On quantum event structures. III. Object of truth values. (English. English summary)

Found. Phys. Lett. **17** (2004), *no.* 5, 403–432.

Summary: "In this work we expand the foundational perspective of category theory on quantum event structures by showing the existence of an object of truth values in the category of quantum event algebras, characterized as subobject classifier. This object plays the corresponding role that the two-valued Boolean truth values object plays in a classical event structure. We construct the object of quantum truth values explicitly and argue that it constitutes the appropriate choice for the valuation of propositions describing the behavior of quantum systems."

{For Part II see [E. Zafiris, Found. Phys. Lett. **14** (2001), no. 2, 167–177; MR1830487].}

MR2078221 (2006b:58006) 58B20 18D30 18F05 58A03 81P10 81P15

Zafiris, Elias (BG-SOFIM)

Boolean coverings of quantum observable structure: a setting for an abstract differential geometric mechanism. (English. English summary)

J. Geom. Phys. **50** (2004), *no.* 1-4, 99–114.

The paper offers a new and quite interesting and promising modeling of fundamental processes in theoretical physics, as they actually are, including measurement situations and observables, along with potential "dynamicalization" of the entire setup. In this connection, one may still remark that "quantization" might be viewed as "dynamical relativization". Now, the novelty of the paper under review lies in its sheaf-theoretic framework, in point of fact, a topos-theoretic one, hence the appearance of sheaf theory, a central aspect, of course, of the former. In addition, there is its potential entanglement with the "dynamical" perspective as above; this is indicated below. Thus, in the author's terminology, Boolean observational contexts, the framework of measurement/experimental (pre)localizations, are appropriately organized into presheaves of Boolean observables, which further entail through the associated sheaves the final piece of information, consisting in an adjunction between the previous presheaves and those corresponding to quantum observables, the latter being finally represented as colimits of objects of the initial presheaves (Yoneda-Mac Lane-Grothendieck construction). What is realized here is another aspect of the so-called "Bohr correspondence principle". The same "adjunction", as before, becomes finally an "isomorphism", providing an invariance of communication/information between "Boolean windows and quantum systems". To this point, the previous setting might be associated, quite well, with "relativization", according to the above phraseology. On the other hand, in view of the very substance of the whole setup of the paper, no "space", in the classical sense, therefore, no "(smooth)manifold" concept, is present at all! Accordingly, to be able to have a confrontation with the "dynamical" part of the entire issue, the author proposes the employment of the so-called "abstract (or even, 'modern') differential geometry" (ADG) [see A. Mallios, *Geometry of vector sheaves. Vol. I*, Kluwer Acad. Publ., Dordrecht, 1998; MR1628461; *Vol. II*; MR1628457; *Modern differential geometry in gauge theories. Vol. I, II*, Birkhäuser Boston, Boston, MA, to appear]. Now, the latter point of view is of a similar, viz. sheaf-theoretic (algebraic/categorical) nature, as that of the paper under review, hence very likely applicable in the quantum regime as well, where already the classical perspective, connected with smooth manifold theory also known as "classical differential geometry" (CDG) has proven to be ineffective, when confronted, for instance, with "singularities/infinities" in the deep quantum ("sub-Planck" territory). In that context, the author supplies here only hints, reserving the right of coming back with details "at a later stage". Yet, he also finally remarks that "the real significance of a quantum structure proves to be not at the level of events, but at the level of gluing together [appropriately compatible] observational contexts". So, in other words, one looks at/understands not the objects, but the functions/laws describing them; indeed, one has here an entirely categorical situation that is also in the spirit of ADG, as before, in the sense that one actually has, therein, simply functors and natural transformations between them. *A. Mallios*

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MR2074656 (2006b:81024) 81P10 18B99

Zafiris, Elias (BG-SOFIM)

Interpreting observables in a quantum world from the categorial standpoint. (English. English summary)

Internat. J. Theoret. Phys. **43** (2004), *no.* 1, 265–298.

Summary: "We develop a relativistic perspective on structures of quantum observables in terms of localization systems of Boolean coordinatizing charts. This perspective implies that the quantum world is comprehended via Boolean reference frames for measurement of observables, pasted together along their overlaps. The scheme is formalized categorically as an instance of the adjunction concept. The latter is used as a framework for the specification of a categorical equivalence signifying an invariance in the translational code of communication between Boolean localizing contexts and quantum systems. Aspects of the scheme semantics are discussed in relation to logic. The interpretation of coordinatizing localization systems, as structure sheaves, provides the basis for the development of an algebraic differential geometric machinery suited to the quantum regime."

MR1830486 (2003a:81008a) 81P05 18C99

Zafiris, Elias (4-LNDIC-TB)

On quantum event structures. I. The categorical scheme. (English. English summary)

Found. Phys. Lett. **14** (2001), *no.* 2, 147–166.

MR1830487 (2003a:81008b) 81P05 18C99

Zafiris, Elias (4-LNDIC-TB)

On quantum event structures. II. Interpretational aspects. (English. English summary)

Found. Phys. Lett. **14** (2001), *no.* 2, 167–177.

Summary of Part I: "In this paper a mathematical scheme for the analysis of quantum event structures is proposed based on category theoretical methods. It is shown that there exists an adjunctive correspondence between Boolean presheaves of event algebras and quantum event algebras. The adjunction permits a characterization of quantum event structures as Boolean manifolds of event structures."

Summary of Part II: "In this paper we analyze the physical semantics and propose an interpretation of quantum event structures from the perspective offered by the categorical scheme of Part I."

MR1805053 (2001k:81018) 81P10 18B25

Zafiris, Elias (4-LNDIC-TB)

Probing quantum structure with Boolean localization systems. (English. English summary)

Internat. J. Theoret. Phys. **39** (2000), *no.* 12, 2761–2778.

The interpretation of quantum mechanics is a topic that continues to intrigue many of us. The author of this paper continues previous research based on the use of category theory, of which he first gives a brief but readable introduction to the essential concepts. We are accustomed to think in terms of classical theory, which in terms of logic translates into Boolean algebras. Yet, the underlying quantum reality is non-Boolean. The task set by the author is then to reconcile this apparent contradiction, by providing a description of quantum systems (here modelled as quantum logics) in terms of "local" Boolean reference frames, in a way similar to that of Minkowski reference frames in general relativity. He thus introduces the notion of a Boolean manifold. The paper is an interesting addition to the literature on the introduction of methods of categorical logic or topos theory to theoretical physics and to the interpretation of quantum theory.

Frank Antonsen

MR1679136 (99m:81033) 81P99 81S99

Halliwell, J. J. (4-LNDIC-TB); **Zafiris, E.** (4-LNDIC-TB)

Decoherent histories approach to the arrival time problem. (English. English summary)

Phys. Rev. D (3) **57** (1998), *no.* 6, 3351–3364.

Summary: "What is the probability of a particle entering a given region of space at any time between t_1 and t_2? Standard quantum theory assigns probabilities to alternatives at a fixed moment of time and is not immediately suited to questions of this type. We use the decoherent histories approach to quantum theory to compute the probability of a nonrelativistic particle crossing $x=0$ during an interval of time. For a system consisting of a single nonrelativistic particle, histories coarse grained according to whether or not they pass through spacetime regions are generally not decoherent, except for very special initial states, and thus probabilities cannot be assigned. Decoherence may, however, be achieved by coupling the particle to an environment consisting of a set of harmonic oscillators in a thermal bath. Probabilities for spacetime coarse grainings are thus calculated by considering restricted density operator propagators of the quantum Brownian motion model. We also show how to achieve decoherence by replicating the system N times and then projecting onto the number density of particles that cross during a given time interval, and this gives an alternative expression for the crossing probability. The latter approach shows that the relative frequency for histories is approximately decoherent for sufficiently large N, a result related to the Finkelstein-Graham-Hartle theorem."

MR1678167 (99k:83130) 83E30 81T30

Zafiris, Elias (4-LNDIC-TB)

Generalized Raychaudhuri and area change equations for classical brane models. (English. English summary)

Phys. Rev. D (3) **58** (1998), *no.* 4, 043509, 8 *pp.*

Summary: "In this paper we derive generalizations of the Raychaudhuri and area change equations for brane worldsheet congruences embedded in spacetimes with torsion. The kinematical description we develop is fully covariant and based on the use of projection tensors. We apply our results to the case of string models."

MR1658739 (99m:53139) 53C80 83C99

Zafiris, Elias (4-LNDIC-TB)

Covariant generalisation of Codazzi-Raychaudhuri and area change equations for relativistic branes. (English. English summary)

J. Geom. Phys. **28** (1998), *no.* 3-4, 271–288.

Summary: "In this paper we derive the generalisation of Gauss-Codazzi, Raychaudhuri and area change equations for classical relativistic branes and multi-dimensional fluids in arbitrary background manifolds with metricity and torsion. The kinematical description we develop is fully covariant and based on the use of projection tensors tilted with respect to the brane worldsheets." *Riccardo Capovilla*

MR1613857 (99k:83129) 83E30 83C99

Zafiris, Elias (4-LNDIC-TB)

Kinematical approach to brane worldsheet deformations in spacetime. (English. English summary)

Ann. Physics **264** (1998), *no.* 1, 75–92.

This paper deals with a formulation of the geometry of the trajectories (or worldsheet) in spacetime of relativistic extended objects (or branes).

MR1613692 (2000c:83015) 83C05 83C20

Zafiris, Elias (4-LNDIC-TB)

Irreducible decomposition of Einstein's equations in spacetimes with symmetries. (English. English summary)

Ann. Physics **263** (1998), *no.* 2, 155–178.

Summary: "In this paper we develop a method to write Einstein's field equations for general matter in an irreducible form that fully incorporates the character of a symmetry introduced in spacetime. We express our method using the framework of double congruences and thus the formalism is covariant. The basic notion on which it is based is that of the geometrisation of a general symmetry. We apply our general scheme to the case of a spacelike conformal Killing vector field on the spacetime manifold regarding the situations of a perfect fluid and anisotropic matter."

MR1480834 (98m:83018) 83C20

Zafiris, Elias (4-LNDIC-TB)

Incorporation of space-time symmetries in Einstein's field equations. (English. English summary)

J. Math. Phys. **38** (1997), *no.* 11, 5854–5877.

In this article, Zafiris develops a method of writing the Einstein field equations in a form that fully incorporates the characteristics of a given symmetry.

He starts off by giving a short review on the different types of symmetries which arise from the existence of a Lie algebra of vector fields on the spacetime manifold invariant under certain geometrical objects. Next the idea is to geometrize the symmetry, i.e., employ the description of it as a set of conditions on the kinematic quantities that characterize the congruenge of the integral lines of the vector field that generate the symmetry. This geometrization enables the author to express the symmetry in a form that makes it possible to incorporate it in the field equations in a straightforward and intrinsic way. The incorporation is obtained by performing Lie derivation of the field equations. The situation of a spacelike conformal Killing vector symmetry is described, mainly in the perfect fluid case. Finally, the advantages of writing out the field equations in such a form are discussed. *Maria Piedade Ramos*