Chapter 0

Prolegomena

0.1 Exordium

The major aim of the application of Analysis and Differential Geometry in Physics is the setting up of a mechanism providing a precise description of the emergence of geometric spectrums due to dynamical interactions, which can be further used for making predictions. In this manner, the notion of a geometric spectrum is considered as the outcome of a physical law or more generally of a dynamical interaction of a particular form. This raises immediately the question if there exists an approach to physical geometric spectrums that is independent of any coordinate point manifold background, in the sense that it refers directly to the physical relations causing the appearance of these spectrums without the intervention of any *ad hoc* coordinate choices. The answer provided to this question in this book is that the theory of *differential vector sheaves*, that is geometric vector sheaves equipped with a connectivity structure and obeying appropriate cohomological conditions, provides the sought after functorial tool for a *universal* and *natural* approach to *physical geometric spectrums*. The major difference of the proposed approach in comparison to the traditional ones based on classical differential calculus and differential geometry of smooth manifolds consists in the realization that a classical analytic technique is susceptible to a natural background-independent generalization if it is *localizable* by sheaf-theoretic means. In this case the technique can be expressed functorially, that is by means of natural transformations of sheaf functors via the machinery of homological algebra. This is of crucial significance for setting up a mechanism describing the emergence of physical geometric spectrums where the notion of background smoothness is inapplicable.

The meticulous development of the concepts of general relativity theory as well as of gauge theories has already taught us an extremely important lesson in this quest. This lesson may be condensed in the following form: Starting from a local description of physical phenomena, one should always seek for a *covariant* or *gauge invariant* formulation of the associated physical laws according to some criterion of *local symmetry* or *local equivalence*. Then, the specification conditions of the corresponding physical geometry, in each particular case, emerge together with the *invariant* formulation and are independent of the local gauges used in the first place. Several issues of an interpretational nature arise together with the requirement of naturality, which force a critical re-examination of the conceptual choices and assumptions initially endorsed from a physical viewpoint. The two most significant ones in relation to a natural approach to physical geometry are the concepts of *locality* and *differentiability*. We will attempt to comment briefly on the related problems so as to motivate conceptually the focus of the whole treatise.

First, the notion of *locality*, or more broadly *localization*, subsumed in the development of both general relativity and classical gauge theories is based on the notion of *spacetime localization*. In other words, at least a local geometric coordinate spacetime background is initially assumed for the local description of physical phenomena. Not only this, but a further more decisive assumption regarding the global nature of this background as a *smooth manifold* is traditionally imposed, so that all physical quantities are subordinate to a covariant description subject precisely to this underlying global background manifold. Of course, from a gauge-theoretic point of view, the major physical role is played by the *local gauge freedom* in the definition of physical quantities, which is represented or realized by means of internal spaces or fibers of local degrees of freedom over each point of the underlying spacetime manifold. In this manner, it seems unavoidable to have an ontology of dependence on two distinct notions of localization. The first or external one is the localization of spatiotemporal variables being subordinate to the global smooth manifold determination, and the second or internal one is the fiber-wise localization of physical quantities. The advent of quantum theory has posed serious difficulties in interpreting the former notion of external localization. Whereas the internal notion of gauge-theoretic localization *persists upon quantization*, the external one is deprived of its original spatiotemporal semantics. Various attempts at quantizing the classical spacetime manifold geometry have not stood up to the expectations. The success and experimental validation of quantum theory requires, first of all, a careful reconsideration of what we mean by the notion of localization of physical quantities. The pertinent question is if it is possible to have a working notion of localization *independently* of the existence of a geometrical spatiotemporal background. Moreover, since all gauge-theoretic concepts are still physically relevant upon quantization, why should their local formulation be dependent on an underlying smooth manifold base space? Addressing these questions has serious consequences and manifests the significance of a critical remark of Hermann Weyl, who noted: "*While topology has succeeded fairly well in mastering continuity, we do not yet understand the inner meaning of the restriction to differential manifolds. Perhaps one day physics will be able to discard it*".

Reflecting on this remark, it becomes questionable why the physical notion of gauge-theoretic localization should be implementable exclusively over a smooth manifold. In other words, why should physical quantities be representable as smooth functions up to gauge scaling factor over a smooth manifold? There is *no intrinsic* *reason* that a local gauge theoretic argument requires the assumption of global smoothness in the functional representation of physical quantities. Notwithstanding this fact, the assumption of continuity seems to be necessary. After all, continuity is a *local* notion and the concept of a topological base space has been precisely formulated in order to express what a continuous function actually is. Thus, from a physical perspective, it would be natural to consider the notion of gauge-theoretic localization for *globally non-smooth* or *distribution-like* continuous physical quantities. Given that these physical quantities form commutative algebras, which can be evaluated in a number field through measurement, the topological representation theorems of Stone and Gelfand provide a concrete *topological realization* of these algebras as algebras of continuous functions over a topological space. Most important, this topological space is neither postulated ad hoc, nor constructed by imposing metrical relations, but is defined by *prime* or indecomposable constituents of the algebras themselves, at least locally, forming in this sense its spectrum.

There are two *caveats* of this modeling philosophy, which should be considered carefully from a physical perspective. The first is that the notion of local gauge theoretic localization *does not* really require, neither the imposition of a global base topological space, nor the assumption that physical quantities should be representable as global continuous functions over such a space. It actually requires much weaker assumptions: First, the notion of a *local scaffolding of control variables*, which may be even determined via operational arguments depending on the experimental procedures for recording events. In a formalization of this type, a global topological space can be only *implicitly* assumed. Put differently, a global topological base space can be only *indirectly* implicated via these local scaffoldings of control parameters, which may be thought of as forming local *covering families* of the former, subject to some well-defined conditions. For example, the open sets of a topological space qualify as local covering families. The most general notion of local covering scaffoldings of this form is provided by the notion of a *Grothendieck topology*. Although the latter notion seems to be very abstract for concrete physical applications, it turns out that precisely the opposite is the case. More concretely, a local scaffolding of control parameters, considered as the locus of gauge localization of a physical quantity, plays the role of a *hole* in a *covering sieve*, viz. an opening through which by specifying the control parameters locally, information can be obtained via measurement. The fact that these openings are localized holes in sieves is of physical significance. This is the case because sieves are *not ad hoc* coordinate objects but they are *functorial*. This is a technical category-theoretic term whose precise meaning in a physical context meets the requirement of *covariance* of local descriptions *independently* of the existence of an underlying global background. Thus, it addresses the need for a natural approach to physical geometry in a background independent manner, and most important, it provides the most appropriate concept of grounding the notion of gauge-theoretic localization of physical quantities. The instantiation of a covering sieve via an efficient physical measurement procedure qualifies the physical quantities as *elements of a presheaf*. The functoriality here means that physical information should be consistent under the operation of *restriction* with respect to the openings of a covering sieve.

Now, there is a technical process called *sheafification* which turns these elements into sections of a sheaf *completing* in this manner the localization procedure. Sheaf functoriality means that physical information should be consistent not only under the operation of restriction, but also under the operation of *extension* with respect to the openings of a covering sieve. There are several consequences of the sheaf-theoretic conceptualization of physical quantities. First, they are not considered as globally defined continuous functions over a global base space, but as *continuous sections* of sheaves, meaning that they are defined locally with respect to the openings employed in each case. Second, physical quantities are not specified by their measured values like globally defined functions do, but they are determined by their *germs*, that is appropriate compatible equivalence classes of sections with respect to these openings. Third, the requirement of compatible amalgamation of local sections under extension with respect to the openings, is expressed through the *gluing* condition characterizing a sheaf of physical quantities. In turn, the gluing condition can be expressed cohomologically in terms of Čech 1-cocycles and their respective *cohomology classes*, which forces a cohomological interpretation of gauge theoretic localization in sheaf-theoretic terms. The latter in conjunction with the above, leads naturally to a *cohomological classification* of state spaces by invariants of a global character. Thus, it paves the way for a coordinate-free description of physical geometry in a background-independent manner. Therefore, the sheaf-theoretic conception of gauge theoretic localization is in accordance to the constraints imposed by quantization.

It is instructive to point out that even in the case of quantum mechanics, all the above aspects become physically relevant. We know that the state of a quantum system is completely determined with respect to some maximal algebra of commuting, and thus, co-measurable observables. Of course, there exist many different commutative algebras of this form, but the effectuation of a measurement procedure always requires the choice of one of them. In this context, the requirement of functoriality is met by considering covering sieves, whose openings see only such local maximal commutative algebras of observables of a quantum system. Moreover, the determination of the state, with respect to a local commutative algebra of observables, is taking place only up to an arbitrary phase, which in turn, specifies the gauge freedom of the state vector of a quantum system. This leads naturally to a cohomological description of quantum state spaces and prepares the ground for a deeper understanding of quantization in relation to observable physical geometries of some form. More precisely, a quantum state space becomes a locally free sheaf of modules of finite rank 1 with respect to a maximal commutative algebra of co-measurable observables, and thus, is described only up to equivalence with respect to isomorphism classes of such sheaves. Thus, the cohomological classification of these sheaves is of utmost physical importance for interpreting the *global aspects of quantization*, for instance Dirac's global quantization condition and the role of the integers.

We may recapitulate the above, by stating the conclusion that the notion of sheaf-theoretic localization of observables and states along the previous lines constitutes a functorial and background manifold independent approach to the localization issue, which is perfectly compatible with the requirements of local gauge freedom and invariance. We stress the fact that according to this approach, the state space *locally* is not only a vector space over some number field but a *free module* over a corresponding local algebra of observables. Thus, the local multiplying coefficients are sections from a commutative algebra of observables and not just numbers from the evaluation field. Still, we can define all the usual notions locally with respect to them, for example we obtain well-defined notions of algebra-valued bilinear forms, inner products and metrics. Notice again that these sections are *not* required to be sections of the sheaf of germs of smooth functions. In contradistinction, the type of sheaf sections should not be chosen ad hoc, but should be specified by other means. Reflecting on this point, there appears the idea that the type of sheaf sections fitting to a particular physical problem should be actually determined by *dynamical means*. Put differently, the notion of invariant sheaf-theoretic localization should not be considered in isolation from the relevant dynamical aspects of a physical theory. After all, a physical theory is a theory of interactions of some kind, and it should be only these interactions that can determine naturally a corresponding physical geometry.

The spectral conception of physical geometry as a result of a dynamical physical law, dictates that locally by duality, the sheaf of observable coefficients should be actually determined as a *local solution* of the dynamical equations of motion. In other words, locally the algebra sheaf of coefficients should be the sole *dynamical variable* determinable as a solution of the equations of motion. It is important to note that this idea is compatible with both the conception of geometry according to Einstein's field equations in the general theory of relativity and the gauge-theoretic principle dictating that the requirement of invariance implies dynamical interactions. Actually, the postulate of invariance, regarding the natural implementation of localization in a physical environment, constitutes the *bridge* between the general mathematical notion of sheaf-theoretic localization and the dynamical constraints imposed by a physical theory. Put concisely, we claim that a process of localization expressed in terms of an algebra sheaf of coefficients in the context of a physical theory should not be chosen ad hoc, but it should be determined as a solution by dynamical or differential means. This claim encapsulates precisely the physical meaning of invariant sheaf-theoretic localization. Notice that we do not distinguish between kinematical and dynamical aspects of a physical theory, since no geometric background substratum is assumed a priori even locally. This implies that even the inertial structure of a physical theory is considered to be of a dynamical character, since it can be formulated by differential means.

The above brings us to the second fundamental issue in relation to a natural approach to physical geometry, viz. the issue of *differentiability*. The pertinent question refers to the problem of formulating what we mean by differentiability in a background manifold independent manner. Clearly, this is the most crucial aspect of a physical theory, which aims to provide predictions about the behavior of a physical system by expressing locally the laws of motion in differential terms. We would dare to say that the *sole purpose* of the background smooth manifold background in physical theories is to afford a *local analytic* mechanism of differentiation, which originates from the locally Euclidean structure. Therefore, the standard differential calculus of Euclidean vector spaces is transferred locally to a smooth manifold, which paves the way for the local analytic formulation of the dynamical laws of motion in terms of smooth functions. Furthermore, the natural physical requirement of independence of geometric objects from the description in a local coordinate system in this context, necessitates the modification of the standard differentiation to the *covariant* one, that is to the notion of the *covariant derivative* along a smooth curve in the manifold. In a nutshell, covariant differentiation provides the means of parallel transport along a smooth curve in the background manifold, and thus the means of *infinitesimal comparison* formulated locally in terms of smooth function coefficients. Despite the immense success of this analytic differentiation concept in theoretical physics, there are serious problems to be addressed: First, is the general notion of covariant differentiation dependent on the smoothness of the underlying curve and the smoothness of the local coefficients used for its formulation? Second, is the above notion of covariant differentiation the natural infinitesimal localization of a general parallel transport process? In other words, does it provide the means of infinitesimal comparison in a local context functorially? Third, the usual description of parallel transport is expressed as a rule of lifting curves from the base manifold to the tangent bundle. From a physical viewpoint, this is problematic because curves on the spacetime manifold are not given a priori, but actually emerge as solutions of the dynamics and are interpreted in this way as smooth histories of observed events.

In the following Section we will explain how all the above issues are addressed from the viewpoint of *Abstract Differential Geometry*, which constitutes the basic conceptual and technical framework of this work. It is instructive at this stage to point out that according to this framework, the general notion of variation or differentiability caused by a dynamical interaction should be expressed intrinsically in terms of a *connection*. The notion of a connection is formulated functorially as a natural transformation from the vector sheaf of states over the observable algebra sheaf of a physical system to the sheaf of vector-valued algebraic differential forms obtained by the sheaf-theoretic localization of the standard algebraic Kähler-de Rham construction. This natural transformation should obey the Leibniz rule, so that in effect can be interpreted physically as a covariant derivative of the sections of the vector sheaf of states according to the pertinent law of variation. From a gauge-theoretic viewpoint, a connection on a vector sheaf of states is interpreted locally as the *potential* of a gauge field, which causes the variation or differentiability of its sections. In this manner, the pair constituted by a vector sheaf of states together with a connection, viz. a differential sheaf, represents the *dynamical variation mechanism* induced by a gauge field, where the connection provides the natural coordinate-free generalization of the notion of a differential equation. The integrability properties of a connection under extension of this mechanism from the local to the global level are derived by means of *sheaf cohomology* theory and not by the application of the usual analytic techniques of the differential geometry of smooth manifolds. The *curvature* of a connection is a tensorial homological observable quantity, which is measured by the gauge field *strength* of the associated local potentials in a *covariant way* with respect to the corresponding observable algebra sheaf. Most important, the formulation of the dynamical variation mechanism should be *functorial* with respect to any utilized observable algebra sheaf. This is expressed as the principle of *invariance* of the differential geometric mechanism with respect to cohomologically suitable algebra sheaves of coefficients. The requirement of functoriality is the crux of a natural approach to physical geometry determined by dynamical means, in the sense that invariance determines dynamical interactions, and thus constitutes the unique natural way to detect a physical geometric spectrum as an outcome of a physical variation law.

0.2 Basic Working Notions

Ever since Leray and Cartan in the 1950's formulated the concept of a sheaf, the various examples and applications of sheaves have come to play a major role in such diverse fields of mathematics as several complex variables, algebraic geometry and algebraic topology. In particular, the work of Grothendieck in algebraic geometry and the generalization of the notion of sheaves beyond the realm of topological spaces, making it applicable over general category-theoretic sites (categories equipped with covering systems called Grothendieck topologies), has created a wealth of new models enriched with the power of the methods of homological algebra.

The concept of a *sheaf* expresses essentially *gluing* conditions, namely the way by which local structural algebraic data (like groups or modules) can be collated compatibly into global ones over a covering system, like the one defined by the open sets of a topological space. In general, a sheaf may be thought of as a *continuously variable* algebraic structure, whose continuous variation is considered over specified local covering domains interlocking together non-trivially. More recently it was discovered that sheaves also have important applications in logic and model theory. For example, the category of all sheaves on a fixed topological space can be viewed as a model of constructive set theory. One of the most interesting aspects of the sheaf logic formalism is that it provides a framework in which a sheaf can be viewed as a model of a space which does *not* have a local structure defined by points but a local structure which is intrinsically continuous with respect to a family of covers, for example the open sets covering a topological space. More concretely, from this perspective the main difference from the usual notion of a set (or an algebraic structure defined set-theoretically) is that in a sheaf we have an extra intrinsic *internal* structure of *granulation* of the elements (called sections) by the partially ordered set of the covers. Heuristically, we think of the covers as measures of definability and the elements of a sheaf restricted to a cover are exactly the members defined to the *extent* provided by this cover.

From a physical viewpoint, sheaf-theoretic ideas pertain to the problem of *extendibility* of *local* observable properties into *global* descriptions, to the problem of *gauge-invariance*, as well as to the problem of *structural order*. Initially, the foundational significance of sheaf theory for physics is related with the possibility of re-examining the problem of locality, together with the inter-relation between local and global physical descriptions under a new rigorous theoretical perspective. More precisely, sheaf-theoretic methods allow the *disassociation* of localization processes from *locally Euclidean* metrical spatiotemporal contexts, which are based on the *point*-localization structure of the real line. This becomes possible via the replacement of the classical metrical gauge of point localization on a background smooth manifold by a sheaf gauge of algebraic-topological localization on an appropriate categorical site. A site, for example a covering system of open sets on a topological space, should be solely used to *support* the relational local-to-global information pertaining to the corresponding scale of observable behavior. This strategy permits the effectuation of generalized localization schemes being capable of capturing even the microscopic non-smooth observable scale of quantum phenomena, where a locally Euclidean model is totally inapplicable. Of course, the proper implementation of these localization schemes should be supplemented by an appropriate sheaf-theoretic differential mechanism generalizing the usual analytic differential geometric framework of smooth manifolds. This requirement has been met by the successful integration of the methods of homological algebra into sheaf theory.

More concretely, since it is possible to interpret standard algebraic notions as homomorphism, kernel, image, subobject and quotient object for sheaves, in such a way that these concepts have essentially the same meaning as in algebra, one can consider them from a categorical standpoint and apply all the constructions of homological algebra to sheaf theory. The resulting category of sheaves has the same classical properties as the category of Abelian groups or the category of modules; in particular, one can define for sheaves direct sums, direct products, tensor products, inductive limits, and other concepts. In this way, the apparatus of sheaf theory has penetrated into various fields of mathematics providing an effective algebraic tool especially in those areas which ask for global solutions to problems whose hypotheses are local. This is due to the fact that there is a *natural* definition of the cohomology of a site (for example, a topological space) with *coefficients in a sheaf.* In particular, the development of algebraic geometry by Grothendieck led to the crystallization of the idea that the natural argument of a cohomology theory is a *pair* consisting of a space (or more generally a category-theoretic site) *together* with a sheaf defined over it, rather than just a space. This realization has been transferred to the field of complex analysis and more recently to the field of differential geometry by the development of the geometric theory of vector sheaves equipped with a connection *à la Mallios*, called Abstract Differential Geometry (*ADG*). Now, the sheaf gauge of algebraic-topological localization and extendibility sought-after is provided by *sheaf cohomology*.

Cohomology can be viewed as a method of assigning *global invariants* to a topological space (or more generally a categorical site) in a *homotopy-invariant* way. The cohomology groups measure the *global obstructions* for extending sections from the local to the global level (for example extending local solutions of a differential equation to a global solution). For instance, de Rham cohomology measures the extent that closed differential forms fail to be exact, and thus the obstruction to integrability (according to the lemma of Poincaré every closed differential form is locally exact). The *de Rham theorem* asserts that the homomorphism from the de Rham cohomology ring to the differentiable singular cohomology ring given by integration of closed forms over differentiable singular cycles is a ring isomorphism. The sheaf-theoretic understanding of this deep result came after the realization that both the de Rham cohomology and the differentiable singular cohomology are actually *special* isomorphic cases of sheaf cohomology. In particular, it has been also clarified that the de Rham cohomology of a differential manifold depends *only* on the property of paracompactness of the underlying topological space. The generalization of the usual de Rham theory on manifolds in sheaf-theoretic terms is most welcome also from a physical point of view, since it is a fact that the structure underlying an *intrinsic* approach to differential calculus, and in particular, differential equations in physics in terms of differential forms is actually *de Rham-cohomology*.

From an algebraic standpoint, the conceptual categorical skeleton of the *generative* mechanism of differential calculus and, *in extenso*, differential geometry is a consequence of the existence of a pair of adjoint functors, or equivalently an *adjunction* expressing the *inverse* algebraic processes of extending and restricting the scalars. Classically, the differential mechanism is based on the fundamental notion of a *connection* on a *projective module* over a commutative unital ring of scalars (or algebra over a field) considered together with the associated de Rham complex. A connection on a module induces a process of *infinitesimal extension* of the scalars of the underlying ring, which is interpreted geometrically as a process of *first-order parallel transport* along infinitesimally variable paths in the spectrum-space of this ring. The next stage of development of the differential mechanism involves the satisfaction of appropriate global requirements referring to the transition from the infinitesimal to the global level. These requirements are of a *homological* nature and characterize the integrability property of the variation process induced by a connection. Moreover, they are properly addressed by the construction of the *de Rham complex* associated to an integrable connection. The non-integrability of a connection is characterized by the notion of *curvature* bearing the semantics of observable disturbances to the process of cohomologically unobstructed variation induced by the corresponding connection.

It is instructive to emphasize that the conceptualization of the classical differential mechanism along these lines *does not* presuppose the existence of an underlying differential manifold. This observation is particularly important because it provides the possibility of abstracting in functorial terms both the definition of a connection and the associated de Rham complex as well. The functorial recasting of the differential mechanism becomes possible by the appropriate qualification of the information encoded in the *extension/restriction of scalars adjunction*. This information permits the definition of the universal object of differential 1-forms, and subsequently, the definition of a connection, together with the associated de Rham complex. The benefits we obtain from the functorial reformulation of the differential mechanism are multiple. First, the differential mechanism becomes explicitly *non-dependent* on the existence of a smooth underlying manifold. Second, it can be applied effectively in a variety of *topological* and *topos-theoretic* localization environments, conceived as categories of sheaves, under the satisfaction of the appropriate cohomological requirements. Finally, the functorial recasting of the generative mechanism of differential calculus and differential geometry is of paramount importance for theoretical physics, because it can be properly used for the functorial formulation of dynamics taking into account the particular *localization properties* of the *observables* in the regime of each physical theory, which are modeled by sheaf-theoretic means. It is precisely such a functorial formulation of differential geometry which has been accomplished by ADG.

The development of the differential geometry of vector sheaves over an abstract topological space has given rise to a fully-fledged theory, which has been conceived and formulated in detail by Mallios [5-7], see also Vassiliou [1]. The significance of ADG for physics has been also shown by an explicit reconstruction and generalization of the framework of gauge theories (Maxwell and Yang-Mills) in sheaf cohomological terms by Mallios [15, 21], see also Mallios [14, 16, 18]. The main conceptual lesson obtained by this generalization is that any physical problem, which is based on the principle of *local gauge invariance*, should be formulated directly in terms of the relevant physical-geometric objects (sections of sheaves) without the intervention of any locally Euclidean space. ADG generalizes the classical differential geometric mechanism of smooth manifolds, by proving explicitly that most of the usual differential geometric constructions on manifolds can be carried out by purely sheaf-theoretic means without any use of any sort of smoothness or any of the conventional analytic methods of spatial differential calculus that is associated with it. The technique used for the generalization of the analytic methods of Classical Differential Geometry (CDG) consists in the following: Initially, a concept of CDG is *suitable* for extension to a broader differential context (beyond the smooth context of manifolds) if it is liable to a process of sheaf-theoretic *localization* explicated by Mallios [13]. Then, this concept is expressed by means of a *natural transformation* of sheaves. The prominent role is played by the *algebra sheaf* of coefficients playing the role of a *"functional arithmetic"*, see e.g. Mallios [17-18], meaning that all geometric objects involved in the formalism are locally expressed in terms of its sections. In physical terms, we interpret the sheaf of coefficients as a *sheaf of observable algebras*, see Zafiris [3, 7]. In this way, an algebra sheaf of coefficients is *not* constrained *ab initio* to be a smooth one (which is used to model an underlying smooth manifold as its spectrum), but it may be for instance a *singular sheaf* of generalized functions including distributions, defined by Rosinger, see e.g. Mallios-Rosinger [1, 2]. In the most general case, we may consider as algebra sheaves of coefficients those of *soft topological algebra sheaves*, which are "closed" with respect to complete projective topological tensor products, see Mallios [4, 6]. We mention parenthetically that the interested reader may consult the monographs by Mallios [1] and Fragoulopoulou [1] for pure mathematical expositions of the theory of topological algebras, the details of which nevertheless will be mostly not needed in the present work. In particular, topological algebra sheaves fitting in the above characterization, which are also *self-adjoint*, are suitable for the development of an extended differential geometric mechanism generalizing the one based on the non-normed topological algebra sheaf of smooth coefficients with interesting physical applications, see e.g. Mallios-Zafiris [1-3], Mallios-Raptis [1-3] and Mallios-Rosinger [1, 2]. This generalization has been motivated to a large extent from previous work carried out in the context of Banach-*Š*ilov algebras having the Banach approximation property by Selesnick [5, 6]. Thus, concerning physical issues, we may employ for example a soft self-adjoint Fréchet-*Š*ilov locally m-convex topological algebra as an algebra of observables, whose *Gelfand spectrum* is then a (Hausdorff) paracompact space. Then, any vector sheaf of rank 1, called a line sheaf, on the topological spectrum of this algebra of observables can be equipped with a connection, whose curvature defines an *integral cohomology class*.

We stress again the prominent physical role of an appropriate sheaf of observable algebras by the fact that all vector sheaves (of states), group sheaves of automorphisms (symmetries) and sheaves of differential forms (gauge potentials and field strengths) partaking in ADG are typically required to be *locally free* sheaves of modules and of *finite rank* over the sheaf of observable algebras. The concept of differentiability of the sections of vector sheaves (covariant derivative of sheaf sections) is formulated in terms of connections. A connection on a vector sheaf bears the dynamical semantics of a potential locally, which causes the differentiability of its sections. In this way, a vector sheaf equipped with a connection may be thought of as a *coordinate-free* generalization of the notion of a differential equation. The integrability properties of a connection under extension of the differential mechanism from the local to the global level are considered from the prism of sheaf cohomology and not by the typical analytic methods. The curvature of a connection is measured by the strength of the associated potential covariantly with respect to the corresponding sheaf of observable algebras.

One of the most important conclusions crystallized by ADG is a new conceptual perspective on the notion of a *base geometric space* for the formulation of differential geometry. A base geometric space provides merely a *"localization basis"* for the formulation of the differential mechanism, whereas the mechanism itself is *not* subordinate to this basis, meaning that it is not dependent on any particular localization basis, although it should respect it. Thus, the object of primary significance in ADG is *not* the base space itself but the sheaf of observable algebras localized over it. Put differently, the differential geometric mechanism is completely expressed locally in terms of the sections of the observable algebra sheaf, whereas the base space itself is only the *carrier* of the sections defined over it, or else it provides only the *means of localization* of these sections. Then, it is precisely the functoriality of ADG with respect to cohomologically appropriate observable algebra sheaves, which permits the detection of dynamical interactions, identified for instance in the case of line sheaves in terms of the integral cohomology class of the curvature. This is a crucial insight and provides a *justification* of the gauge principle and its consequences used successfully in theoretical physics. A particular manifestation of this principle in quantum mechanics is related to the local abelian *phase invariance* in the specification of the state vector of a quantum system. The locality demand in the formulation of the gauge principle is crucial because although it is true that the phase of a quantum system can be always gauged away locally, this is not the case globally.

The use of the differential geometry of vector sheaves and in particular sheaf cohomology, according to the functorial framework of ADG, is novel among the techniques of theoretical physics and arises from a sheaf cohomological local-to-global representation of physical properties in contrast to the conventional description in terms of local differential equations on a smooth spacetime manifold. This departure is necessitated to a certain extent by problems related with the modeling of the spacetime continuum. The usual spacetime manifold model based locally on the properties of Euclidean spaces and ultimately on the properties of the *real line* presupposes an underlying structure of point-events which can be resolved with infinite precision. This model was associated operationally with idealized measurement procedures based on the assumption that it is the local Euclidean geometry that decides which objects do and which do not qualify as *infinitesimal measuring rods* at each point of a 4-d smooth spacetime manifold. Developments in quantum mechanics, quantum field theories, and attempts at quantizing gravity encounter serious difficulties in reconciling with the local Euclidean manifold structure, which seem to be related conceptually to this classically idealized small-scale structure. Thus, it is of great interest to consider theories which represent the physical *event continuum* in an essentially different way without losing at the same time the differential geometric framework of dynamics provided by smooth manifolds.

For instance, the general theory of relativity is formulated in terms of differential equations involving *smooth functions* (local components of the metric tensor participating in Einstein's field equations), interpreted as providing a description of the local *metrical* properties of the spacetime manifold around any specific point, depending on the distribution of the energy-momentum tensor. Notwithstanding this fact, all of the observable cosmological predictions of the theory derive not from local solutions of the field equations, per se, but from the inferred global metric spacetime structure generated by the method of *analytic continuation* of some local solution over an extended region. It is precisely from these global solutions that we become able to make predictions. Now, the technique of analytic continuation is mathematically of a sheaf-theoretic nature and *not* dependent at all on both the hypothesis of the local Euclidean structure of spacetime, and the use of smooth functions. This fact points to the conjecture that other non-smooth or distribution-like sheaves of coefficients should be relevant for the continuation of some local solution over extended regions in case that the smooth one is obstructed or becomes singular, *preventing* at the same time the field equations from breaking down at *singularities*, given that these equations can be formulated appropriately in terms of these generalized sheaves of coefficients.

Regarding the above demand, it is instructive to stress again that the development of ADG, viz. the geometric theory of vector sheaves equipped with a connective structure (covariant differentiation), in the form of a functorial generalization of the differential geometric framework of smooth manifolds, has led to the realization that the mechanism underlying an intrinsic approach to modeling and addressing dynamical problems *independently* of the local Euclidean structure is essentially de Rham cohomology, and in particular *sheaf cohomology.* More precisely, the validity of the de Rham complex in its sheaf-theoretic guise is *not* restricted to the consideration of smooth coefficients (at it is the case when the spectrum of the coefficients is a smooth manifold). Thus, we may consider more general distribution-like sheaves of coefficients fitting in the de Rham complex and formulate the field equations in terms of these sheaves instead of the smooth ones. In this way, there exists the possibility of *re-assessing* the global or large-scale problems of general relativity related with the intrinsic smooth spacetime manifold singularities and anomalies from the perspective of appropriate generalized sheaves of coefficients. From a physical viewpoint, this approach would allow to consider distribution-like sheaves corresponding to *non-punctual* localization properties, which would nevertheless still satisfy the field equations (in a sheaf-theoretic form). Beside the technical flexibility provided by the geometry of differential vector sheaves in tackling efficiently these physical problems, an important reason to consider as a clear indication that such an approach is worth pursuing further in relation to physical geometry, stems from its harmonic resonance with the already mentioned critical remark of the father of the *"gauge principle"* H. Weyl, which has actually played a major role in motivating this work: *"While topology has succeeded fairly well in mastering continuity, we do not yet understand the inner meaning of the restriction to differential manifolds. Perhaps one day physics will be able to discard it"*.

0.3 Observables and States

0.3.1 *Sheaf-Theoretic Observable Localization*

Procedures of *physical measurement* presuppose the existence of *localization processes* for extracting information related with the local behavior of physical systems. A localization process is implemented by the preparation of *local reference frames* for the measurement of *observables*. It is important to realize that the effectuation of a localization process is not always equivalent to conferring a numerical identity to observables, expressed in terms of some value corresponding to a physical attribute. On the contrary, the latter is only a limited case under the assumption that all local reference frames can be contracted to points. In this limited case, points constitute unique measures of localization in the *physical continuum*. This is the crucial assumption underlying the employment of the set-theoretic structure of the real line as a model of the physical event continuum, understood as a set of independent points possessing the property of infinite distinguishability with absolute precision. The success of this localizing philosophy in classical physics is due to the association of the notion of physical continuum with the attribute of position and the fact that all classical observables can be potentially determined precisely and simultaneously at the unique measure of localization of this attribute, namely at a spatial point parameterized by the field of real numbers. Nevertheless, the major foundational difference of quantum systems from classical ones is a consequence of the incompatibility of precise determination of all quantum observables within a single local measurement frame, meaning that the simultaneous sharp specification of all observables is impossible within a single local measurement frame.

Thus, a foundational strategy implied by quantum theory should fulfill the following objectives: First, it should disassociate the physical meaning of localization from its *restricted spatial* connotation context. Secondly, it should allow the functional dependence of observables on generalized localization measures, not necessarily based on the existence of an underlying structure of points on the real line. For implementing this strategy, it should be essential to interpret any local observable as a relational algebraic *partial information carrier* with respect to a corresponding local frame of measurement. Next, it should be necessary to establish appropriate compatibility conditions for *gluing* the information content of local observables globally. Mathematically, the implementation of this localization strategy is precisely captured by the concept of a *sheaf-theoretic localization* structure referring to an *algebra of observables*. The nature of sheaf-theoretic localization induces i) a shift from point-set to topological localization models of quantum algebraic observable structures, and ii) a replacement of the classical metrical gauge of point localization on a background manifold by a sheaf gauge of algebraic topological localization on a categorical site capturing the relational information of observables.

Initially, it is important to indicate that both of our fundamental physical theories, namely general relativity and quantum mechanics, can be characterized in general terms as special instances in the process of replacement of the *constant* by the *variable*. More concretely, in the case of general relativity this process takes place through the negation of the fixed kinematical structure of spacetime, by making the *metric* into a *dynamical* object determined solely by the solution of *Einstein's field equations*. This essentially means that the geometrical relations defined on a four dimensional manifold, making it into a spacetime, become variable. The dynamically induced infinitesimal variability of the metric constitutes the means of making geometrical relations variable in the descriptive terms of general relativity. The intelligibility of the framework is enriched by the imposition of the principle of *general covariance* of the field equations under arbitrary smooth coordinate transformations of the points of the underlying spacetime manifold. As an immediate consequence, the points of the manifold are *not dynamically localizable* entities in the theory. Thus, it is *illegitimate* to use the set-theoretic point structure of the manifold for individuating directly *spacetime events*, independently of the localization properties of the metric determined dynamically. In this way, it can be asserted that manifold points bear only an indirect reference as indicators of spacetime events, and most importantly, only after the dynamical specification of variable geometrical relations among them, obtained as particular *solutions* of the generally covariant field equations in terms of the metric. We conclude that the conceptualization of the physical event continuum in general relativity requires explicit reference to the *relational localization* properties of the metric, which is determined dynamically in the context of this theory.

In the case of quantum theory, the process of replacement of the constant by the variable is signified by the imposition of *Heisenberg's uncertainty relations*, which determine the limits for *simultaneous* measurements of certain pairs of *conjugate* physical observables, like position and momentum. In a well-defined sense, the uncertainty relations may be interpreted as measures of the *valuation vagueness* associated with the simultaneous determination of conjugate pairs of observables within the same experimental context. This is in *contradistinction* to all classical theories, where the valuation algebra is fixed once and for all to be the field of real numbers, reflecting the fact that values admissible as measured results must be real numbers, *irrespectively* of the measurement context and simultaneously for all physical observables. The above problem of observable valuation vagueness in quantum theory can be resolved *topos-theoretically* via a relational process of topological localization of the *global non-commutative* algebra of quantum observables with respect to a *categorical diagram* of interlocking families of *local commutative* subalgebras of observables. Each maximal local commutative subalgebra contains all observables that can be simultaneously measured within the same experimental context.

From a mathematical point of view, the general process of semantic transition from the constant to the variable along the previous lines takes place by passing from rigid set-theoretic algebraic structures to *continuous sheaf-theoretic* algebraic structures. In general, a *sheaf* may be thought of as a *continuously variable set*, whose continuous variation is considered over specified local frames interlocking together non-trivially. The local frames are required to obey certain topological closure conditions, which generalize the definition of a topology formulated in terms of open sets in classical topological spaces. In this way, we obtain a *local relativization of physical representability* by shifting the semantics of observables from the topos of sets to a topos of sheaves for an appropriate topology. Intuitively speaking, this amounts to a relativization with respect to the local behavior of physical observables as opposed to their point behavior. We remind that in the guiding instance of a topological space, a *topology* is defined by a collection of *open loci* (to be thought of as *probes* or *covers* of a space) which is closed by the formation of arbitrary unions and finite intersections of open loci. The definition of a topology on a space provides the means to talk about the notion of a *continuous function* between topological spaces. A function is said to be continuous if and only if the inverse image of every open locus of the range is an open locus of the domain topological space. This is an attempt to capture the intuition that there are no breaks or separations in a continuous function. It is important to realize the following: i) The definition of a topology on a space is *solely* used for the formalization of what a continuous function is on that space, ii) the continuity of a function is a property which is determined *locally*, that is only by reference to open loci. This means that due to the variability of open loci in the topology the property of continuity of a function should respect the inverse algebraic operations of *restriction* and *extension* (uniquely) with respect to open loci. Thus, a continuous function can be restricted consistently to open subloci of any open locus in the topology and inversely extended by gluing uniquely together all its local restrictions. This is the *crucial conceptual insight* which is incorporated in the technical definition of a sheaf.

This insight referring to the precise formulation of the property of continuity may be generalized in two directions: Firstly, instead of open loci or open regions covering a topological space we may consider generalized local covering frames under the constraint that they collectively obey topological *closure* conditions *analogous* to the ones used for open loci. Secondly, instead of functions varying continuously over local frames we may consider *generalized functional relations*, called technically *sections* of a sheaf. Conceptually speaking, local sections of a sheaf describe relations relatively to a local frame. A sheaf is essentially the totality of its sections (both local and global if the latter exist) and their consistent interrelations with respect to the local-global distinctions subsumed by the *interlocking properties* of the underlying frames. In this sense, the operation of restriction of sections acquires a meaning with respect to *nesting* local frames, whereas the operation of extension or gluing of sections with respect to compatibly pairwise *overlapping* local frames. We note that if only the operation of restriction for nested local frames is satisfied, then we obtain a structure called a *presheaf*. In case that the operation of extension is also satisfied, then we obtain a *separated presheaf*. Only in case that the operation of extension can be *uniquely* carried out, that is local sections can be uniquely glued together, a separated presheaf becomes a sheaf. We note that, in general, there will be more locally defined or partial sections than globally defined ones, since not all partial sections need be extendible to global ones, but a compatible family of partial sections uniquely extends to a global one, or in other words, any presheaf uniquely defines a sheaf. More precisely, for the leading example of localization with respect to the open covers of a topological space we have the following basic notions:

A presheaf $\mathbb{F}$ of sets on a topological space X, consists of the following data:

I. For every open set U of X, *a set* denoted by $\mathbb{F}(U)$, and

II. For every inclusion $V \hookrightarrow U$ of open sets of X, a *restriction morphism* of sets in the opposite direction:

$$r(U|V) : \mathbb{F}(U) \to \mathbb{F}(V) \tag{0.3.1}$$

such that:

a. $r(U|U) =$ identity at $\mathbb{F}(U)$ for all open sets U of X.

b. $r(V|W) \circ r(U|V) = r(U|W)$ for all open sets $W \hookrightarrow V \hookrightarrow U$.

Usually, the following simplifying notation is used: $r(U|V)(s) := s|_V$.

A presheaf $\mathbb{F}$ of sets on a topological space X, is defined to be a *sheaf* if it satisfies the following two conditions, for every family V_a, $a \in I$, of local open covers of V, where V open set in X, such that $V = \cup_a V_a$:

I. *Local identity* axiom of sheaf:

Given $s, t \in \mathbb{F}(V)$ with $s|_{V_a} = t|_{V_a}$ for all $a \in I$, then $s = t$.

II. *Gluing* axiom of sheaf:

Given $s_a \in \mathbb{F}(V_a)$, $s_b \in \mathbb{F}(V_b)$, $a, b \in I$, such that:

$$s_a|_{V_a \cap V_b} = s_b|_{V_a \cap V_b}, \tag{0.3.2}$$

for all $a, b \in I$, then there exists a *unique* $s \in \mathbb{F}(V)$, such that: $s|_{V_a} = s_a \in F(V_a)$ and $s|_{V_b} = s_b \in F(V_b)$.

If we consider the *partially ordered set* of open subsets of a topological space X as a category denoted by $\mathcal{O}(X)$, then $\mathbb{F}$ denotes the *contravariant* presheaf/sheaf functor that assigns to each open set $U \subset X$ a set in the category **Sets**. We note that the above definitions hold if instead of presheaves/sheaves of sets we consider presheaves/sheaves of algebraic structures, for example groups, algebras over a field, vector spaces over a field or modules over an algebra. For detailed pure mathematical expositions of sheaf theory the reader may consult Mallios [6], Mac Lane - Moerdijk [1], Tennison [1] and Vassiliou [1].

As a basic example, if $\mathbb{F}$ denotes the contravariant presheaf functor that assigns to each open set $U \subset X$, the *commutative unital* algebra of all real-valued continuous functions on U, then $\mathbb{F}$ is actually a sheaf. This is intuitively clear since the specification of a topology on X (and hence, of a topological localization system on $\mathcal{O}(X)$) is solely used for the definition of the continuous functions on X, thought of as observables on X. Thus, the continuity of each function can be determined *locally*. This means that continuity respects the operation of restriction to open sets, and moreover that continuous functions can be collated in a unique manner, as it is required for the satisfaction of the sheaf condition. More precisely, the sheaf condition in this case means that the following *sequence* of commutative algebras is *left exact*;

$$0 \to \mathbb{F}(U) \to \prod\nolimits_a \mathbb{F}(U_a) \to \prod\nolimits_{a,b} \mathbb{F}(U_a \cap U_b). \tag{0.3.3}$$

Let us further consider that x is a point of a topological space of control variables X. Moreover, let T be a set consisting of open subsets of X, containing x, such that the following condition holds: For any two open sets U, V, containing x, there exists an open set $W \in T$, contained in the intersection $U \cap V$. We may say that T constitutes *a basis* for the system of open sets around x. We form the *disjoint union* of all $\mathbb{F}(U)$, denoted by;

$$\mathbb{D}(x) := \bigcup_{U \in T} \mathbf{F}(U). \tag{0.3.4}$$

Then we can define an *equivalence relation* in $\mathbb{D}(x)$, by requiring that $f \sim g$ for $f \in \mathbb{F}(U)$, $g \in \mathbb{F}(V)$, provided that they have the *same restriction* to a smaller open set contained in T. Then we define;

$$colim_T[\mathbb{F}(U)] := \mathbb{D}(x)/\sim_T. \tag{0.3.5}$$

It is clear that the above definition is independent of the chosen basis of open covers T, and thus corresponds to an *inductive limit* construction. The inductive limit obtained is denoted by $\mathbb{F}_x$, and referred to as the *stalk* of $\mathbb{F}$ at the point $x \in X$. Physically, we identify an element of $\mathbb{F}$ of sort U with a *local section* of $\mathbb{F}$ over the open set U. Then the equivalence relation, used in the definition of the stalk $\mathbb{F}_x$ at the point $x \in X$ is interpreted as follows: Two local sections $f \in \mathbb{F}(U)$, $g \in \mathbb{F}(V)$, induce the *same contextual information* at x in X, provided that they have the same restriction to a smaller open set contained in the basis T. Then, the stalk $\mathbb{F}_x$ is the set containing all contextual information at x, that is the set of all equivalence classes. Moreover, the image of a local section $f \in \mathbf{F}(U)$ at the stalk $\mathbb{F}_x$, denoted by f_x, that is the equivalence class of this local section f, is precisely the *germ* of f at the point x. We deduce that the *fibration* corresponding to a sheaf of sets $\mathbb{F}$ is a *topological bundle*, called an *ètale bundle*, defined by the continuous mapping $\varphi : F \to X$, where;

$$F = \bigcup_{x \in X} \mathbb{F}_x, \tag{0.3.6}$$

$$\varphi^{-1}(x) = \mathbb{F}_x. \tag{0.3.7}$$

The mapping φ is locally a *homeomorphism* of topological spaces. The topology in F is defined as follows: for each local section $f \in \mathbb{F}(U)$, the set $\{f_x, x \in U\}$ is open, and moreover, an arbitrary open set is a union of sets of this form. Obviously, the same arguments hold in the case of a sheaf of sets $\mathbb{F}$ endowed with some algebraic structure, for example $\mathcal{F}$-algebras (where $\mathcal{F}$ is a field). Finally, the sheaf $\mathbb{F}$ can be canonically identified as the *sheaf of cross-sections* of the corresponding ètale bundle F.

We stress that the notion of a sheaf depends on the fact that we require the gluing condition with respect to all local covering frames, that is to *all covers* of any local frame. In principle, one could select some covers of a local frame and require the gluing condition only with respect to the selected covers. In this way, the notion of sheaf would be meant with respect to the selected family of covers. On the other hand, there is no restriction in considering only *hereditary covers*, that is covers containing all their subcovers. More precisely, any cover can be made hereditary (by adding to each cover all its considered subcovers) and compatible (uniquely glued) families of sections on the original cover are in *bijective* correspondence with compatible families of sections on the new one. This fact provides an important insight on the nature of the *topological localization process* of observables implicated by the concept of sheaf. First, let us think of local frames as partial information carriers related to measurement of observables. Now, the idea of considering hereditary covers can be implemented physically via a procedure of localization by *sieving*. More concretely, a *sieve* on a local frame can be thought of as consisting of *spectrum holes* distributed in different nested layers, such that every partial information carrier which is in a sense "smaller" than one of its holes *passes through* it but not otherwise. In this way, the notion of a sieve incorporates an *asymmetric logical order*. Moreover, since the partial information carriers should bear the semantics of local covers, certain conditions should hold distinguishing some sieves as *covering* ones within the class of all possible sieves. These conditions are the following: α) The covering sieves should be *stable under intersection*, meaning that the intersection of covering sieves should also be a covering sieve, for each local frame; β) the covering sieves should be *transitive*, such that covering sieves of subframes of a frame in covering sieves of this frame, should also be covering sieves of this frame themselves and, γ) the covering sieves should be *stable under change* of a local frame.

From a physical perspective, the consideration of covering sieves as hereditary covers used for localizing observable information with respect to local frames, gives rise to *localization systems* of global observable structures which induce a semantic transition of events from a set-theoretic to a sheaf-theoretic level. The operation which assigns to each local frame a collection of covering sieves satisfying the stability and transitivity conditions described qualitatively above, defines a topology, which is technically called a *Grothendieck topology*. The notion of a Grothendieck topology formulated in terms of covering sieves is significant for the following reasons: First, it elucidates precisely the topological concept of *locality* in relational *logical order*-theoretic terms, such that this concept becomes distinguished from its usual metrical spatiotemporal connotation. Second, it permits the *collation* of local observable information into global ones by utilization of the notion of sheaf with respect to a defined topology. In more detail, the extension of observable information from the local to the global takes place via a compatibly glued family of sections over a covering sieve (localization system) constituted of covering local frames (local covers). A sheaf assigns a set of sections to each local frame of a localization system, representing a set of observables which can be measured with respect to this local frame. A selection of sections from these sets, one for each local frame, forms a *compatibly glued family* with respect to a localization system, if the selection respects the operation of restriction, and additionally, if the sections selected agree whenever two local frames of the localization system overlap. If such a locally compatible selection of sections *extends uniquely* to a global one, then the sheaf conditions are satisfied. We note that in general, there will be more local sections than global ones (if they exist), since not all local observables need be extendible to global ones, but a compatible family of local observables uniquely extends to a global one with respect to a localization system. We conclude from the previous analysis that sheaf-theoretic gluing provides the means of *extensive connection*, that is extending a local section over a local frame to a higher level local frame and so on towards approximating the global with respect to some covering sieve.

0.3.2 *Vector Sheaves of States and Local Gauge Invariance*

From a physical viewpoint, the construction of a sheaf constitutes the natural outcome of a complete localization process. For example, in quantum mechanics we may consider the localization process of the set of the *Hamiltonian eigenstates* of an electron (energy eigenfunctions) constituting a *Hilbert space basis of vectors* with respect to some base space of control parameters.

We consider a pair $(X, \mathcal{A})$ consisting of a *paracompact* (*Hausdorff*) topological space X and a *soft sheaf* of commutative rings $\mathcal{A}$ localized over X. According to our general philosophy, we consider the above pair as the *Gelfand spectrum* of an appropriate algebra of observables A. We remind that if $\mathbb{C}$ is the field of complex numbers, then an $\mathbb{C}$-algebra A is a ring A together with a morphism of rings $\mathbb{C} \to A$ (making A into a vector space over $\mathbb{C}$) such that, the morphism $A \to \mathbb{C}$ is a linear morphism of vector spaces. Notice that the same holds if we substitute the field $\mathbb{C}$ with any other field, for instance, the field of real numbers $\mathbb{R}$. We also assume that the stalk $\mathcal{A}_x$ of germs is a *local commutative* $\mathbb{C}$-algebra for any point $x \in X$. A typical example is the case, where X is a smooth manifold of and $\mathcal{A}$ is the $\mathbb{C}$-algebra sheaf of germs of smooth functions localized over X. Together with a $\mathbb{C}$-algebra sheaf $\mathcal{A}$ we also consider the *abelian group* sheaf of invertible elements of $\mathcal{A}$, denoted by $\mathcal{A}^{\bullet} := \tilde{\mathcal{A}}$.

An $\mathcal{A}$-module $\mathcal{E}$ is called a *locally free $\mathcal{A}$-module of states of finite rank m*, or simply a *vector sheaf of states*, if for any point $x \in X$ there exists an open set U of X such that:

$$\mathcal{E} \mid_U \cong \bigoplus^m (\mathcal{A} \mid_U) := (\mathcal{A} \mid_U)^m, \tag{0.3.8}$$

where $(\mathcal{A} \mid_U)^m$ denotes the m-terms *direct sum* of the sheaf of $\mathbb{C}$-observable algebras $\mathcal{A}$ restricted to U, for some $m \in \mathbb{N}$. We call $(\mathcal{A} \mid_U)^m$ the *local sectional frame* or equivalently *local gauge* of states of $\mathcal{E}$ associated via the open covering $\mathcal{U} = \{U\}$ of X.

In case that the rank is 1, the corresponding vector sheaf is called a *line sheaf of states*, that is locally for any point $x \in X$ there exists an open set U of X such that:

$$\mathcal{E} \mid_U \cong \mathcal{A} \mid_U . \tag{0.3.9}$$

Furthermore, if for any point $x \in X$ there exists an open set U of X such that:

$$\mathbb{S} \mid_U \cong \bigoplus^m (\mathbb{C} \mid_U) := (\mathbb{C} \mid_U)^m, \tag{0.3.10}$$

then we call any locally free $\mathbb{C}$-module $\mathbb{S}$ of finite rank m, for some $m \in \mathbb{N}$, a *complex linear local system* of rank m.

The notion of a vector sheaf of states generalizes the notion of a vector space of states when there exists for instance a *parametric dependence* of a system from a topological space of control variables. The crucial point is that locally every section of a finite rank vector sheaf can be written as a *finite linear combination of a basis of sections with coefficients from the local observable algebra*. For example, if X is a smooth manifold and $\mathcal{A}$ is the $\mathbb{C}$-algebra or $\mathbb{R}$-algebra sheaf of germs of smooth functions on X, then every section can be locally written as a *finite* linear combination or *superposition* of a basis of sections with coefficient being real-valued smooth functions. We note that the physical notion of *time dependence* may be implicitly introduced via *time-parameterized paths* on X. It is also the case that the set of sections of every *vector bundle* on a topological space (not necessarily a smooth manifold) forms a vector sheaf of sections localized over this space. Thus, the notion of a vector sheaf of states provides the most *general* conceptual and technical framework to consider in a unifying way all cases, where there exists a parametric dependence of some observable by a topological space of control variables X.

Given a vector sheaf of states $\mathcal{E}$, there is specified a *Čech* 1-*cocycle with respect to a covering* $\mathcal{U} = \{U\}$ of X, called a *coordinate* 1-*cocycle* in $Z^1(\mathcal{U}, GL(m, \mathcal{A})$ (with values in the sheaf of germs of sections into the group $GL(m, \mathbb{C})$), as follows:

$$\eta_\alpha : \mathcal{E} \mid_{U_\alpha} \cong (\mathcal{A} \mid_{U_\alpha})^m, \tag{0.3.11}$$

$$\eta_\beta : \mathcal{E} \mid_{U_\beta} \cong (\mathcal{A} \mid_{U_\beta})^m. \tag{0.3.12}$$

Thus, for every $x \in U_\alpha$ we have a *stalk isomorphism*:

(0.3.13) $$\eta_\alpha(x) : \mathcal{E}_x \cong \mathcal{A}_x{}^m,$$

and similarly for every $x \in U_\beta$. If we consider that $x \in U_\alpha \cap U_\beta$, then we obtain the isomorphism:

(0.3.14) $$g_{\alpha\beta}(x) = \eta_\alpha(x)^{-1} \circ \eta_\beta(x) : \mathcal{A}_x{}^m \cong \mathcal{A}_x{}^m.$$

The $g_{\alpha\beta}(x)$ is thought of as an *invertible matrix of germs* at x. Consequently, $g_{\alpha\beta}$ is an invertible matrix section in the sheaf of germs $GL(m, \mathcal{A})$ (taking values in the *general linear group* $GL(m, \mathbb{C})$). Moreover, $g_{\alpha\beta}$ satisfy the *cocycle conditions* $g_{\alpha\beta} \circ g_{\beta\gamma} = g_{\alpha\gamma}$ on triple intersections whenever they are defined.

It is clear in this way that we obtain a vector bundle with typical fiber $\mathbb{C}^m$, structure group $GL(m, \mathbb{C})$, whose sections form the vector sheaf of states we started with. In particular, for $m = 1$, we obtain a *line bundle* L with fiber $\mathbb{C}$, structure group $GL(1, \mathbb{C}) \cong \tilde{\mathbb{C}}$ (the non-zero complex numbers), whose sections form a *line sheaf* of states $\mathcal{L}$. Clearly, by imposing a *unitarity* condition the structure group is reduced to $U(1)$. Thus, particularly in the case of a line sheaf of states we have a bijective correspondence:

(0.3.15) $$\mathcal{L} \leftrightarrow (g_{\alpha\beta}) \in Z^1(\mathcal{U}, \tilde{\mathcal{A}})$$

where $GL(1, \mathcal{A}) \cong \mathcal{A}^\bullet := \tilde{\mathcal{A}}$ is the group sheaf of invertible elements of $\mathcal{A}$ (taking values in $\tilde{\mathbb{C}}$), and $Z^1(\mathcal{U}, \tilde{\mathcal{A}})$ is the set of coordinate 1-cocycles. In physical terminology, a coordinate 1-cocycle effects a *local frame transformation*, or equivalently a *local gauge transformation* of a vector sheaf of states $\mathcal{E}$.

Every 1-cocycle can be conjugated with a 0-*cochain* (t_α) in the set $C^0(\mathcal{U}, \tilde{\mathcal{A}})$ to obtain another equivalent 1-cocycle:

(0.3.16) $$g'_{\alpha\beta} = t_\alpha \cdot g_{\alpha\beta} \cdot t_\beta^{-1}.$$

If we consider the *coboundary operator* :

(0.3.17) $$\Delta^0 : C^0(\mathcal{U}, \tilde{\mathcal{A}}) \to C^1(\mathcal{U}, \tilde{\mathcal{A}}),$$

then the image of Δ^0 gives the set of *1-coboundaries* of the form $\Delta^0(t_\alpha^{-1})$ in $B^1(\mathcal{U}, \tilde{\mathcal{A}})$:

(0.3.18) $$\Delta^0(t_\alpha^{-1}) := t_\alpha \cdot t_\beta^{-1}.$$

Thus, we obtain that the 1-cocycle $g'_{\alpha\beta}$ is equivalent to the 1-cocycle $g_{\alpha\beta}$ in $Z^1(\mathcal{U}, \tilde{\mathcal{A}})$ if and only if there exists a 0-cochain (t_α) in the set $C^0(\mathcal{U}, \tilde{\mathcal{A}})$, such that:

(0.3.19) $$g'_{\alpha\beta} \cdot g_{\alpha\beta}^{-1} = \Delta^0(t_\alpha^{-1})$$

where, the multiplication above is meaningful in the *abelian group* of 1-cocycles $Z^1(\mathcal{U}, \tilde{\mathcal{A}})$.

Due to the bijective correspondence of line sheaves with coordinate 1-cocycles with respect to an open covering $\mathcal{U}$, we deduce immediately the following:

Theorem 0.1. *The set of isomorphism classes of line sheaves of states over the same topological space X, denoted by $Iso(\mathcal{L})(X)$, is in bijective correspondence with the set of cohomology classes $H^1(X, \tilde{\mathcal{A}})$:*

(0.3.20) $$Iso(\mathcal{L})(X) \cong H^1(X, \tilde{\mathcal{A}}).$$

The above proposition interpreted in physical terms constitutes the *cohomological formulation of the principle of local gauge (phase) invariance* in the description of the state space of a system described by a line sheaf. Furthermore, each equivalence class $[\mathcal{L}] \equiv \mathcal{L}$ in $Iso(\mathcal{L})(X)$ has an inverse, defined by:

$$\mathcal{L}^{-1} := Hom_{\mathcal{A}}(\mathcal{L}, \mathcal{A}) \tag{0.3.21}$$

where, $Hom_{\mathcal{A}}(\mathcal{L}, \mathcal{A}) := \mathcal{L}^*$ denotes the *dual line sheaf* of $\mathcal{L}$. This is actually deduced from the fact that we can define the *tensor product* of two equivalence classes of line sheaves over $\mathcal{A}$ so that:

$$\mathcal{L} \otimes_{\mathcal{A}} \mathcal{L}^* \cong Hom_{\mathcal{A}}(\mathcal{L}, \mathcal{L}) \equiv \mathcal{E}nd_{\mathcal{A}}\mathcal{L} \cong \mathcal{A}. \tag{0.3.22}$$

Hence, we obtain the following refinement in the cohomological formulation of local gauge invariance:

Theorem 0.2. *The set of isomorphism classes of line sheaves of states over the same topological space X, $Iso(\mathcal{L})(X)$, has an abelian group structure with respect to the tensor product over the observable algebra sheaf $\mathcal{A}$, and analogously the set of cohomology classes $H^1(X, \tilde{\mathcal{A}})$ is also an abelian group, where the tensor product of two line sheaves of states corresponds to the product of their respective coordinate 1-cocycles.*

0.3.3 *Exponential Short Exact Sequence*

In all cases of physical interest, the topological space X of control variables is considered to be paracompact, while the observable algebra sheaf $\mathcal{A}$ is a soft sheaf meaning that every section over some closed subset in X can be *extended* to a section over X. More importantly, the following *short sequence of abelian group sheaves is exact*, which models sheaf-theoretically the process of *exponentiation*, for details see Mallios [7, p.330]:

$$0 \to \mathbb{Z} \xrightarrow{\iota} \mathcal{A} \xrightarrow{e} \tilde{\mathcal{A}} \to 1, \tag{0.3.23}$$

where $\mathbb{Z}$ is the constant abelian group sheaf of integers (sheaf of locally constant sections valued in the group of integers), such that:

$$Ker(e) = Im(\iota) \cong \mathbb{Z}. \tag{0.3.24}$$

Clearly, due to the *canonical imbedding* of the constant abelian group sheaf $\mathbb{C}$ into $\mathcal{A}$ as well as of $\tilde{\mathbb{C}}$ into $\tilde{\mathcal{A}}$, we have the validity of the *short exact sequence of constant abelian group sheaves*:

$$0 \to \mathbb{Z} \xrightarrow{\iota} \mathbb{C} \xrightarrow{exp} \tilde{\mathbb{C}} \to 1, \tag{0.3.25}$$

$$Ker(exp) = Im(\iota) \cong \mathbb{Z}. \tag{0.3.26}$$

The above exponential short exact sequence can be specialized further to the following short exact sequence of constant abelian group sheaves:

$$0 \to \mathbb{Z} \xrightarrow{\iota} \mathbb{R} \xrightarrow{exp(2\pi i)} \mathbb{U}(1) \to 1 \tag{0.3.27}$$

where $\mathbb{R}$ is the constant abelian group sheaf of reals and $\mathbb{U}(1)$ is the abelian group sheaf of unit modulus complexes (phases).

The fundamental significance of these three exponential short exact sequences of abelian group sheaves cannot be overestimated. In particular, we obtain a further *refinement* of the cohomological formulation of local phase (gauge) invariance pertaining to the state space, according to the following:

Theorem 0.3. *Each equivalence classes of line sheaves of states in $Iso(\mathcal{L})(X)$ is in bijective correspondence with a cohomology class in the integral 2-dimensional cohomology group of X.*

If we consider the exponential short exact sequence of abelian group sheaves, we immediately obtain a *long exact sequence in sheaf cohomology*, which due to paracompactness of X is reduced to *Čech cohomology.* Thus, we have:

(0.3.28)

$$\ldots \to H^1(X,\mathbb{Z}) \to H^1(X,\mathcal{A}) \to H^1(X,\tilde{\mathcal{A}}) \to H^2(X,\mathbb{Z}) \to H^2(X,\mathcal{A}) \to H^2(X,\tilde{\mathcal{A}}) \to \ldots$$

Because of the fact that $\mathcal{A}$ is a soft sheaf, we have:

$$H^1(X,\mathcal{A}) = H^2(X,\mathcal{A}) = 0 \tag{0.3.29}$$

and consequently we obtain:

$$0 \to H^1(X,\tilde{\mathcal{A}}) \xrightarrow{\delta_c} H^2(X,\mathbb{Z}) \to 0. \tag{0.3.30}$$

Thus, equivalently we obtain the following isomorphism of abelian groups (*Chern isomorphism*) :

$$\delta_c : H^1(X,\tilde{\mathcal{A}}) \xrightarrow{\cong} H^2(X,\mathbb{Z}). \tag{0.3.31}$$

Since we have shown previously that the set of isomorphism classes of line sheaves of states over X, viz. $Iso(\mathcal{L})(X)$, is in bijective correspondence with the abelian group of cohomology classes $H^1(X,\tilde{\mathcal{A}})$, we deduce that:

$$Iso(\mathcal{L})(X) \cong H^2(X,\mathbb{Z}). \tag{0.3.32}$$

Thus, each equivalence classes of line sheaves of states is in bijective correspondence with a cohomology class in the integral 2-dimensional cohomology group of X.

0.4 Connections and Differential Analysis

0.4.1 *Kähler-de Rham Paradigm*

Differential geometry is traditionally associated with the notion of *smooth manifolds.* Smooth manifolds are geometric spaces which are *locally homeomorphic to*

Euclidean spaces of an appropriate dimension. The topological (actually metrical) linear vector space structure of a Euclidean space in the neighborhood of a manifold point is used for transferring locally the methods of analytic calculus to smooth manifolds, and thus generate locally the whole differential geometric framework. Conclusively, this *local linear structure* provides the seed of generation of *spatially determined differential analysis* on a smooth manifold, which is interpreted physically as a process of dynamical extensive connection caused by an interaction field. The localization scheme provided by the notion of a smooth manifold is very *restrictive* not only for the incorporation of quantum phenomena (where the local Euclidean linear space structure is not tenable) but even for an efficient treatment of spacetime *singularities* obstructing the validity of the field equations formulated in terms of the corresponding algebra of smooth functions.

In this manner, the pertinent question is the following: Is the generation of local differential analysis, and thus differential geometry, *solely dependent*, and thus spatially determined, by the local linear space structure of Euclidean spaces or there exists a *universal algebraic mechanism* of generation of differentiability not dependent on this particular metrical Euclidean localization scheme? If such an algebraic mechanism of differential analysis exists, then the whole differential geometric framework of smooth manifolds would constitute a particular manifestation shaped by the modeling Euclidean linear space localization scheme.

A crucial observation comes from the fact that Kähler's *purely algebraic* theory of generation of differential forms does not involve *any* assumptions involving Euclidean spaces and is solely based on the properties of the *inverse processes* of extension and restriction of commutative algebras. Concomitantly, De Rham's *algebraic-topological* theory of cohomology is based on the existence of modules of *exterior differential n-forms* (admitting an appropriate characterization as closed or exact by intrinsic means), and the formation of *complexes* by means of differential operators acting on these modules. The *Kähler-de Rham homological algebraic mechanism of differential analysis* is *not dependent* on any local Euclidean vector space structure for its function, and thus provides an excellent candidate for the generalization of the differential geometric framework of smooth manifolds to more general localization schemes specified by *dynamical* means (in analogy to the case of General Relativity). In order to be applicable for modeling the process of extensive connection dynamically (that is via the action of a physical field) in the observability scale of a physical theory (defined by the corresponding reference frames) it should be susceptible to *sheaf-theoretic localization.*

The fact that the homological Kähler-de Rham differential mechanism actually admits a sheaf-theoretic localization with respect to appropriate localization schemes is not a trivial issue, for details see Mallios[7, p.326], Vassiliou [1, p.57], Abel-Ntumba[1], Fragoulopoulou-Papatriantafillou [1], and for a physical interpretation and applications Mallios-Zafiris [1, 2]. What it shows is that the concepts of differential geometry which remain logically invariant under abstraction from the smooth manifold scaffolding are the ones which can be *localized by sheaf-theoretic means*. Accordingly, the sheaf gauge of localization of observables shows that the intrinsic mechanism of differential observability induced by the action of a physical field is of a *sheaf cohomological* nature. As we have already mentioned, cohomology can be conceived as an effective method of assigning *global invariants* to a localization scheme (for example to a topological space, or more generally to a categorical site) in a *homotopy-invariant* way. The cohomology groups measure the *global obstructions* for extending sections from the local to the global level. From a physical perspective, these obstructions have global observable effects and may be associated with *field sources* preventing the uniform and maximally undisturbed dynamical process of extensive connection. It is precisely these obstructions detected by cohomology groups, which make sheaf cohomology the proper *measurement tool* of global observability.

0.4.2 *Kähler's Algebraic Extension Method*

It is instructive to describe briefly the purely algebraic form of Kähler's construction from a physical viewpoint. In general algebraic terms, the process of extending the form of observation with respect to an algebra of scalars, for instance an $\mathbb{R}$-algebra $\mathcal{A}$, due to field interactions, is described by means of a *fibering*, defined as an injective morphism of $\mathbb{R}$-algebras $\iota : A \hookrightarrow B$. Thus, the $\mathbb{R}$-algebra B is considered as a module over the algebra A. A section of the fibering $\iota : A \hookrightarrow B$, is represented by a morphism of $\mathbb{R}$-algebras $s : B \to A$ such that $\iota \circ s = id_B$. The fundamental extension of scalars of the $\mathbb{R}$-algebra A is obtained by *tensoring* A with itself over the subalgebra of the base field (for instance the real numbers), that is $\iota : A \hookrightarrow A \otimes_{\mathbb{R}} A$. Trivial cases of scalars extensions, in fact isomorphic to A, induced by the fundamental one, are obtained by tensoring A with $\mathbb{R}$ from both sides, that is $\iota_1 : A \hookrightarrow A \otimes_{\mathbb{R}} \mathbb{R}$, $\iota_2 : A \hookrightarrow \mathbb{R} \otimes_{\mathbb{R}} A$.

The basic idea of Riemann that has been incorporated for instance in the context of Einstein's general theory of relativity is that *geometry* should be built from the *infinitesimal to the global.* Geometry in this context is understood in terms of metric structures that can be defined on a differential spacetime manifold. If we adopt Kähler's algebraic viewpoint, then the effectuation of geometry as a result of interactions, requires first of all the *extension of scalars* of the algebra A by *infinitesimal quantities*, defined as a fibration:

$$d_* : A \hookrightarrow A \oplus M \cdot \epsilon, \tag{0.4.1}$$

$$f \mapsto f + d(f) \cdot \epsilon \tag{0.4.2}$$

where $d(f) =: df$ is considered as the infinitesimal part of the extended scalar, and ϵ denotes the infinitesimal unit obeying $\epsilon^2 = 0$. The algebra of infinitesimally extended scalars $A \oplus M \cdot \epsilon$ is called the algebra of *dual numbers* over A with coefficients in the A-module M. It is immediate to see that the algebra $A \oplus M \cdot \epsilon$, as an abelian group is just the direct sum $A \oplus M$, whereas the multiplication is defined by:

$$(f + df \cdot \epsilon) \bullet (f^{'} + df^{'} \cdot \epsilon) = f \cdot f^{'} + (f \cdot df^{'} + f^{'} \cdot df) \cdot \epsilon. \tag{0.4.3}$$

Note that, we further require that the composition of the augmentation $A \oplus M \cdot \epsilon \to A$, with d_* is the identity. Equivalently, the above fibration, viz., the homomorphism of algebras $d_* : A \hookrightarrow A \oplus M \cdot \epsilon$, can be formulated as a *derivation*, that is in terms of an additive $\mathbb{R}$-linear morphism:

$$d : A \to M, \tag{0.4.4}$$

$$f \mapsto df \tag{0.4.5}$$

that, moreover, satisfies the Leibniz rule :

$$d(f \cdot g) = f \cdot dg + g \cdot df. \tag{0.4.6}$$

Since the formal symbols of differentials $\{df, f \in A\}$, are reserved for the *universal derivation*, the A-module M is identified as the free A-module $\Omega := \Omega^1(A)$ of 1-*forms* generated by these formal symbols, modulo the Leibniz constraint, where the scalars of the distinguished subalgebra $\mathbb{R}$, that is the real numbers, are treated as constants.

Kähler's fundamental insight consists in the realization that the free A-module Ω can be constructed explicitly from the fundamental form of scalars extension of A, that is $\iota : A \hookrightarrow A \otimes_{\mathbb{R}} A$ by considering the morphism:

$$\mu : A \otimes_{\mathbb{R}} A \to A, \tag{0.4.7}$$

$$\sum_i f_i \otimes g_i \mapsto \sum_i f_i \cdot g_i. \tag{0.4.8}$$

Then, by taking the *kernel* of this morphism of algebras, that is, the ideal :

$$I = Ker\mu = \{\theta \in A \otimes_{\mathbb{R}} A : \mu(\theta) = 0\} \subset A \otimes_{\mathbb{R}} A \tag{0.4.9}$$

we obtain the following:

Theorem 0.4. *The morphism of A-modules:*

$$\Sigma : \Omega \to \frac{I}{I^2}, \tag{0.4.10}$$

$$df \mapsto 1 \otimes f - f \otimes 1 \tag{0.4.11}$$

is an isomorphism.

In order to show this, we notice that the fractional object $\frac{I}{I^2}$ has an A-module structure defined by:

$$f \cdot (f_1 \otimes f_2) = (f \cdot f_1) \otimes f_2 = f_1 \otimes (f \cdot f_2) \tag{0.4.12}$$

for $f_1 \otimes f_2 \in I$, $f \in A$. We can check that the second equality is true by proving that the difference of $(f \cdot f_1) \otimes f_2$ and $f_1 \otimes (f \cdot f_2)$ belonging to I, is actually an element of I^2, viz., the equality is true modulo I^2. So we have:

$$(f \cdot f_1) \otimes f_2 - f_1 \otimes (f \cdot f_2) = (f_1 \otimes f_2) \cdot (f \otimes 1 - 1 \otimes f). \tag{0.4.13}$$

The first factor of the above product of elements belongs to I by assumption, whereas the second factor also belongs to I, since we have that:

$$\mu(f \otimes 1 - 1 \otimes f) = 0. \tag{0.4.14}$$

Hence, the product of elements above belongs to $I \cdot I = I^2$. Consequently, we can define a morphism of A-modules:

$$\Sigma : \Omega \to \frac{I}{I^2}, \tag{0.4.15}$$

$$df \mapsto 1 \otimes f - f \otimes 1. \tag{0.4.16}$$

Now, we construct the inverse of that morphism as follows: The A-module Ω can be made an *ideal* in the algebra of dual numbers over A, viz., $A \oplus \Omega \cdot \epsilon$. Moreover, we can define the morphism of algebras:

$$A \times A \to A \oplus \Omega \cdot \epsilon, \tag{0.4.17}$$

$$(f_1, f_2) \mapsto f_1 \cdot f_2 + f_1 \cdot df_2\epsilon. \tag{0.4.18}$$

This is an $\mathbb{R}$-bilinear morphism of algebras, and thus, it gives rise to a morphism of algebras:

$$\Theta : A \otimes_{\mathbb{R}} A \to A \oplus \Omega \cdot \epsilon. \tag{0.4.19}$$

Then, by definition we have that $\Theta(I) \subset \Omega$, and also, $\Theta(I^2) = 0$. Hence, there is obviously induced a morphism of A-modules:

$$\Omega \leftarrow \frac{I}{I^2}, \tag{0.4.20}$$

which is the *inverse* of Σ. Consequently, we conclude that:

$$\Omega \cong \frac{I}{I^2}. \tag{0.4.21}$$

Thus, the free A-module Ω of 1-forms is isomorphic with the free A-module $\frac{I}{I^2}$ of *Kähler differentials* of the algebra of scalars A over $\mathbb{R}$, conceived as distinguished ideals within the algebra of infinitesimally extended scalars $[A \oplus \Omega \cdot \epsilon]$ due to interaction, according to the following split short exact sequence :

$$\Omega \hookrightarrow A \oplus \Omega \cdot \epsilon \twoheadrightarrow A, \tag{0.4.22}$$

or equivalently formulated as:

$$0 \to \Omega \to A \otimes_{\mathbb{R}} A \to A. \tag{0.4.23}$$

By dualizing, we obtain the *dual* A-module of Ω, that is, $\Xi := Hom(\Omega, A)$. Consequently, we have at our disposal, expressed in terms of infinitesimal scalars extension of the algebra $\mathcal{A}$, semantically intertwined with the generation of geometry as a result of interaction, new types of observables related with the incorporation of differentials and their duals, called vectors.

Let us now explain the functional role of geometry in relation to the infinitesimally extended rings of scalars. This is of particular significance for *dynamical physical theories*, where the ring of scalars stands for an algebra of *observable attributes* taking values in the field of real numbers. This *absolute valuation requirement* necessitates that our form of observation is tautological with real numbers representability. This means that all types of physical quantities should possess *uniquely* defined dual types, such that their representability can be made possible by means of real numbers. This is exactly the role of a *geometry induced by a metric.* Concretely, a metric structure assigns a unique dual to each physical quantity, by effectuating an isomorphism $\tilde{g}$ between the A-module Ω and its dual A-module $\Xi = Hom(\Omega, A)$, that is:

$$\tilde{g} : \Omega \longrightarrow \Xi, \tag{0.4.24}$$

such that:

$$\tilde{g} : \Omega \cong \Xi, \tag{0.4.25}$$

$$df \mapsto v_f := \tilde{g}(df). \tag{0.4.26}$$

Equivalently, a metric g stands for an $\mathbb{R}$-valued *symmetric bilinear form* on Ω, that is $g : \Omega \times \Omega \to \mathbb{R}$, yielding an invertible $\mathbb{R}$-linear morphism $\tilde{g} : \Omega \to \Xi$. Notice that for df, $dh \in \Omega$, a symmetric bilinear form g acts, via $\tilde{g}$, on df to give an element of the dual, $\tilde{g}(df) \in \Xi$, which then acts on dh to give $(\tilde{g}(df))(dh) = (\tilde{g}(dh))(df)$, or equivalently, $v_f(dh) = v_h(df) \in \mathbb{R}$. Also note that the invertibility of $\tilde{g}$ amounts to the property of *non-degeneracy* of g, meaning that for each $df \in \Omega$, there exists $dh \in \Omega$, such that $(\tilde{g}(df))(dh) = v_f(dh) \neq 0$.

Thus the functional role of a metric geometry induced by a bilinear form forces the observation of extended scalars by means of representability in the field of real numbers and is reciprocally conceived as the result of interactions causing infinitesimal variations in the scalars of an $\mathbb{R}$-algebra A of observables.

In order that Kähler's algebraic extension method becomes suitable as a generator of a universal mechanism of differential analysis it should be susceptible to the process of sheaf-theoretic localization of an observable algebra. This would allow the applicability of this mechanism in conjunction with the principle of local gauge invariance in a sheaf-theoretic context suited to the localization requirements of a corresponding dynamical physical theory. Most important, it would allow the complete disassociation of the differential mechanism of any underlying spatial substratum opening up the way for a *functorial* and *topos-theoretic* formulation of differential geometry.

0.4.3 *Connections and the Sheaf-Theoretic de Rham Complex*

The localization of Kähler's algebraic extension method in sheaf theoretic terms requires first of all the notion of a *universal derivation* of the observable algebra sheaf $\mathcal{A}$ considered for generality as an algebra sheaf over the constant sheaf of complexes $\mathbb{C}$.

The *universal* $\mathbb{C}$-*derivation* of the observable algebra sheaf $\mathcal{A}$ to the universal $\mathcal{A}$-module sheaf $\Omega^1(\mathcal{A}) := \Omega^1$, called the $\mathcal{A}$-*module sheaf of differential* 1-*forms*, is the universal $\mathbb{C}$-linear sheaf morphism (natural transformation) $\partial := d^0$:

$$d^0 : \mathcal{A} \to \Omega^1 \tag{0.4.27}$$

such that the *Leibniz condition* is satisfied:

$$d^0(s \cdot t) = s \cdot d^0(t) + t \cdot d^0(s), \tag{0.4.28}$$

for any continuous local sections s, t belonging to $\mathcal{A}(U)$, with U open set in X. We may also use the following notational convention: $\mathcal{A} := \Omega^0$.

Associated with the universal $\mathbb{C}$-derivation d^0 on $\mathcal{A}$, there also exists the *logarithmic universal derivation* defined as follows:

The *logarithmic universal derivation* of the observable abelian group sheaf $\tilde{\mathcal{A}}$ to the abelian group sheaf of 1-forms Ω^1 is the universal morphism of abelian group sheaves $\tilde{\partial} := \tilde{d}^0$:

$$\tilde{d}^0 : \tilde{\mathcal{A}} \to \Omega^1, \tag{0.4.29}$$

defined by the relation:

$$\tilde{d}^0(u) := u^{-1} \cdot d^0(u) \tag{0.4.30}$$

for any continuous *invertible* local section $u \in \tilde{\mathcal{A}}(U)$, with U open set in X.

Given the validity of the *Poincaré Lemma*, we also have:

$$Ker(\tilde{d}^0) = \tilde{\mathbb{C}}. \tag{0.4.31}$$

If we consider a complex vector sheaf of states $\mathcal{E}$, then we define an $\mathcal{A}$-connection on $\mathcal{E}$ as follows:

A connection $\mathcal{D}_{\mathcal{E}} := \nabla_{\mathcal{E}}$ on the vector sheaf $\mathcal{E}$ is a $\mathbb{C}$-*linear* sheaf morphism:

$$\nabla_{\mathcal{E}} : \mathcal{E} \to \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E}, \tag{0.4.32}$$

referring to $\mathbb{C}$-vector space sheaves, such that the corresponding Leibniz condition is satisfied:

$$\nabla_{\mathcal{E}}(a \cdot s) = a \cdot \nabla_{\mathcal{E}}(s) + s \otimes d^0(a). \tag{0.4.33}$$

Moreover, for each $n \in N$, $n \geq 2$, the n-fold *exterior product* is defined as follows: $\Omega^n(\mathcal{A}) = \wedge^n \Omega^1(\mathcal{A})$ where $\Omega(\mathcal{A}) := \Omega^1(\mathcal{A})$. We notice that there exists a $\mathbb{C}$-linear sheaf morphism:

$$d^n : \Omega^n(\mathcal{A}) \to \Omega^{n+1}(\mathcal{A}) \tag{0.4.34}$$

for all $n \geq 0$. Let $\omega \in \Omega^n(\mathcal{A})$, then ω has the form:

$$\omega = \sum f_i(dl_{i1} \wedge \ldots \wedge dl_{in}) \tag{0.4.35}$$

with f_i, l_{ij}, $\in \mathcal{A}$ for all integers i, j. Further, we define:

$$d^n(\omega) = \sum df_i \wedge dl_{i1} \wedge \ldots \wedge dl_{in}. \tag{0.4.36}$$

From the above, we immediately obtain that the composition of two consecutive $\mathbb{C}$-linear sheaf morphisms vanishes, that is $d^{n+1} \circ d^n = 0$.

The sequence of $\mathbb{C}$-linear sheaf morphisms:

$$\mathcal{A} \to \Omega^1(\mathcal{A}) \to \ldots \to \Omega^n(\mathcal{A}) \to \ldots \tag{0.4.37}$$

is a *complex* of $\mathbb{C}$-vector space sheaves, called the *sheaf-theoretic de Rham complex* of $\mathcal{A}$.

Moreover, given the validity of the Poincaré Lemma, $Ker(d^0) = \mathbb{C}$, and the fact that $\mathcal{A}$ is a soft observable algebra sheaf, we obtain:

The sequence of $\mathbb{C}$-vector space sheaves is exact:

$$\mathbf{0} \to \mathbb{C} \to \mathcal{A} \to \Omega^1(\mathcal{A}) \to \ldots \to \Omega^n(\mathcal{A}) \to \ldots \tag{0.4.38}$$

Thus, the sheaf-theoretic de Rham complex of the observable algebra sheaf $\mathcal{A}$ constitutes a *resolution of the constant sheaf* $\mathbb{C}$.

If we assume that the pair $(\mathcal{E}, \nabla_{\mathcal{E}})$ denotes a complex vector sheaf of states equipped with a connective structure, defined by a connection $\nabla_{\mathcal{E}}$ on $\mathcal{E}$, then $\nabla_{\mathcal{E}}$ induces a sequence of $\mathbb{C}$-linear morphisms:

$$\mathcal{E} \to \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E} \to \ldots \to \Omega^n(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E} \to \ldots \tag{0.4.39}$$

or equivalently:

$$\mathcal{E} \to \Omega^1(\mathcal{E}) \to \ldots \to \Omega^n(\mathcal{E}) \to \ldots \tag{0.4.40}$$

where the morphism:

$$\nabla^n : \Omega^n(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E} \to \Omega^{n+1}(\mathcal{A}) \otimes_{\mathbf{A}} \mathcal{E}, \tag{0.4.41}$$

is given by the formula:

$$\nabla^n(\omega \otimes v) = d^n(\omega) \otimes v + (-1)^n \omega \wedge \nabla(v), \tag{0.4.42}$$

for all $\omega \in \Omega^n(\mathcal{A})$, $v \in \mathcal{E}$. It is immediate to see that $\nabla^0 = \nabla_{\mathcal{E}}$.

The composition of $\mathbb{C}$-linear morphisms $\nabla^1 \circ \nabla^0$ is called the *curvature* of the connection $\nabla_{\mathcal{E}}$:

$$\nabla^1 \circ \nabla^0 := R_{\nabla} : \mathcal{E} \to \Omega^2(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E} = \Omega^2(\mathcal{E}). \tag{0.4.43}$$

As a straightforward consequence of the above definition we obtain:

The curvature R_{∇} of a connection $\nabla_{\mathcal{E}}$ on the vector sheaf of states $\mathcal{E}$ is an $\mathcal{A}$-linear sheaf morphism, that is a $\mathcal{A}$-covariant, or equivalently an $\mathcal{A}$-*tensor*.

The $\mathcal{A}$-covariant nature of the curvature R_{∇} is to be contrasted with the connection $\nabla_{\mathcal{E}}$ which is only $\mathbb{C}$-covariant.

The sequence of $\mathbb{C}$-linear sheaf morphisms

$$\mathcal{E} \to \Omega^1(\mathcal{A})\otimes_{\mathcal{A}}\mathcal{E} \to \ldots \to \Omega^n(\mathcal{A})\otimes_{\mathcal{A}}\mathcal{E} \to \ldots \tag{0.4.44}$$

is a *complex* of $\mathbb{C}$-vector space sheaves if and only if:

$$R_\nabla = 0. \tag{0.4.45}$$

Thus, the curvature $\mathcal{A}$-linear sheaf morphism R_∇ expresses the *obstruction* for the above sequence to qualify as a complex. We say that the connection $\nabla_{\mathcal{E}}$ is *integrable* if $R_\nabla = 0$, and we refer to the obtained complex as the sheaf-theoretic de Rham complex of the integrable connection $\nabla_{\mathcal{E}}$ on the vector sheaf $\mathcal{E}$ in that case. It is also usual to call a connection $\nabla_{\mathcal{E}}$ *flat* if $R_\nabla = 0$. We note that the universal $\mathbb{C}$-derivation d^0 on $\mathcal{A}$ defines an integrable or flat connection.

A flat connection defines a *maximally undisturbed process of dynamical variation* caused by the corresponding physical field. In this sense, a non-vanishing curvature signifies the existence of disturbances from the maximally symmetric state of that variation. These disturbances can be cohomologically identified as obstructions to deformation caused by physical sources. In this case, the sheaf-theoretic de Rham complex is *not acyclic*, viz. it has non-trivial cohomology groups. These groups measure the obstructions caused by sources and are responsible for a non-vanishing covariantly observable curvature of the connection.

In the context of a dynamical physical theory, where the representability principle over the field of real numbers is demanded, the existence of uniquely defined duals is necessary. Thus, the action of a field is identified with a *linear connection* on the $\mathcal{A}$-vector sheaf of states $\Xi = Hom(\Omega^1, \mathcal{A})$, made isomorphic with Ω^1, by means of a bilinear form playing the role of a *metric*:

$$g : \Omega^1 \simeq \Xi = \Omega^{1^*}. \tag{0.4.46}$$

We stress the fact that a physical observable geometry in this case is considered with respect to a metric. Consequently, the *physical field* may be represented by the pair (Ξ, ∇_Ξ). The required *metric compatibility* of the connection is expressed as follows:

$$\nabla_{Hom_{\mathcal{A}}(\Xi,\Xi^*)}(g) = 0. \tag{0.4.47}$$

Taking into account the requirement of representability over the real numbers, and thus considering the relevant *evaluation trace operator* by means of the metric, we arrive at the analogue of *Einstein's field equations*, which in the absence of matter sources with respect to $\mathcal{A}$ read:

$$\mathcal{R}(\nabla_\Xi)(\Xi) = 0. \tag{0.4.48}$$

where $\mathcal{R}(\nabla_\Xi)$ denotes the relevant *Ricci scalar curvature*. In more detail, we first define the curvature endomorphism $\mathfrak{R}_\nabla \in End(\Xi)$, called *Ricci curvature operator*. Since the Ricci curvature $\mathfrak{R}_\nabla$ is locally matrix-valued, by taking its trace using the metric, that is by considering its evaluation or contraction by means of the metric, we arrive at the definition of the Ricci scalar curvature $\mathcal{R}(\nabla_{\Xi})(\Xi)$ obeying the above equation.

Thus, the metric describing the physical geometry, as a result of field interactions, is *dynamically* determined as a solution of the above equation in relation to the metric compatible connection on the vector sheaf $Hom_{\mathcal{A}}(\Xi, \Xi^*)$.

0.4.4 *Local Forms of Connection and Curvature on Vector Sheaves of States*

We proceed by a direct calculation to show that every connection $\nabla_{\mathcal{E}}$, where $\mathcal{E}$ is a finite rank-n vector sheaf of states on X, can be *decomposed locally* as follows:

$$\nabla_{\mathcal{E}} = d^0 + \omega, \tag{0.4.49}$$

where $\omega = \omega_{\alpha\beta}$ denotes an $n \times n$ matrix of sections of local 1-forms, called the *matrix potential* of $\nabla_{\mathcal{E}}$. Moreover, under a change of local frame matrix $g = g_{\alpha\beta}$ the matrix potentials transform as follows:

$$\acute{\omega} = g^{-1}\omega g + g^{-1}d^0 g. \tag{0.4.50}$$

If we consider a coordinatizing basis of sections defined over an open cover U of X, denoted by:

$$e^U \equiv \{U;\ (e_\alpha)_{1 \leq \alpha \leq n}\} \tag{0.4.51}$$

of the vector sheaf $\mathcal{E}$ of rank-n, called a local sectional frame or a local gauge of $\mathcal{E}$, then every continuous local section $s \in \mathcal{E}(U)$, where, $U \in \mathcal{U}$, can be expressed uniquely with respect to this local frame as a *superposition*:

$$s = \sum_{\alpha=1}^{n} s_\alpha e_\alpha, \tag{0.4.52}$$

with coefficients s_α in $\mathcal{A}(U)$. The action of $\nabla_{\mathcal{E}}$ on these sections of $\mathcal{E}$ is expressed as follows:

$$\nabla_{\mathcal{E}}(s) = \sum_{\alpha=1}^{n} (s_\alpha \nabla_{\mathcal{E}}(e_\alpha) + e_\alpha \otimes d^0(s_\alpha)), \tag{0.4.53}$$

$$\nabla_{\mathcal{E}}(e_\alpha) = \sum_{\alpha=1}^{n} e_\alpha \otimes \omega_{\alpha\beta},\ 1 \leq \alpha, \beta \leq n, \tag{0.4.54}$$

where $\omega = \omega_{\alpha\beta}$ denotes an $n \times n$ matrix of sections of local 1-forms. Consequently we have;

$$\nabla_{\mathcal{E}}(s) = \sum_{\alpha=1}^{n} e_\alpha \otimes (d^0(s_\alpha) + \sum_{\beta=1}^{n} s_\beta \omega_{\alpha\beta}) \equiv (d^0 + \omega)(s). \tag{0.4.55}$$

Thus, every connection $\nabla_{\mathcal{E}}$, where $\mathcal{E}$ is a finite rank-n vector sheaf on X, can be decomposed locally as follows:

$$\nabla_{\mathcal{E}} = d^0 + \omega. \tag{0.4.56}$$

In this context, $\nabla_{\mathcal{E}}$ is identified as a *covariant derivative* operator acting on sections of the vector sheaf of states $\mathcal{E}$, and being decomposed locally as a sum consisting of a flat or integrable part identical with d^0, and a generally non-integrable part ω, called the local frame (gauge) matrix potential of the connection.

The behavior of the local potential ω of $\nabla_{\mathcal{E}}$ under local frame transformations constitutes the *'transformation law of local potentials'* and is obtained as follows: Let $e^U \equiv \{U;\ e_{\alpha=1\cdots n}\}$ and $h^V \equiv \{V;\ h_{\beta=1\cdots n}\}$ be two local frames of $\mathcal{E}$ over the open sets U and V of X, such that $U \bigcap V \neq \emptyset$. Let us denote by $g = g_{\alpha\beta}$ the following change of local frame matrix:

$$h_\beta = \sum_{\alpha=1}^{n} g_{\alpha\beta} e_\alpha. \tag{0.4.57}$$

Under such a local frame transformation $g_{\alpha\beta}$, it is straightforward to obtain that the local potential ω of $\nabla_{\mathcal{E}}$ transforms as follows in matrix form:

$$\acute{\omega} = g^{-1}\omega g + g^{-1} d^0 g. \tag{0.4.58}$$

Furthermore, it is instructive to find the local form of the curvature R_∇ of a connection $\nabla_{\mathcal{E}}$, where $\mathcal{E}$ is a locally free finite rank-n sheaf of modules (vector sheaf) $\mathcal{E}$ on X, defined by the following $\mathcal{A}$-linear morphism of sheaves:

$$R_\nabla := \nabla^1 \circ \nabla^0 : \mathcal{E} \to \Omega^2(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E} := \Omega^2(\mathcal{E}). \tag{0.4.59}$$

Due to its property of $\mathcal{A}$-*covariance*, a non-vanishing curvature represents in this context, the $\mathcal{A}$-covariant, and thus *geometric observable* deviation from the *monodromic* form of variation corresponding to an integrable connection. Moreover, since the curvature R_∇ is an $\mathcal{A}$-linear morphism of sheaves of $\mathcal{A}$-modules, that is an $\mathcal{A}$-tensor, R_∇ may be thought of as an element of $End(\mathcal{E}) \otimes_{\mathcal{A}} \Omega^2(\mathcal{A}) := \Omega^2(End(\mathcal{E}))$, that is:

$$R_\nabla \in \Omega^2(End(\mathcal{E})). \tag{0.4.60}$$

Hence, the local form of the curvature R_∇ of a connection $\nabla_{\mathcal{E}}$, consists of local $n \times n$ matrices having for entries local 2-forms. In particular, the local form of the curvature $R_\nabla|_U$, where U open in X, in terms of the local potentials ω is expressed by:

$$R_\nabla|_U = d^1\omega + \omega \wedge \omega, \tag{0.4.61}$$

at it can be easily shown by substitution of the local potentials in the composition $\nabla^1 \circ \nabla^0$. Furthermore, by application of the differential operator d^2 on the above we obtain:

$$d^2 R_\nabla|_U = R_\nabla|_U \wedge \omega - \omega \wedge R_\nabla|_U. \tag{0.4.62}$$

The behavior of the curvature R_{∇} of a connection $\nabla_{\mathcal{E}}$ under local frame transformations constitutes the '*transformation law of potentials' strength*'. If we agree that $g = g_{\alpha\beta}$ denotes the change of local frame matrix, we have previously considered in the discussion of the transformation law of local connection potentials, we deduce the following local transformation law:

$$R_{\nabla} \overset{g}{\mapsto} \acute{R}_{\nabla} = g^{-1}(R_{\nabla})g, \tag{0.4.63}$$

that is they transform covariantly by *conjugation* with respect to a local frame transformation. It is useful to collect the above findings in the form of the following proposition:

Theorem 0.5.

I. The local form of the curvature $R_{\nabla}|_U$, where U open in X, in terms of the local matrix potentials ω is given by:

$$R_{\nabla}|_U = d^1\omega + \omega \wedge \omega. \tag{0.4.64}$$

II. Under a change of local frame matrix $g = g_{\alpha\beta}$ the local form of the curvature transforms by conjugation:

$$R_{\nabla} \overset{g}{\mapsto} \acute{R}_{\nabla} = g^{-1}(R_{\nabla})g. \tag{0.4.65}$$

We note that the above holds for any complex vector sheaf $\mathcal{E}$. We may now specialize in the case of a *line sheaf* of states $\mathcal{L}$ equipped with a connection, denoted by the pair $(\mathcal{L}, \nabla)$. In this case, due to the isomorphism:

$$\mathcal{L} \otimes_{\mathcal{A}} \mathcal{L}^* \cong Hom_{\mathcal{A}}(\mathcal{L}, \mathcal{L}) \equiv End_{\mathcal{A}}\mathcal{L} \cong \mathcal{A}, \tag{0.4.66}$$

we obtain the following simplifications: the local form of a connection over an open set is just a local 1-form or *local potential* (that is a local continuous section of Ω^1), whence the local form of the curvature of the connection over an open set is a local 2-form. The significant result obtained by the local transformation law in this case is that the curvature is *local frame invariant*, that is it does not change under any local frame transformation.

$$R_{\nabla} \overset{g}{\mapsto} R_{\nabla}' = R_{\nabla} = d^1\omega. \tag{0.4.67}$$

Thus, we obtain a *global 2-form* R_{∇} defined over X, which is also a *closed* 2-form because of the fact that:

$$d^2 R_{\nabla} = 0. \tag{0.4.68}$$

Therefore, we conclude that for a line sheaf of states $\mathcal{L}$ equipped with a connection ∇, denoted by the pair $(\mathcal{L}, \nabla)$, the curvature R_{∇} of the connection is a global closed 2-form.

0.4.5 *Gauge Equivalence Classes of Differential Line Sheaves*

We consider two line sheaves which are equivalent via an isomorphism $h : \mathcal{L} \xrightarrow{\cong} \mathcal{L}'$. We would like to extend the notion of equivalence for two line sheaves equipped with a connective structure $(\mathcal{L}, \nabla)$ and $(\mathcal{L}', \nabla')$. We proceed as follows:

Given an isomorphism $h : \mathcal{L} \xrightarrow{\cong} \mathcal{L}'$ of line sheaves of states, we say that ∇ is *frame or gauge equivalent* to ∇' if they are *conjugate connections* under the action of h:

$$\nabla' = h \cdot \nabla \cdot h^{-1}. \tag{0.4.69}$$

Thus, we may consider the set of equivalence classes on pairs of the form $(\mathcal{L}, \nabla)$ under an isomorphism h as previously, denoted by $Iso(\mathcal{L}, \nabla)$. It is now important to find the relation between $Iso(\mathcal{L}, \nabla)$ and the *abelian group* $Iso(\mathcal{L})$. For this purpose, we need to explore the local form of the pair $(\mathcal{L}, \nabla)$.

We call a line sheaf $\mathcal{L}$ equipped with a connection ∇ a *differential line sheaf* and we denote it by the pair $(\mathcal{L}, \nabla)$. Then, we have the following:

Theorem 0.6. *I. The local form of a differential line sheaf is given by:*

$$(\mathcal{L}, \nabla) \leftrightarrow (g_{\alpha\beta}, \omega_\alpha) \in Z^1(\mathcal{U}, \tilde{\mathcal{A}}) \times C^0(\mathcal{U}, \Omega^1). \tag{0.4.70}$$

II. An arbitrary pair $(g_{\alpha\beta}, \omega_\alpha) \in Z^1(\mathcal{U}, \tilde{\mathcal{A}}) \times C^0(\mathcal{U}, \Omega^1)$ *determines a differential line sheaf if*

$$\delta^0(\omega_\alpha) = \tilde{d^0}(g_{\alpha\beta}). \tag{0.4.71}$$

In order to elucidate the above, we note that a line sheaf is expressed in local coordinates bijectively in terms of a Čech coordinate 1-cocycle $g_{\alpha\beta}$ in $Z^1(\mathcal{U}, \tilde{\mathcal{A}})$ associated with the covering $\mathcal{U}$. A connection ∇ is expressed bijectively in terms of a 0-cochain of 1-forms, called the local potentials of the connection, and denoted by ω_α with respect to the covering $\mathcal{U}$ of X, that is $\omega_\alpha \in C^0(\mathcal{U}, \Omega^1)$. We deduce that the local form of a differential line sheaf is the following:

$$(\mathcal{L}, \nabla) \leftrightarrow (g_{\alpha\beta}, \omega_\alpha) \in Z^1(\mathcal{U}, \tilde{\mathcal{A}}) \times C^0(\mathcal{U}, \Omega^1). \tag{0.4.72}$$

Conversely, an arbitrary pair $(g_{\alpha\beta}, \omega_\alpha) \in Z^1(\mathcal{U}, \tilde{\mathcal{A}}) \times C^0(\mathcal{U}, \Omega^1)$ determines a differential line sheaf if the 'transformation law of local potentials' is satisfied by this pair, that is:

$$\omega_\beta = g_{\alpha\beta}^{-1} \omega_\alpha g_{\alpha\beta} + g_{\alpha\beta}^{-1} d^0 g_{\alpha\beta}, \tag{0.4.73}$$

$$\omega_\beta = \omega_\alpha + g_{\alpha\beta}^{-1} d^0 g_{\alpha\beta}. \tag{0.4.74}$$

Thus, given an open covering $\mathcal{U} = \{U_\alpha\}$, a 0-cochain (ω_α) valued in the sheaf Ω^1 determines the local form of a connection ∇ on the line sheaf $\mathcal{L}$, where the latter is expressed in local coordinates bijectively in terms of a Čech coordinate 1-cocycle $(g_{\alpha\beta})$ valued in $\tilde{\mathcal{A}}$ with respect to $\mathcal{U}$, if and only if the corresponding local 1-forms

ω_α of the 0-cochain with respect to $\mathcal{U}$ are pairwise inter-transformable (local frame-equivalent) on overlaps $U_{\alpha\beta}$ via the local frame transition functions (isomorphisms) $g_{\alpha\beta} \in \tilde{\mathcal{A}}(U_{\alpha\beta})$ according to the 'transformation law of local potentials':

$$\delta^0(\omega_\alpha) = \tilde{d^0}(g_{\alpha\beta}), \tag{0.4.75}$$

where,

$$\tilde{d^0}(g_{\alpha\beta}) = g_{\alpha\beta}^{-1} \cdot d^0 g_{\alpha\beta}, \tag{0.4.76}$$

and δ^0 denotes the 0-th coboundary operator $\delta^0 : C^0(\mathcal{U}, \Omega^1) \to C^1(\mathcal{U}, \Omega^1)$, such that:

$$\delta^0(\omega_\alpha) = \omega_\beta - \omega_\alpha. \tag{0.4.77}$$

Next, we consider two line sheaves which are equivalent via an isomorphism $h : \mathcal{L} \xrightarrow{\cong} \mathcal{L}'$, such that their corresponding connections are conjugate under the action of h:

$$\nabla' = h \nabla h^{-1}. \tag{0.4.78}$$

Under these conditions the differential line sheaves $(\mathcal{L}, \nabla)$ and $(\mathcal{L}', \nabla')$ are called *gauge or frame equivalent*. Thus, we may consider the set of *gauge equivalence classes* $[(\mathcal{L}, \nabla)]$ of differential line sheaves as above, denoted by $Iso(\mathcal{L}, \nabla)$. Remarkably, the following holds:

Theorem 0.7. *The set of gauge equivalence classes of differential line sheaves $Iso(\mathcal{L}, \nabla)$, is an abelian subgroup of the abelian group $Iso(\mathcal{L})$.*

If we consider the local form of the tensor product of two gauge equivalent differential line sheaves we have:

$$(\mathcal{L}, \nabla) \otimes_{\mathcal{A}} (\mathcal{L}', \nabla') \leftrightarrow (g_{\alpha\beta} \cdot g'_{\alpha\beta}, \omega_\alpha + \omega'_\alpha), \tag{0.4.79}$$

which satisfies the 'transformation law of local potentials':

$$\tilde{d^0}(g_{\alpha\beta} \cdot g'_{\alpha\beta}) = \tilde{d^0}(g_{\alpha\beta}) + \tilde{d^0}(g'_{\alpha\beta}) = \delta^0(\omega_\alpha) + \delta^0(\omega'_\alpha) = \delta^0(\omega_\alpha + \omega'_\alpha). \tag{0.4.80}$$

Moreover, the inverse of a pair $(g_{\alpha\beta}, \omega_\alpha)$ is given by $(g_{\alpha\beta}^{-1}, -\omega_\alpha)$, whereas the neutral element in the group $Iso(\mathcal{L}, \nabla)$ is given by $(id_{\alpha\beta}, 0)$, which corresponds to the trivial differential line sheaf $(\mathcal{A}, d^0)$.

Now, if we consider a line sheaf $\mathcal{L}$ of states, it is classified up to isomorphism by a *cohomology class* $\delta_c(\mathcal{L})$ in the abelian group $H^2(X, \mathbb{Z})$. A natural question arising in this setting is if it is possible to express the *global invariant information* provided by the equivalent line sheaves' classifying integer cohomology class by means of a *differential cohomological invariant*. This differential invariant should be associated with a *differential equation*, or more concretely with the *parallel transport* rule imposed by a physically induced connectivity structure on the line sheaf. The possibility of expressing the classifying integer cohomology class of isomorphic line sheaves of states by means of a differential invariant would also provide a cohomological justification of the physical gauge principle, according to which local gauge invariance implies the existence of local gauge potentials, and thus of a connection on a line sheaf of states. Given that local gauge invariance is depicted by means of coordinate 1-cocycles with respect to a covering $\mathcal{U}$ of X and the fact that the localization of a connection is expressed in terms of local gauge potentials with respect to $\mathcal{U}$, it is natural to seek for a relation between the invariant of the former, that is an integer cohomology class, and the differential invariant of the latter, that is the class of the curvature of the connection.

0.4.6 *Quantization Condition via Cohomology*

An immediate important consequence of the above characterization of gauge equivalent differential line sheaves in local terms with respect to an open covering $\mathcal{U}$, is that all of them have the same curvature. Indeed, by applying the differential operator d^1 on the above relation and using the fact that:

$$d^1 \circ d^0 = 0, \tag{0.4.81}$$

from which we have that:

$$d^1 \circ \tilde{d}^0 = 0, \tag{0.4.82}$$

we obtain:

$$d^1(\omega'_\alpha) = d^1(\omega_\alpha). \tag{0.4.83}$$

Hence, we formulate the following:

Theorem 0.8. *Gauge equivalent differential line sheaves always have the same curvature.*

We denote the *curvature of a gauge equivalence class* of differential line sheaves by R. We notice that R is a global 2-form on X since it is local frame-change invariant, which is also closed because of the fact that:

$$d^2 \circ d^1 \omega_\alpha = d^2 R = 0. \tag{0.4.84}$$

Thus, the global 2-form R, which belongs to $Ker(d^2) : \Omega^2 \to \Omega^3$, called Ω^2_c, being a $\mathbb{C}$-vector sheaf subspace of Ω^2, determines a global differential invariant of gauge equivalent differential line sheaves. This the case because the global 2-form R determines a *2-dimensional de Rham cohomology class* $[R]$, identified as a *2-dimensional complex Čech cohomology class* in $H^2(X, \mathbb{C})$. In particular, if we consider a differential line sheaf the differential invariant de Rham cohomology class $[R]$ is *independent of the connection* used to represent R locally. In other words, a particular connection of a differential line sheaf provides the means to express this global differential invariant locally, whereas the latter is independent of the particular means used to represent it locally.

Furthermore, since any two gauge equivalent differential sheaves have the same curvature, we conclude that they are *physically indistinguishable*. This means that the abelian group $Iso(\mathcal{L},\nabla)$ is partitioned into *orbits* over the image of $Iso(\mathcal{L},\nabla)$ into Ω^2_c, where each orbit (fiber) is labeled by a closed 2-form R of Ω^2_c:

$$Iso(\mathcal{L},\nabla) = \sum_R Iso(\mathcal{L},\nabla)_R. \tag{0.4.85}$$

Thus, the abelian group of equivalence classes of differential line sheaves fibers over those elements of Ω^2_c, viz. over those closed global 2-forms in Ω^2_c which can be identified as curvatures. In this way, a pertinent problem is the concrete identification of the image of $Iso(\mathcal{L},\nabla)$ into Ω^2_c. Put differently, we are looking for an *intrinsic* characterization of those global closed 2-forms in Ω^2_c, which are instantiated as curvatures of gauge equivalence classes of differential line sheaves. The resolution of this problem is provided by the sheaf-theoretic formulation of the *Chern-Weil integrality theorem.* In physical terms, the integrality theorem provides the general and intrinsic cohomological formulation of *Dirac's quantization condition.* According to this, the de Rham cohomology class $[R]$ of any differential line sheaf in $H^2(X,\mathbb{C})$ is in the image of a cohomology class in the integral 2-dimensional cohomology group $H^2(X,\mathbb{Z})$ into $H^2(X,\mathbb{C})$. More precisely, the Chern-Weil integrality theorem may be formulated as follows in our context:

Theorem 0.9. *A global closed 2-form is the curvature R of a differential line sheaf if and only if its 2-dimensional de Rham cohomology class is integral, viz.* $[R] \in Im(H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{C}))$.

It is instructive to discuss briefly the basic concepts of this fundamental theorem providing the cohomological justification of the quantization condition. We know that each equivalence classes of line sheaves is in bijective correspondence with a cohomology class in the integral 2-dimensional cohomology group of X. Thus, we have to show that a 2-dimensional de Rham cohomology class is a curvature differential invariant class of gauge equivalent differential line sheaves if and only if it is an integral 2-dimensional cohomology class. First, by the natural injection $\mathbb{Z} \hookrightarrow \mathbb{C}$ we obtain:

$$H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{C}), \tag{0.4.86}$$

where any cohomology class belonging to the image of the above map is called an *integral cohomology class.* Next, we consider the following exact sequences:

$$0 \to \mathbb{Z} \xrightarrow{\iota} \mathcal{A} \xrightarrow{exp} \tilde{\mathcal{A}} \to 1, \tag{0.4.87}$$

which is the exponential short exact sequence of abelian group sheaves, such that:

$$Ker(exp) = Im(\iota) \cong \mathbb{Z} \tag{0.4.88}$$

and the short exact sequence of $\mathbb{C}$-vector sheaves,

$$0 \to \mathbb{C} \xrightarrow{\varepsilon} \mathcal{A} \xrightarrow{d^0} d^0\mathcal{A} \to 0, \tag{0.4.89}$$

which is a fragment of the *de Rham resolution of the constant sheaf* $\mathbb{C}$, such that:

$$Ker(d^0) = Im(\varepsilon) \cong \mathbb{C}. \tag{0.4.90}$$

Furthermore, we consider the following commutative diagram:

$$\begin{tikzcd} \mathcal{A} \arrow[r, "exp"] \arrow[d, "id"'] & \tilde{\mathcal{A}} \arrow[d, "\frac{1}{2\pi i}\tilde{d}^0"] \\ \mathcal{A} \arrow[r, "d^0"] & d^0\mathcal{A} \end{tikzcd}$$

Thus we obtain the relation:

$$2\pi i \cdot d^0 = \tilde{d}^0 \circ exp. \tag{0.4.91}$$

If we take the relevant fragments of the corresponding long exact sequences in cohomology we obtain the following commutative diagram:

$$\begin{tikzcd} H^1(X, \tilde{A}) \arrow[r, "\delta_c"] \arrow[d, "\frac{1}{2\pi i}\tilde{d}^0"'] & H^2(X, \mathbb{Z}) \arrow[d, "\iota^*"] \\ H^1(X, d^0\mathcal{A}) \arrow[r, "\cong"] & H^2(X, \mathbb{C}) \end{tikzcd}$$

Thus, the image of a cohomology class of $H^2(X, \mathbb{Z})$ into an integral cohomology class of $H^2(X, \mathbb{C})$ corresponds to the cohomology class specified by the image of the 1-cocycle $g_{\alpha\beta}$ into $H^1(X, d^0\mathcal{A})$, viz. by the 1-cocycle $\frac{1}{2\pi i}\tilde{d}^0(g_{\alpha\beta})$.

In more detail, we first have that due to the exactness of the exponential sheaf sequence of abelian group sheaves:

$$0 \to \mathbb{Z} \xrightarrow{\iota} \mathcal{A} \xrightarrow{exp} \tilde{\mathcal{A}} \to 1, \tag{0.4.92}$$

$$g_{\alpha\beta} = exp(w_{\alpha\beta}), \tag{0.4.93}$$

where $(w_{\alpha\beta}) \in C^1(\mathcal{U}, \mathcal{A})$ as well as

$$\delta_c(w_{\alpha\beta}) = (z_{\alpha\beta\gamma}) \in Z^2(\mathcal{U}, \mathbb{Z}). \tag{0.4.94}$$

Explicitly, we may consider $w_{\alpha\beta} = ln(g_{\alpha\beta})$, so that:

$$\delta_c(w_{\alpha\beta}) = (z_{\alpha\beta\gamma}) := \frac{1}{2\pi i}(ln(g_{\alpha\beta}) + ln(g_{\beta\gamma}) - ln(g_{\alpha\gamma})) \in Z^2(\mathcal{U}, \mathbb{Z}). \tag{0.4.95}$$

Now, we need to locate the curvature 2-dimensional complex de Rham cohomology class differential invariant, viz. the complex 2-dimensional cohomology class $[R]$ in $H^2(X, \mathbb{C})$. For this purpose, regarding the curvature we have that:

$$R = (d\omega_\alpha) \in Z^0(\mathcal{U}, d\Omega^1). \tag{0.4.96}$$

By the transformation law of local potentials, it holds:

$$\delta^0(\omega_\alpha) = \tilde{d^0}(g_{\alpha\beta}), \tag{0.4.97}$$

where,

$$\tilde{d^0}(g_{\alpha\beta}) = g_{\alpha\beta}^{-1} \cdot d^0 g_{\alpha\beta} \tag{0.4.98}$$

and $g_{\alpha\beta} \in Z^1(\mathcal{U}, \tilde{\mathcal{A}})$, whence δ^0 denotes the 0-th coboundary operator $\delta^0 : C^0(\mathcal{U}, \Omega^1) \to C^1(\mathcal{U}, \Omega^1)$, such that:

$$\delta^0(\omega_\alpha) = \omega_\beta - \omega_\alpha. \tag{0.4.99}$$

Thus, we obtain:

$$\delta^0(\omega_\alpha) = 2\pi i \cdot d^0(w_{\alpha\beta}), \tag{0.4.100}$$

where $d^0(w_{\alpha\beta}) \in Z^1(\mathcal{U}, d^0\mathcal{A})$. Therefore, we deduce that:

$$\frac{1}{2\pi i}(\omega_\beta - \omega_\alpha) = d^0(ln(g_{\alpha\beta})). \tag{0.4.101}$$

If we inspect the short exact sequence of $\mathbb{C}$-vector sheaves:

$$0 \to \mathbb{C} \xrightarrow{\varepsilon} \mathcal{A} \xrightarrow{d^0} d^0\mathcal{A} \to 0, \tag{0.4.102}$$

we immediately deduce that:

$$\delta(ln(g_{\alpha\beta})) = (ln(g_{\alpha\beta}) + ln(g_{\beta\gamma}) - ln(g_{\alpha\gamma})) \in Z^2(\mathcal{U}, \mathbb{C}). \tag{0.4.103}$$

Thus, finally we obtain:

$$[R] = 2\pi i \cdot [(z_{\alpha\beta\gamma})] \in H^2(X, \mathbb{C}), \tag{0.4.104}$$

$$[(g_{\alpha\beta})] = \frac{1}{2\pi i}[R] = [(z_{\alpha\beta\gamma})] \in H^2(X, \mathbb{Z}), \tag{0.4.105}$$

$$[R] \in Im(H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{C})). \tag{0.4.106}$$

Thus, we have obtained an intrinsic characterization of the subset of those global closed 2-forms in Ω_c^2, which are instantiated as curvatures of gauge equivalence classes of differential line sheaves, denoted by $\Omega_{c,\mathbb{Z}}^2$. Therefore, a global closed 2-form is the curvature R of a differential line sheaf if and only if its 2-dimensional de Rham cohomology class is integral, viz. $[R] \in Im(H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{C}))$. This constitutes the generalized cohomological formulation of Dirac's quantization condition.

0.4.7 *Integrable Differential Line Sheaves*

A very important observation is that a *constant* 1-*cocycle* $(\xi_{\alpha\beta}) \in Z^1(\mathcal{U}, \tilde{\mathbb{C}})$ can be interpreted as the coordinate 1-cocycle of a line sheaf of states with respect to an open covering $\mathcal{U}$. Since the coordinate 1-cocycle $(\xi_{\alpha\beta})$ is constant, the bijectively corresponding to it line sheaf is a complex linear local system of rank 1, abbreviated as a *line local system.* The significance of this notion in relation to the integrability properties of a differential line sheaf lies on the following theorem:

Theorem 0.10. *There exists a bijective correspondence between differential line sheaves equipped with an integrable connection and line local systems.*

In order to clarify the above assertion, let us consider any differential line sheaf $(\mathcal{L}, \nabla)$ whose connection ∇ is integrable, viz. its curvature is zero. Then, the set of sections of $\mathcal{L}$ being in the kernel of the connection morphism ∇, that is:

$$Ker(\nabla) := \{s \in \mathcal{L} : \nabla(s) = 0\} \tag{0.4.107}$$

forms a line local system. We call the sections of $Ker(\nabla)$ *covariantly constant sections* of $\mathcal{L}$ with respect to ∇. Inversely, given a line local system, denoted by Λ, we may define a differential line sheaf $(\mathcal{L}, \nabla)$ as follows:

$$\mathcal{L} := \mathcal{A} \otimes_{\mathbb{C}} \Lambda, \tag{0.4.108}$$

and for every pair of local sections $a \in \mathcal{A}(U)$, $s \in \Lambda(U)$:

$$\breve{\nabla}(a \otimes s) := da \otimes s. \tag{0.4.109}$$

Clearly, the above defined connection is integrable and leads to the conclusion that there exists a *bijective* correspondence between differential line sheaves with an integrable connection, denoted by $(\mathcal{L}, \breve{\nabla})$ and line local systems. Moreover, every line local system is identified with the sheaf of covariantly constant sections of an integrable differential line sheaf, meaning of a line sheaf $\mathcal{L}$ with respect to an integrable connection $\breve{\nabla}$ on $\mathcal{L}$, viz. with $Ker(\breve{\nabla})_{\mathcal{L}}$.

Therefore, we immediately obtain the following equivalences:

$$Z^1(\mathcal{U}, \tilde{\mathbb{C}}) \ni (\xi_{\alpha\beta}) \cong (\mathcal{L}, \breve{\nabla}) \cong Ker(\breve{\nabla})_{\mathcal{L}} \cong \Lambda, \tag{0.4.110}$$

$$H^1(X, \tilde{\mathbb{C}}) \cong [(\mathcal{L}, \breve{\nabla})] \cong [Ker(\breve{\nabla})_{\mathcal{L}}] \cong [\Lambda]. \tag{0.4.111}$$

If we also assume that the underlying topological space X of control variables is locally path-connected, then the above series of equivalences is refined as follows:

$$Hom(\pi_1(X), \tilde{\mathbb{C}}) \cong H^1(X, \tilde{\mathbb{C}}) \cong [(\mathcal{L}, \breve{\nabla})] \cong [Ker(\breve{\nabla})_{\mathcal{L}}] \cong [\Lambda], \tag{0.4.112}$$

where the first term denotes the set of representations of the *fundamental group* of the topological space X of control variables to $\tilde{\mathbb{C}}$.

0.4.8 *Quantum Unitary Rays*

For reasons implicated by the probabilistic interpretation of states in quantum mechanics, which utilizes the *inner product* structure of the state space, we need to focus on the case of isomorphism classes of Hermitian line sheaves, defined as follows:

Given a line sheaf $\mathcal{L}$ on X, an $\mathcal{A}$*-valued Hermitian inner product* on $\mathcal{L}$ is a skew-$\mathcal{A}$-bilinear sheaf morphism :

$$\varrho : \mathcal{L} \oplus \mathcal{L} \to \mathcal{A}, \tag{0.4.113}$$

$$\varrho(\alpha s, \beta t) = \alpha \cdot \bar{\beta} \cdot \varrho(s,t), \tag{0.4.114}$$

for any s, $t \in \mathcal{L}(U)$, α, $\beta \in \mathcal{A}(U)$, U open in X. Moreover, $\varrho(s,t)$ is skew-symmetric, viz. $\varrho(s,t) = \overline{\varrho(s,t)}$.

A line sheaf $\mathcal{L}$ on X, together with an $\mathcal{A}$-valued Hermitian inner product on $\mathcal{L}$ constitute a *Hermitian line sheaf.*

A line sheaf is expressed in local coordinates bijectively in terms of a Čech coordinate 1-cocycle $g_{\alpha\beta}$ in $Z^1(\mathcal{U}, \tilde{\mathcal{A}})$ associated with the open covering $\mathcal{U}$. A Čech coordinate 1-cocycle $g_{\alpha\beta}$ corresponding to a Hermitian line sheaf consists of local sections of $\mathcal{S}U(1,\mathcal{A})$, viz. the *special unitary* group sheaf of $\mathcal{A}$ of order 1. This is simply a coordinate 1-cocycle $g_{\alpha\beta}$ in $Z^1(\mathcal{U}, \tilde{\mathcal{A}})$, such that the unitarity condition $|g_{\alpha\beta}| = 1$ is satisfied. Clearly, in the case that a coordinate 1-cocycle $g_{\alpha\beta}$ is constant, we have $g_{\alpha\beta}$ in $Z^1(\mathcal{U}, U(1))$, or equivalently $g_{\alpha\beta}$ in $Z^1(\mathcal{U}, \mathbb{S}^1)$.

Next, we need to define the notion of a connection which is compatible with the $\mathcal{A}$-valued Hermitian inner product on $\mathcal{L}$. A connection ∇ on $\mathcal{L}$ is called *Hermitian* if it is compatible with ϱ:

$$d^0 \varrho(s,t) = \varrho(\nabla s, t) + \varrho(s, \nabla t) \tag{0.4.115}$$

for any s, $t \in \mathcal{L}(U)$, U open in X.

A Hermitian differential line sheaf of states of a quantum system, or equivalently a *quantum unitary ray*, denoted by $(\mathcal{L}, \nabla, \varrho)$, is a Hermitian line sheaf equipped with a Hermitian connection.

First, we note that although we use the same symbol ϱ in the second part of the above, it refers to an extension of the $\mathcal{A}$-valued Hermitian inner product on $\mathcal{L}$, defined as follows:

$$\varrho : \Omega(\mathcal{L}) \oplus \mathcal{L} \to \Omega(\mathcal{L}), \tag{0.4.116}$$

$$\varrho(s \otimes t, s') := \varrho(s, s') \cdot t \tag{0.4.117}$$

for any s, $s' \in \mathcal{L}(U)$, $t \in \Omega(U)$, U open in X. Furthermore, Ω is considered to be a vector sheaf on X.

Second, if we have a Hermitian line sheaf $(\mathcal{L}, \varrho)$, which is also equipped with a connection ∇, described in terms of local potentials $\omega = (\omega_\alpha)$, it is straightforward to see that the compatibility condition of the connection with the Hermitian inner product ϱ is satisfied, such that $(\mathcal{L}, \nabla, \varrho)$ is a unitary ray, if and only if:

$$\omega + \overline{\omega} = \tilde{d}^0(\varrho), \tag{0.4.118}$$

where $\tilde{d}^0$ denotes the logarithmic universal derivation of the observable abelian group sheaf $\tilde{\mathcal{A}}$ to the abelian group sheaf of 1-forms Ω^1, that is the universal morphism of abelian group sheaves:

$$\tilde{d}^0 : \tilde{\mathcal{A}} \to \Omega^1 \tag{0.4.119}$$

defined by the relation:

$$\tilde{d}^0(u) := u^{-1} \cdot d^0(u) \tag{0.4.120}$$

for any continuous invertible local section $u \in \tilde{\mathcal{A}}(U)$, with U open set in X. Given the validity of the Poincaré Lemma, we also have:

$$Ker(\tilde{d}^0) = \tilde{\mathbb{C}}. \tag{0.4.121}$$

Third, by considering the Hermitian line sheaf $(\mathcal{L}, \varrho)$ and a local frame of $\mathcal{L}$, we may apply locally the Gram-Schmidt *orthonormalization* procedure in our context, such that $\mathcal{L}$ has locally an *orthonormal frame*. Thus, with respect to this orthonormal frame of $\mathcal{L}$, we obtain:

$$\tilde{d}^0(\varrho) = 0, \tag{0.4.122}$$

and finally we deduce that $\omega = -\overline{\omega}$.

Consequently, given that every Hermitian line sheaf $(\mathcal{L}, \varrho)$ of states admits a Hermitian connection, we have specified completely the notion of a quantum unitary ray, denoted by $(\mathcal{L}, \nabla, \varrho)$, according to the above. In order to simplify further the notation, we denote a quantum unitary ray, viz. a Hermitian differential line sheaf of states, by $(\mathcal{L}, \nabla_\varrho)$. The next important task is to establish the local form of a quantum unitary ray with respect to an open covering of X, and inversely determine the conditions that local components have to satisfy so that they constitute a quantum unitary ray. This is a corollary of the theorem formulated in 0.4.5.

Theorem 0.11. *i. The local form of a unitary ray is given by:*

$$(\mathcal{L}, \nabla) \leftrightarrow (g_{\alpha\beta}, \omega_\alpha) \in Z^1(\mathcal{U}, \tilde{\mathcal{A}}) \times C^0(\mathcal{U}, \Omega^1), \tag{0.4.123}$$

where $|g_{\alpha\beta}| = 1$.

ii. An arbitrary pair $(g_{\alpha\beta}, \omega_\alpha) \in Z^1(\mathcal{U}, \tilde{\mathcal{A}}) \times C^0(\mathcal{U}, \Omega^1)$ *determines a unitary ray if*

$$\delta^0(\omega_\alpha) = \tilde{d}^0(g_{\alpha\beta}), \tag{0.4.124}$$

$$\omega + \overline{\omega} = \tilde{d}^0(\varrho). \tag{0.4.125}$$

A Čech coordinate 1-cocycle $g_{\alpha\beta}$ of a unitary ray consists of local sections of $\mathcal{SU}(1, \mathcal{A})$, viz. the special unitary group sheaf of $\mathcal{A}$ of order 1. This is simply a coordinate 1-cocycle $g_{\alpha\beta}$ in $Z^1(\mathcal{U}, \tilde{\mathcal{A}})$, such that the unitarity condition $|g_{\alpha\beta}| = 1$ is satisfied. Clearly, in the case that a coordinate 1-cocycle $g_{\alpha\beta}$ is constant, we have $g_{\alpha\beta}$ in $Z^1(\mathcal{U}, U(1))$, or equivalently $g_{\alpha\beta}$ in $Z^1(\mathcal{U}, \mathbb{S}^1)$. Moreover, given a Hermitian line sheaf $(\mathcal{L}, \varrho)$, which is also equipped with a connection ∇, described in terms of local potentials $\omega = (\omega_\alpha) \in C^0(\mathcal{U}, \Omega^1)$, the compatibility condition of the connection with the Hermitian inner product ϱ is satisfied if and only if $\omega + \overline{\omega} = \tilde{d}^0(\varrho)$.

0.4.9 *Gauge Equivalence of Quantum Unitary Rays*

We remind that if we consider two line sheaves which are equivalent via an isomorphism $h : \mathcal{L} \xrightarrow{\cong} \mathcal{L}'$, such that their corresponding connections are conjugate under the action of h:

$$\nabla' = h \nabla h^{-1}, \tag{0.4.126}$$

then the corresponding differential line sheaves $(\mathcal{L}, \nabla)$ and $(\mathcal{L}', \nabla')$ are *gauge or frame equivalent*, where the set of *gauge equivalence classes* $[(\mathcal{L}, \nabla)]$ of differential line sheaves is denoted by $Iso(\mathcal{L}, \nabla)$. Further, the following proposition holds:

Theorem 0.12. *The set of gauge equivalence classes of unitary rays $Iso(\mathcal{L}, \nabla_\varrho)$ is an abelian subgroup of the set of gauge equivalence classes of differential line sheaves $Iso(\mathcal{L}, \nabla)$, which in turn, is an abelian subgroup of the abelian group $Iso(\mathcal{L})$.*

This is an immediate corollary of Theorem 0.7. More precisely, since a unitary ray is a differential line sheaf such that $|g_{\alpha\beta}| = 1$ the set of gauge equivalence classes of quantum unitary rays $Iso(\mathcal{L}, \nabla_\varrho)$ is an abelian subgroup of the set of gauge equivalence classes of differential line sheaves $Iso(\mathcal{L}, \nabla)$.

Next, we need to specify the local conditions under which pairs of the form $(g_{\alpha\beta}, \omega_\alpha)$ and $(g'_{\alpha\beta}, \omega'_\alpha)$ determine gauge equivalent differential line sheaves, and concomitantly gauge equivalent quantum unitary rays.

Theorem 0.13. *For any two gauge equivalent differential line sheaves $(\mathcal{L}, \nabla)$ and $(\mathcal{L}', \nabla')$ in $Iso(\mathcal{L}, \nabla)$, the following holds: $(\mathcal{L}, \nabla)$ is equivalent to $(\mathcal{L}', \nabla')$, meaning that $h : \mathcal{L} \xrightarrow{\cong} \mathcal{L}'$ and $\nabla' = h \nabla h^{-1}$, if and only if there exists a 0-cochain $(t_a) \in C^0(\mathcal{U}, \tilde{\mathcal{A}}) = \prod_\alpha \tilde{\mathcal{A}}(U_\alpha)$, such that:*

$$g'_{\alpha\beta} \cdot g^{-1}_{\alpha\beta} = \Delta^0(t_a^{-1}), \tag{0.4.127}$$

$$\omega'_\alpha - \omega_\alpha = \tilde{d}^0(t_a^{-1}), \tag{0.4.128}$$

where by definition the coordinate 1-cocycles of the line sheaves $\mathcal{L}$ and $\mathcal{L}'$ associated with the common open covering $\mathcal{U}$ are given by $g_{\alpha\beta} = \phi_\alpha \circ {\phi_\beta}^{-1} \in Z^1(\mathcal{U}, \tilde{\mathcal{A}})$ and $g'_{\alpha\beta} = \psi_\alpha \circ {\psi_\beta}^{-1} \in Z^1(\mathcal{U}, \tilde{\mathcal{A}})$, according to the following diagram:

$$\begin{tikzcd}
\mathcal{L}|_{U_\alpha} \arrow[r, "h_\alpha"] \arrow[d, "\phi_\alpha"'] & \mathcal{L}'|_{U_\alpha} \arrow[d, "\psi_\alpha"] \\
\mathcal{A}|_{U_\alpha} \arrow[r, "t_\alpha"] & \mathcal{A}|_{U_\alpha}
\end{tikzcd}$$

It is clear that an 1-cocycle $g'_{\alpha\beta}$ is equivalent to another 1-cocycle $g_{\alpha\beta}$ in $Z^1(\mathcal{U},\tilde{\mathcal{A}})$ if and only if there exists a 0-cochain (t_α) in the set $C^0(\mathcal{U},\tilde{\mathcal{A}})$, such that:

$$g'_{\alpha\beta} = t_\alpha \cdot g_{\alpha\beta} \cdot t_\beta^{-1}, \tag{0.4.129}$$

that is $g'_{\alpha\beta}$ is similar to $g_{\alpha\beta}$ under conjugation with (t_α). This is equivalently written as an action of the image of the coboundary operator, that is of the abelian group of 1-coboundaries in $B^1(\mathcal{U},\tilde{\mathcal{A}})$ on the abelian group of 1-cocycles in $Z^1(\mathcal{U},\tilde{\mathcal{A}})$:

$$\Delta^0 : C^0(\mathcal{U},\tilde{\mathcal{A}}) \to C^1(\mathcal{U},\tilde{\mathcal{A}}), \tag{0.4.130}$$

$$g'_{\alpha\beta} = \Delta^0(t_\alpha^{-1}) \cdot g_{\alpha\beta}, \tag{0.4.131}$$

$$g'_{\alpha\beta} \cdot g_{\alpha\beta}^{-1} = \Delta^0(t_\alpha^{-1}). \tag{0.4.132}$$

Thus, the hypothesis of an isomorphism $h : \mathcal{L} \xrightarrow{\cong} \mathcal{L}'$ is equivalent to the existence of a 0-cochain $(t_a) \in C^0(\mathcal{U},\tilde{\mathcal{A}}) = \prod_\alpha \tilde{\mathcal{A}}(U_\alpha)$ such that the above condition is satisfied. At a further stage, the gauge equivalence of the differential line sheaves $(\mathcal{L},\nabla)$ and $(\mathcal{L}',\nabla')$ induced by $h \leftrightarrow (t_\alpha)$ implies that $\nabla' = h\nabla h^{-1}$. Hence, by using the local expressions of the connections ∇' and ∇ by means of the local 1-forms ω'_α and ω_α correspondingly, we immediately obtain:

$$\omega'_\alpha = t_\alpha \omega_\alpha t_\alpha^{-1} + t_\alpha d^0(t_\alpha^{-1}), \tag{0.4.133}$$

$$\omega'_\alpha = \omega_\alpha + \tilde{d}^0(t_\alpha^{-1}). \tag{0.4.134}$$

Thus, we have shown that:

$$\omega'_\alpha - \omega_\alpha = \tilde{d}^0(t_a^{-1}). \tag{0.4.135}$$

0.4.10 *Spectral Beams and Polarization Symmetry*

The global 2-form R, which belongs to $Ker(d^2) : \Omega^2 \to \Omega^3$, called Ω^2_c, being a $\mathbb{C}$-vector sheaf subspace of Ω^2, determines a global differential invariant of gauge equivalent quantum unitary rays. This the case because the global 2-form R determines a 2-dimensional de Rham cohomology class $[R]$, identified as a 2-dimensional complex Čech cohomology class in $H^2(X,\mathbb{C})$. The Hermitian connection of a quantum unitary ray provides the means to express this global differential invariant locally, whereas the latter is actually independent of the connection utilized to represent it locally.

From the *curvature recognition integrality theorem*, we have concluded that the abelian group $Iso(\mathcal{L},\nabla)$ is partitioned into orbits over $\Omega^2_{c,\mathbb{Z}}$, where each orbit (fiber) is labeled by an integral global closed 2-form R of $\Omega^2_{c,\mathbb{Z}}$, providing the differential invariant $[R]$ of this orbit in de Rham cohomology:

$$Iso(\mathcal{L},\nabla) = \sum_{R\in\Omega^2_{c,\mathbb{Z}}} Iso(\mathcal{L},\nabla)_R. \tag{0.4.136}$$

If we restrict the abelian group $Iso(\mathcal{L}, \nabla)$ to gauge equivalent quantum unitary rays we obtain an abelian subgroup of the former, denoted by $Iso(\mathcal{L}, \nabla_\varrho)$. Clearly, the latter abelian group is also partitioned into orbits over $\Omega^2_{c,\mathbb{Z}}$, where each *orbit* is labeled by an *integral global closed* 2-form R of $\Omega^2_{c,\mathbb{Z}}$, where R is the curvature of the corresponding gauge equivalence class of quantum unitary rays.

We define each Hermitian differential line sheaf of states $(\mathcal{L}, \nabla_\varrho)$ belonging to an orbit $Iso(\mathcal{L}, \nabla_\varrho)_R$ a *quantum unitary R-ray*, whence the orbit itself is called a *spectral $[R]$-beam* and is characterized by the integral differential invariant $[R]$.

Each spectral $[R]$-beam consists of gauge equivalent quantum unitary R-rays, which are *indistinguishable* from the perspective of their common curvature integral differential invariant $[R]$. A natural question arising in the context of gauge equivalent quantum unitary R-rays is how they are related to each other. In other words, although all gauge equivalent unitary R-rays cannot be distinguished from the perspective of their curvature differential invariant, is there any other intrinsic way that we can distinguish among them? From a quantum physical perspective, if such an intrinsic and invariant way of distinguishing among gauge equivalent unitary R-rays exists, it means that there exists a *global kinematical symmetry* of spectral $[R]$-beams.

Theorem 0.14. *There exists a free group action of the abelian group $H^1(X, \mathbb{S}^1)$ on the abelian group of unitary rays $Iso(\mathcal{L}, \nabla_\varrho)$, where $\mathbb{S}^1$ denotes the abelian group sheaf $\mathbb{S}^1 \equiv \mathcal{U}(1) \equiv \mathcal{S}U(1, \mathbb{C})$, which is restricted to a free group action on each spectral $[R]$-beam.*

There exists a free group action of $\mathbb{S}^1 \hookrightarrow \mathbb{C} \hookrightarrow \mathcal{A}$ on the group sheaf $\tilde{\mathcal{A}}$ of invertible elements of $\mathcal{A}$:

$$\mathbb{S}^1 \times \tilde{\mathcal{A}}(U) \to \tilde{\mathcal{A}}(U), \tag{0.4.137}$$

$$(\xi, f) \mapsto \xi \cdot f. \tag{0.4.138}$$

with $\xi \in \mathbb{S}^1$ and $f \in \tilde{\mathcal{A}}(U)$ for any open U in X. This action is transferred naturally as a free action to the corresponding groups of coordinate 1-cocycles of the respective abelian group sheaves:

$$Z^1(\mathcal{U}, \mathbb{S}^1) \times Z^1(\mathcal{U}, \tilde{\mathcal{A}}) \to Z^1(\mathcal{U}, \tilde{\mathcal{A}}), \tag{0.4.139}$$

$$(\xi_{\alpha\beta}) \cdot (g_{\alpha\beta}) = (\xi_{\alpha\beta} \cdot g_{\alpha\beta}), \tag{0.4.140}$$

where $(\xi_{\alpha\beta}) \in Z^1(\mathcal{U}, \mathbb{S}^1)$, $(g_{\alpha\beta}) \in Z^1(\mathcal{U}, \tilde{\mathcal{A}})$. This free action can be also extended to the corresponding cohomology groups being still a free action:

$$H^1(X, \mathbb{S}^1) \otimes H^1(X, \tilde{\mathcal{A}}) \to H^1(X, \tilde{\mathcal{A}}), \tag{0.4.141}$$

$$[(\xi_{\alpha\beta})] \cdot [(g_{\alpha\beta})] = [(\xi_{\alpha\beta} \cdot g_{\alpha\beta})], \tag{0.4.142}$$

where $[(\xi_{\alpha\beta})] \in H^1(X, \mathbb{S}^1)$, $[(g_{\alpha\beta})] \in H^1(X, \tilde{\mathcal{A}}) \cong Iso(\mathcal{L})$. Now, we can finally define a group action of $H^1(X, \mathbb{S}^1)$ on the abelian group $Iso(\mathcal{L}, \nabla)$ as follows: We consider $\xi \equiv [(\xi_{\alpha\beta})] \in H^1(X, \mathbb{S}^1)$, $[(\mathcal{L}, \nabla)] \in Iso(\mathcal{L}, \nabla)$, and we define the sought group action as follows:

$$\xi \cdot [(\mathcal{L}, \nabla)] := [(\xi \cdot \mathcal{L}, \nabla)] \equiv [(\mathcal{L}', \nabla)], \tag{0.4.143}$$

$$\mathcal{L}' = \xi \cdot \mathcal{L} \leftrightarrow (\xi_{\alpha\beta}) \cdot (g_{\alpha\beta}) = (\xi_{\alpha\beta} \cdot g_{\alpha\beta}). \tag{0.4.144}$$

It is immediate to show that the pair $(\xi_{\alpha\beta} \cdot g_{\alpha\beta}, \omega_\alpha)$ actually satisfies the 'transformation law of local potentials', that is:

$$\delta(\omega_\alpha) = \tilde{d}^0(\xi_{\alpha\beta} \cdot g_{\alpha\beta}). \tag{0.4.145}$$

Now, given that:

$$Ker(\tilde{d}^0) = \tilde{\mathbb{C}}, \tag{0.4.146}$$

as a consequence of the Poincaré Lemma, we can easily show that the above defined group action of $H^1(X, \mathbb{S}^1)$ on the abelian group $Iso(\mathcal{L}, \nabla)$ is actually free.

For this purpose, we consider the equivalent differential line sheaves $(\mathcal{L}, \nabla)$ and $(\mathcal{L}', \nabla) = \xi \cdot [(\mathcal{L}, \nabla)] = [(\xi \cdot \mathcal{L}, \nabla)]$ in the abelian group $Iso(\mathcal{L}, \nabla)$, where $\xi := [(\xi_{\alpha\beta})] \in H^1(X, \mathbb{S}^1)$, and thus $(\xi_{\alpha\beta}) \in Z^1(\mathcal{U}, \mathbb{S}^1)$, according to the above. Due to the above equivalence, we conclude that there exists a 0-cochain $(t_a) \in C^0(\mathcal{U}, \tilde{\mathcal{A}}) = \prod_\alpha \tilde{\mathcal{A}}(U_\alpha)$, such that:

$$\xi_{\alpha\beta} = \Delta^0(t_\alpha^{-1}), \tag{0.4.147}$$

and hence, equivalently:

$$[(\xi_{\alpha\beta})] = 1 \in H^1(X, \tilde{\mathcal{A}}), \tag{0.4.148}$$

where $(\xi_{\alpha\beta}) \in Z^1(\mathcal{U}, \mathbb{S}^1) \hookrightarrow (\xi_{\alpha\beta}) \in Z^1(\mathcal{U}, \tilde{\mathcal{A}})$. Furthermore, by hypothesis we have:

$$\omega'_\alpha = \omega_\alpha, \tag{0.4.149}$$

and thus:

$$\tilde{d}^0(t_a^{-1}) = -\tilde{d}^0(t_a) = 0, \tag{0.4.150}$$

such that $t_a \in Ker(\tilde{d}^0) = \tilde{\mathbb{C}}$. Thus, we obtain that an automorphism h of a line sheaf $\mathcal{L}$ is also an automorphism of the differential line sheaf $(\mathcal{L}, \nabla)$ if and only if $t_a \in Ker(\tilde{d}^0) = \tilde{\mathbb{C}}$. Hence, in the unitary case considered, we deduce that the 0-cochain $(t_a) \in C^0(\mathcal{U}, \tilde{\mathcal{A}})$ actually belongs to $C^0(\mathcal{U}, \mathbb{S}^1)$. Therefore, we conclude that:

$$[(\xi_{\alpha\beta})] = 1 \in H^1(X, \mathbb{S}^1), \tag{0.4.151}$$

which proves the freeness of the above action.

We note that, if we do not assume the unitarity condition, the same argument shows that the group action of $H^1(X, \tilde{\mathbb{C}})$, induced by the natural injection $\tilde{\mathbb{C}} \hookrightarrow \tilde{\mathcal{A}}$ on the abelian group $Iso(\mathcal{L}, \nabla)$ is actually free, where in this more general case we have that $[(\xi_{\alpha\beta})] = 1 \in H^1(X, \tilde{\mathbb{C}})$, where $(\xi_{\alpha\beta}) \in Z^1(\mathcal{U}, \tilde{\mathbb{C}})$.

Consequently, the free group action of $H^1(X, \mathbb{S}^1)$ on $Iso(\mathcal{L}, \nabla)$ is *restricted* to a free group action on its abelian subgroup of quantum unitary rays $Iso(\mathcal{L}, \nabla_\varrho)$. Since the latter abelian group is partitioned into spectral $[R]$-beams over $\Omega^2_{c,\mathbb{Z}}$, we conclude that the above free group action is finally transferred as a *free group action* of $H^1(X, \mathbb{S}^1)$ on each spectral $[R]$-beam.

A cohomology class in the abelian group $H^1(X, \mathbb{S}^1)$ is called a *polarization phase germ* of a spectral $[R]$-beam.

The intuition behind the above definition is that a cohomology class in the abelian group $H^1(X, \mathbb{S}^1) \cong H^1(X, U(1))$ can be evaluated at a homology cycle $\gamma \in H_1(X)$ by means of the pairing:

$$H_1(X) \times H^1(X, U(1)) \to U(1) \tag{0.4.152}$$

to obtain a global observable gauge-invariant phase factor in $U(1)$. Thus, gauge equivalent unitary R-rays may be intrinsically distinguished by means of a polarization phase germ, identified as a cohomology class in $H^1(X, \mathbb{S}^1)$.

If we assume that the underlying space X of control variables is a *locally path-connected* space, then we obtain the following:

Theorem 0.15. *A polarization phase germ of a spectral $[R]$-beam is realized by a representation of the fundamental group of the topological space X of control variables to $\mathbb{S}^1$.*

As an immediate consequence of the Hurewicz isomorphism we obtain:

$$Hom(\pi_1(X), \tilde{\mathbb{C}}) \cong H^1(X, \tilde{\mathbb{C}}). \tag{0.4.153}$$

By restriction to the unitary case we obtain:

$$Hom(\pi_1(X), \mathbb{S}^1) \cong H^1(X, \mathbb{S}^1). \tag{0.4.154}$$

Thus, a polarization phase germ expressed in terms of a cohomology class in $H^1(X, \mathbb{S}^1)$ is realized by a *representation of the fundamental group* of the topological space X to $\mathbb{S}^1$.

0.4.11 *Affine Structure of Spectral Beams*

We have shown that there exists a free group action of $H^1(X, \mathbb{S}^1) \cong Hom(\pi_1(X), \mathbb{S}^1)$ on each spectral $[R]$-beam. A natural problem in this context is the specification of the appropriate conditions, which would qualify this free action as a *transitive* one as well.

For this purpose, we consider the following sequence of abelian group sheaves:

$$1 \to \tilde{\mathbb{C}} \xrightarrow{\epsilon} \tilde{\mathcal{A}} \xrightarrow{\tilde{d}^0} \Omega^1 \xrightarrow{d^1} d^1\Omega^1 \to 0. \tag{0.4.155}$$

As a consequence of the Poincaré Lemma we have that:

$$Ker(\tilde{d}^0) = \tilde{\mathbb{C}}. \tag{0.4.156}$$

Moreover, we have that $d^1 \circ \tilde{d^0} = 0$, and thus $Im(\tilde{d^0}) \subseteq Ker(d^1)$. Now, we consider those closed 1-forms θ_α of Ω^1, for which the following condition is satisfied:

$$Im(\tilde{d^0}) = Ker(d^1). \tag{0.4.157}$$

The closed 1-forms θ_α of Ω^1, satisfying $Im(\tilde{d^0}) = Ker(d^1)$ are called *logarithmically exact closed 1-forms.*

Theorem 0.16. *The following sequence of abelian group sheaves is an exact sequence if restricted to logarithmically exact closed 1-forms:.*

$$1 \to \tilde{\mathbb{C}} \xrightarrow{\epsilon} \tilde{\mathcal{A}} \xrightarrow{\tilde{d^0}} \Omega^1 \xrightarrow{d^1} d^1\Omega^1 \to 0. \tag{0.4.158}$$

Theorem 0.17. *A spectral $[R]$-beam is an affine space with structure group the characters of the fundamental group with respect to logarithmically exact closed 1-forms.*

By the exact sequence of abelian group sheaves of the previous proposition, we obtain a 0-cochain (θ_α) of logarithmically exact closed 1-forms:

$$(\theta_\alpha) \in C^0(\mathcal{U}, Ker(d^1)) = (\theta_\alpha) \in C^0(\mathcal{U}, Im(\tilde{d^0})) = \tilde{d^0}(C^0(\mathcal{U}, \tilde{\mathcal{A}})). \tag{0.4.159}$$

Hence, for a 0-cochain (θ_α) of logarithmically exact closed 1-forms θ_α, there exists a 0-cochain t_α in $\tilde{\mathcal{A}}$, such that:

$$\theta_\alpha = \tilde{d^0}(t_\alpha^{-1}). \tag{0.4.160}$$

This 0-cochain (θ_α) may be considered as representing an integrable connection $\breve{\nabla}$ on a line sheaf $\mathbf{K}$, whose coordinate 1-cocycle with respect to an open covering $\mathcal{U}$ is given by:

$$\zeta_{\alpha\beta} = t_\beta^{-1} t_\alpha. \tag{0.4.161}$$

Clearly, in this case the transformation law of local potentials is satisfied as follows:

$$\delta^0(\theta_\alpha) = \theta_\beta - \theta_\alpha = \tilde{d^0}(t_\beta^{-1}) - \tilde{d^0}(t_\alpha^{-1}) = \tilde{d^0}(t_\beta^{-1} t_\alpha) = \tilde{d^0}(\zeta_{\alpha\beta}). \tag{0.4.162}$$

Next, we consider a spectral $[R]$-beam, viz. an orbit $Iso(\mathcal{L}, \nabla_\varrho)_R$, consisting of gauge equivalent unitary R-rays, which are indistinguishable from the perspective of their common differential invariant $[R]$. Again, we also consider those closed 1-forms θ_α of Ω^1, for which the following condition is satisfied:

$$Im(\tilde{d^0}) = Ker(d^1), \tag{0.4.163}$$

that is the logarithmically exact closed 1-forms. We take a pair of equivalent unitary R-rays, denoted by $(\mathcal{L}, \nabla_\varrho)$, $(\mathcal{L}', \nabla'_\varrho)$ correspondingly. Then, we have that:

$$R = (d\omega_\alpha) = (d\omega'_\alpha), \tag{0.4.164}$$

$$d(\omega_\alpha - \omega'_\alpha) = 0. \tag{0.4.165}$$

We conclude that $(\omega_\alpha - \omega'_\alpha)$ is of the form θ_α of Ω^1, viz. it is a logarithmically exact closed 1-form. This means that we obtain a 0-cochain $(\omega_\alpha - \omega'_\alpha)$ of logarithmically exact closed 1-forms:

(0.4.166)

$$(\omega_\alpha - \omega'_\alpha) \in C^0(\mathcal{U}, Ker(d^1)) = (\omega_\alpha - \omega'_\alpha) \in C^0(\mathcal{U}, Im(\tilde{d}^0)) = \tilde{d}^0(C^0(\mathcal{U}, \tilde{\mathcal{A}})).$$

Hence, for a 0-cochain $(\omega_\alpha - \omega'_\alpha)$ of logarithmically exact closed 1-forms, there exists a 0-cochain t_α in $\tilde{\mathcal{A}}$, such that:

(0.4.167) $$\omega_\alpha - \omega'_\alpha = \tilde{d}^0(t_\alpha^{-1}),$$

(0.4.168) $$\omega_\alpha = \omega'_\alpha + \tilde{d}^0(t_\alpha^{-1}).$$

Furthermore, we may consider another unitary R-ray in the same orbit, denoted by $(\mathcal{L}'', \nabla''_\varrho)$ characterized locally as follows:

(0.4.169) $$g''_{\alpha\beta} := \zeta_{\alpha\beta} \cdot g'_{\alpha\beta} = t_\alpha \cdot g'_{\alpha\beta} \cdot t_\beta^{-1},$$

(0.4.170) $$\omega''_\alpha = \omega_\alpha = \omega'_\alpha + \tilde{d}^0(t_\alpha^{-1}),$$

such that:

(0.4.171) $$\tilde{d}^0(g''_{\alpha\beta}) = \delta^0(\omega''_\alpha).$$

Thus, if we consider any two unitary R-rays $(\mathcal{L}, \nabla_\varrho)$, $(\mathcal{L}', \nabla'_\varrho)$, we can always find another unitary R-ray $(\mathcal{L}'', \nabla''_\varrho = \nabla_\varrho)$ equivalent to both of them, viz. belonging to the *same orbit* over their common differential invariant $[R]$, or equivalently belonging to the same spectral $[R]$-beam. Therefore, for any two unitary R-rays $(\mathcal{L}, \nabla_\varrho)$, $(\mathcal{L}', \nabla'_\varrho)$, we can always *substitute*, for instance, the second one of them $(\mathcal{L}', \nabla'_\varrho)$ with an equivalent unitary R-ray characterized by the same 0-cochain of potentials like the first one, viz. by $(\mathcal{L}'', \nabla_\varrho)$ whose local form is given by the above defined pair $(g''_{\alpha\beta}, \omega_\alpha)$. Consequently, we obtain:

(0.4.172) $$\tilde{d}^0(g''_{\alpha\beta}) = \delta^0(\omega''_\alpha) = \delta^0(\omega_\alpha) = \tilde{d}^0(g_{\alpha\beta}).$$

We conclude that:

(0.4.173) $$\tilde{d}^0(g''_{\alpha\beta}) - \tilde{d}^0(g_{\alpha\beta}) = 0,$$

(0.4.174) $$\tilde{d}^0(g''_{\alpha\beta} \cdot g_{\alpha\beta}^{-1}) = 0.$$

Thus, $(g''_{\alpha\beta} \cdot g_{\alpha\beta}^{-1}) \in Z^1(\mathcal{U}, Ker(\tilde{d}^0))$, or equivalently by the Poincaré Lemma:

(0.4.175) $$(g''_{\alpha\beta} \cdot g_{\alpha\beta}^{-1}) \in Z^1(\mathcal{U}, \tilde{\mathbb{C}}),$$

which in the quantum unitary case considered reduces to:

(0.4.176) $$(g''_{\alpha\beta} \cdot g_{\alpha\beta}^{-1}) \in Z^1(\mathcal{U}, \mathbb{S}^1).$$

The above, can be equivalently formulated as follows:

(0.4.177) $$g''_{\alpha\beta} = \xi_{\alpha\beta} \cdot g_{\alpha\beta},$$

where $\xi_{\alpha\beta} \in Z^1(\mathcal{U}, \mathbb{S}^1)$. By considering the corresponding cohomology classes we obtain:

$$[g''_{\alpha\beta}] = [(\xi_{\alpha\beta})] \cdot [(g_{\alpha\beta})] = [(\xi_{\alpha\beta} \cdot g_{\alpha\beta})], \tag{0.4.178}$$

where $[(\xi_{\alpha\beta})] \in H^1(X, \mathbb{S}^1)$, $[(g_{\alpha\beta})] \in H^1(X, \tilde{\mathcal{A}}) \cong Iso(\mathcal{L})$. Thus, we finally deduce that:

$$[(\mathcal{L}', \nabla'_{\varrho})] = [(\mathcal{L}'', \nabla_{\varrho})] = [(\xi \cdot \mathcal{L}, \nabla)] = [(\xi)] \cdot [(\mathcal{L}, \nabla)], \tag{0.4.179}$$

where $[(\xi)] = [(\xi_{\alpha\beta})] \in H^1(X, \mathbb{S}^1)$. Therefore, we finally arrive at the following conclusion:

The free group action of $H^1(X, \mathbb{S}^1)$ on a spectral $[R]$-beam is also transitive with respect to logarithmically exact closed 1-forms, and therefore a spectral $[R]$-beam becomes a $H^1(X, \mathbb{S}^1) \cong Hom(\pi_1(X), \mathbb{S}^1)$-affine space, or equivalently an affine space with structure group the characters of the fundamental group.

Thus, each partition block or fiber $Iso(\mathcal{L}, \nabla_{\varrho})_R$ labeled by the curvature differential invariant $[R]$, viz. each spectral $[R]$-beam is an *affine space* with structure group $H^1(X, \mathbb{S}^1) \cong Hom(\pi_1(X), \mathbb{S}^1)$. In this manner, any two quantum unitary R-rays differ by an element of $H^1(X, \mathbb{S}^1)$, and conversely any two quantum unitary rays which differ by an element of $H^1(X, \mathbb{S}^1)$ are characterized by the same differential invariant $[R]$, or equivalently are R-rays of the same spectral $[R]$-beam. Hence, although all gauge equivalent quantum unitary R-rays cannot be distinguished from the perspective of their common curvature differential invariant, there exists a *free and transitive action* of the group $H^1(X, \mathbb{S}^1) \cong Hom(\pi_1(X), \mathbb{S}^1)$, characterized as the global kinematical symmetry group of a $[R]$-beam, which completely distinguishes among them by means of characters of the fundamental group of X. Inversely, from any one quantum unitary R-ray we can obtain intrinsically its whole equivalence class by means of the free and transitive action of the abelian group $H^1(X, \mathbb{S}^1)$ on the depicted one. We conclude that whenever two unitary rays are characterized by the same differential invariant $[R]$, viz. they belong to the same equivalence class (orbit) under the action of $H^1(X, \mathbb{S}^1)$ on $Iso(\mathcal{L}, \nabla_{\varrho})_R$ (which is actually the only class due to transitivity of this action, identified as a spectral $[R]$-beam), then they differ by a character of the fundamental group of X.

0.4.12 *Monodromy Group and Integrable Phase Factors*

We have shown in the previous Section that a polarization phase germ of a spectral $[R]$-beam is realized by a representation of the fundamental group of the topological space X of control variables to $\mathbb{S}^1$. As a consequence, we obtain:

Theorem 0.18. *The global polarization symmetry group of a spectral* $[R]$*-beam is realized in terms of unitary line local systems. Equivalently, there exists a bijective correspondence between the polarization phase germs of a spectral* $[R]$*-beam and isomorphism classes of unitary line local systems.*

This is an immediate corollary of the following equivalences:

$$Z^1(\mathcal{U}, \tilde{\mathbb{C}}) \ni (\xi_{\alpha\beta}) \cong (\mathcal{L}, \breve{\nabla}) \cong Ker(\breve{\nabla})_{\mathcal{L}} \cong \Lambda, \tag{0.4.180}$$

$$H^1(X, \tilde{\mathbb{C}}) \cong [(\mathcal{L}, \breve{\nabla})] \cong [Ker(\breve{\nabla})_{\mathcal{L}}] \cong [\Lambda]. \tag{0.4.181}$$

The above equivalences restrict as follows in the case of Hermitian integrable differential line sheaves, or equivalently integrable unitary rays:

$$Z^1(\mathcal{U}, \mathbb{S}^1) \ni (\xi_{\alpha\beta}) \cong (\mathcal{L}, \breve{\nabla}_{\varrho}) \cong Ker(\breve{\nabla}_{\varrho})_{\mathcal{L}} \cong \Lambda_{\varrho}, \tag{0.4.182}$$

$$H^1(X, \mathbb{S}^1) \cong [(\mathcal{L}, \breve{\nabla}_{\varrho})] \cong [Ker(\breve{\nabla}_{\varrho})_{\mathcal{L}}] \cong [\Lambda_{\varrho}]. \tag{0.4.183}$$

A natural question in this setting refers to the *realization* of the polarization symmetry group of a spectral $[R]$-beam at a point of the base space of control variables X.

Theorem 0.19. *At each point of the topological space X, the polarization symmetry group of a spectral $[R]$-beam is realized via the monodromy group of a unitary line local system at the depicted point, or equivalently by the group of monodromies of the corresponding integrable unitary ray whose covariantly constant sections form this unitary line local system.*

We already know that a polarization phase germ of a spectral $[R]$-beam is realized by a representation of the fundamental group of X to $\mathbb{S}^1$. Moreover, $\tilde{\mathbb{C}}$ is identified locally with the group of automorphisms of a line local system, $GL(1, \mathbb{C}) \cong \tilde{\mathbb{C}}$. Thus, in the unitary case, $\mathbb{S}^1$ is identified locally with the group of automorphisms of a unitary line local system Λ_{ϱ}, which may be thought of as the locally constant sheaf of covariantly constant sections of a corresponding Hermitian integrable line sheaf (integrable unitary ray) $(\mathcal{L}, \breve{\nabla}_{\varrho})$.

For clarity, we may fix a base point x_0 of X, and consider a path $\gamma : [0, 1] \to X$, such that $\gamma(0) = x_0$, $\gamma(1) = x_1$. Thus, if Λ is a line local system on X, then it is pulled back to $[0, 1]$ as a constant sheaf, denoted by $\gamma^\star(\Lambda)$. Hence, we have that: $(\gamma^\star(\Lambda))_0 \cong (\gamma^\star(\Lambda))_1$. Therefore, due to the isomorphisms $(\gamma^\star(\Lambda))_0 \cong \Lambda_{\gamma(0)} := \Lambda_{x_0}$ and $(\gamma^\star(\Lambda))_1 \cong \Lambda_{\gamma(1)} := \Lambda_{x_1}$, we obtain a $\mathbb{C}$-vector space isomorphism $\Lambda_{x_0} \cong \Lambda_{x_1}$, which depends only on the homotopy class of γ. Furthermore, this isomorphism may be thought of as being induced by the parallel transport condition of the corresponding integrable connection of the Hermitian integrable line sheaf $(\mathcal{L}, \breve{\nabla}_{\varrho})$. Now, if we consider a loop based at the point x_0 of X, we obtain an abelian group homomorphism:

$$\mu : \pi_1(X, x_0) \to GL(\Lambda_{x_0}) \cong (GL(1, \mathbb{C}))_{x_0} \cong \tilde{\mathbb{C}}_{x_0}. \tag{0.4.184}$$

In the case of a unitary line local system, we obtain the abelian group homomorphism:

$$\mu : \pi_1(X, x_0) \to {\mathbb{S}^1}_{x_0}. \tag{0.4.185}$$

The image of μ in ${\mathbb{S}^1}_{x_0}$ is the *monodromy group of the unitary local system* Λ_{ϱ}. Thus, for each *homotopy class* of loops based at x_0, we obtain an *integrable phase factor*, identified with the *monodromy* of the unitary local system $\Lambda_{\varrho_{x_0}}$. Equivalently, this is the same as the monodromy of the corresponding integrable quantum unitary ray $(\mathcal{L}, \breve{\nabla}_{\varrho})$, acquired by *parallel transport* along a loop based at x_0 and belonging to a homotopy class in $\pi_1(X, x_0)$.

0.4.13 *Aharonov-Bohm Effect*

A concrete observable manifestation of global integrable phase factors in quantum theory is provided by the *Aharonov-Bohm effect*, see Aharonov-Bohm [1, 2]. This effect demonstrates the significance of the local electromagnetic potentials of a *zero curvature spectral beam*. More precisely, the global relative phase factor observed due to the Aharonov-Bohm effect is of a *topological* origin and expressed via an integrable connection. This is the case because the existence of a very long solenoid restricting the magnetic field flux within its borders and making the region it occupies inaccessible to a charged particle, makes the topology of the base localization space *multiple-connected* having the homotopical symmetry of a circle. The evolving states (wave functions), depicted as implicitly parameterized sections of a fibration by a time parameter, are transported by an integrable connection because the propagation takes place in a field-free region. The global phase factor measures the monodromy of the integrable connection due to the topological obstruction imposed to the motion by the inaccessible region enclosed by the boundary of the solenoid, where there exists the flux of the magnetic field.

The Aharonov-Bohm effect constitutes a perfect example of demonstrating the nature and significance of global observable topological phase factors in quantum theory. In particular, the two most important points clarified by the analysis of this effect, pertaining to the whole issue of global quantum topological phases are the following: [1] The *local gauge freedom* of the phase notion, which necessitates the consideration of sheaf-theoretic models of parametric dependence over non-trivial topological spaces; [2] The *mutually implicative* roles of the local and global levels depicted differentially, viz. by means of a connection as follows: (i) An extensive integration process of the contributions of all local gauge potentials to monodromies from the local to the global, and (ii) inversely as a differential localization process of some global topological invariant in terms of a multiplicity of local gauge potentials expressing the allowed contextual variability of the connection with respect to this invariant.

Before we examine in detail this effect from our perspective we formulate the following:

Theorem 0.20. *The abelian group of quantum unitary rays $Iso(\mathcal{L}, \nabla_{\varrho})$ is a central extension of the abelian group of integral global closed 2-forms R of $\Omega^2_{c,\mathbb{Z}}$ by the abelian group of polarization phase germs $H^1(X, \mathbb{S}^1)$.*

The proof of the proposition is straightforward by the exactness of the following sequence of abelian groups:

$$1 \to H^1(X, \mathbb{S}^1) \xrightarrow{\sigma} Iso(\mathcal{L}, \nabla_\varrho) \xrightarrow{\kappa} \Omega^2_{c,\mathbb{Z}} \to 0. \tag{0.4.186}$$

We restrict our focus on spectral $[R]$-beams. Each spectral $[R]$-beam $Iso(\mathcal{L}, \nabla_\varrho)_R$ is an affine space with structure group $H^1(X, \mathbb{S}^1)$ and from Dirac's quantization condition we know that $(2\pi i)^{-1}[R]$ is a 2-dimensional integral cohomology class of X.

Theorem 0.21. *A zero curvature spectral $[R]$-beam $Iso(\mathcal{L}, \nabla_\varrho)_0$ is isomorphic to the abelian group of polarization phase germs $H^1(X, \mathbb{S}^1)$.*

We have that $Iso(\mathcal{L}, \nabla_\varrho)_R = \kappa^{-1}(R)$, where $R \in \Omega^2_{c,\mathbb{Z}}$. Hence, by the fact that $H^1(X, \mathbb{S}^1) \hookrightarrow Iso(\mathcal{L}, \nabla_\varrho)$ and the previous proposition we deduce that:

$$\kappa^{-1}(0) = Iso(\mathcal{L}, \nabla_\varrho)_0 \cong H^1(X, \mathbb{S}^1). \tag{0.4.187}$$

Thus, we obtain immediately the following corollary:

Theorem 0.22. *A spectral $[R]$-beam is an $Iso(\mathcal{L}, \nabla_\varrho)_0$-torsor with respect to logarithmically exact closed 1-forms.*

In physical terminology, an *Aharonov-Bohm phase factor* refers to the realization of a zero curvature spectral $[R]$-beam $Iso(\mathcal{L}, \nabla_\varrho)_0$. By this we mean that an Aharonov-Bohm phase is the experimentally realized global gauge-invariant phase characteristic of a *gauge equivalence class* of integrable Hermitian differential line sheaves, or equivalently integrable quantum unitary rays. From the above, we have that:

$$Iso(\mathcal{L}, \nabla_\varrho)_0 \cong H^1(X, \mathbb{S}^1) \cong Hom(\pi_1(X), \mathbb{S}^1). \tag{0.4.188}$$

Thus, for each point x_0 of the topological space X, we have:

$$\mu : \pi_1(X, x_0) \to {\mathbb{S}^1}_{x_0} \cong U(1). \tag{0.4.189}$$

Therefore, an Aharonov-Bohm phase factor is realized as the global monodromy group element $\mu(\gamma) \in U(1)$ for each homotopy class of loops γ based at x_0. In the original formulation, the topological space X is homotopically contractible to the circle, and hence its second integer cohomology is trivial. So we have a gauge equivalence class of integrable Hermitian differential line sheaves, or equivalently a gauge equivalence class of line local systems on the circle, which form a zero curvature spectral beam. The global gauge-invariant phase factors via which this beam is realized is the monodromy group of the beam, which is identified with the image of the fundamental group of the circle, viz. the integers $\mathbb{Z}$ into $U(1)$. The monodromy depends only on the *winding number* and is observed as a shift in the *interference pattern* of the beam.

We may equivalently provide a simple cohomological argument, which explains the realization of Aharonov-Bohm type of phases for non-simply connected X, which for simplicity here assume to be a manifold. This argument stresses again the fact that the Aharonov-Bohm type of phases is the global gauge-invariant observable factor of a zero curvature beam, viz. of a gauge equivalence class of zero curvature quantum unitary rays. For this purpose, we use the fact that a zero curvature beam is isomorphic to the abelian group $H^1(X, U(1))$. Thus, in classical differential geometric terms may be expressed as follows:

$$[\Psi(\theta_\alpha, -)] = exp\left(\frac{ie}{\hbar c}\oint_{[-]}\theta_\alpha\right), \tag{0.4.190}$$

where we have inserted the corresponding physical units, and we consider θ_α real-valued in the Lie algebra of $U(1)$. Notice that $[\Psi(\theta_\alpha, -)]$ is an element of $H^1(X, U(1))$, which is evaluated at a homology cycle $\gamma \in H_1(X)$ by means of the pairing:

$$H_1(X) \times H^1(X, U(1)) \to U(1), \tag{0.4.191}$$

$$(\gamma, [\Psi(\theta_\alpha, -)]) \mapsto [\Psi(\theta_\alpha, \gamma)] = exp\left(\frac{ie}{\hbar c}\oint_{\gamma}\theta_\alpha\right), \tag{0.4.192}$$

where $[\Psi(\theta_\alpha, \gamma)]$ is identified as a global Aharonov-Bohm observable gauge-invariant phase factor of the beam in $U(1)$. Notice that if we consider any unitary ray of this beam, then we only obtain a real-valued phase, defined by:

$$\Psi(\theta_\alpha, \gamma) = \frac{e}{\hbar c}\oint_{\gamma}\theta_\alpha. \tag{0.4.193}$$

Due to the isomorphism of groups $\mathbb{R}/\mathbb{Z} \cong U(1)$, we have to take the quotient of the set of all $\Psi((\theta_\alpha)_i, \gamma)$ for all unitary rays by the equivalence relation: $\Psi((\theta_\alpha)_1, \gamma) \sim \Psi((\theta_\alpha)_2, \gamma)$ if $((\theta_\alpha)_1 - (\theta_\alpha)_2) \in \mathbb{Z}$. In physical terms, this means that the interference phase patterns of quantum unitary rays differing by an integer *cannot be distinguished* experimentally, and thus the physically meaningful global gauge-invariant information is only the Aharonov-Bohm phase factors of the form $[\Psi(\theta_\alpha, \gamma)] = exp(\frac{ie}{\hbar c}\oint_\gamma \theta_\alpha)$ referring to the *global realization* of the beam, viz. to the global realization of the whole gauge equivalence class of quantum unitary rays.

0.4.14 *Holonomy of Spectral Beams*

In the original *Berry-Simon* formulation of the notion of a global quantum *geometric phase factor* acquired by a quantum system after completion of an *adiabatic cyclic evolution*, the parametric localization of a quantum state from a base space is induced by the functional dependence of the Hamiltonian of the system by a set of *control variables*, see Berry [1], Simon [1], Wilczek-Shapere [1]. Thus, the dynamical evolution is driven via the *implicit temporal dependence* of the Hamiltonian through the control variables. Under the assumption that the set of these variables forms a smooth manifold the time dependence is depicted by means of differentiable paths on this space. The cyclic evolution signifies the *periodic property* of a quantum state with respect to the control variables, whence the adiabatic hypothesis is equivalent to the specification of a parallel transport condition on the evolution of normalized state vectors, which in general is considered to be *path-dependent*. The fibers of the induced *spectral line bundle* stand for the *eigenspaces* of the Hamiltonian operator. The adiabatic rule gives rise to a non-integrable connection, which in turn defines a covariant derivative operator on the sections of the corresponding Hermitian line sheaf, that is the eigenstates of the Hamiltonian. We note that an eigenstate of the Hamiltonian is required to remain in the eigenspace of the same instantaneous eigenvalue during the adiabatic evolution. The non-dynamical (non-Hamiltonian dependent) global phase assembled during a cyclic evolution along a closed path on the base space is thought of as *"memory"* of the evolution, since it encodes the global geometric features of the space of control variables in the algebraic (group-theoretic) form of the *holonomy* of the connection.

The observable global phase factor is of a geometric origin because it depends *solely* on the geometry of the base space pathway along which the state is transported. If the eigenvalues of the Hamiltonian are degenerate or close to each other, then the adiabatic transportation constraint is not realistic and is substituted by another appropriate connection depending on the context considered. Moreover, the gauge freedom of a state vector localized at a fiber over an eigenspace of the Hamiltonian is not an one-dimensional complex phase any more but an n-dimensional complex matrix of phases, called a non-abelian complex phase. Thus, even the adiabatic transportation condition is *not necessary* for the experimental detection of global phase factors. This has been also demonstrated by the line bundle formulation of the complex Hilbert space of states over the complex projective Hilbert space. In this way, the one-dimensional projection operators play the role of control variables. This line bundle is endowed with a natural connection obtained by differentiation of the inner product of normalized sections (quantum state vectors) of the bundle and the adiabatic hypothesis is not involved at all. Then, the global phase factor is obtained as the holonomy transformation of this connection with respect to a closed path on the complex projective space.

We note that most of the discussions about geometric phase factors in quantum mechanics are expressed in the language of *holonomies*. From the abstract perspective of this work, all observable geometric phase factors are consequences of the curvature of spectral $[R]$-beams and in particular of the fact that $(2\pi i)^{-1}[R]$ is a 2-dimensional integral cohomology class of X. As we have already explained, global observable topological phase factors can be completely understood in terms of monodromies of zero curvature beams. Again it needs to be emphasized that observable holonomies refer to spectral $[R]$-beams and *not* to individual unitary R-rays. For simplicity, we restrict our discussion to the case that X is a paracompact smooth manifold.

We notice that for any real valued form ϕ of degree k we may define:

$$\hat{\phi}(\eta) := \Psi(\phi, \eta) = \left(\int_{\eta} \phi \right) + \mathbb{Z}, \tag{0.4.194}$$

where η is a k-chain of X and $\hat{\phi}$ is a k-cochain of X with values in $\mathbb{R}/\mathbb{Z}$. Now, we consider the group morphism:

$$\Xi : Z_k(X) \to \mathbb{R}/\mathbb{Z}, \tag{0.4.195}$$

such that there exists a $k+1$-form τ, which satisfies $\Xi \circ \partial = \hat{\tau}$, or equivalently:

$$\Xi(\partial \zeta) = \left(\int_{\zeta} \tau \right) + \mathbb{Z} = \hat{\tau}(\zeta) \in \mathbb{R}/\mathbb{Z}, \tag{0.4.196}$$

for any smooth map $\zeta : \Delta_{k+1} \to X$. We note that Δ_n stands for the standard n-dimensional *simplex* and the space of n-chains is generated by Δ_n. We also notice that the following holds:

$$d(\hat{\tau}) = \widehat{d\tau} = \Xi \circ \partial^2 = 0, \tag{0.4.197}$$

and therefore τ is a closed $k+1$-form.

Next, we consider a unitary R-ray $(\mathcal{L}, \nabla_{\varrho})$ and use the fact that R is an integral global closed 2-form R of $\Omega^2_{c,\mathbb{Z}}$. Similarly to the definition of gauge potentials in the case of Hermitian differential line sheaves, we consider R as a *purely imaginary* closed 2-form, such that $R = i \cdot \Theta$. Thus, according to the above, and since Θ is an integral global real-valued form of degree 2, we may consider the following group morphism, which we call *holonomy morphism*:

$$\mathbb{H} : Z_1(X) \to \mathbb{R}/\mathbb{Z}, \tag{0.4.198}$$

$$\mathbb{H}(\partial S) = \left(\int_{S} \Theta \right) + \mathbb{Z} = \hat{\Theta}(S) \in \mathbb{R}/\mathbb{Z}, \tag{0.4.199}$$

for any smooth map $S : \Delta_2 \to X$. Equivalently, we have:

$$\mathbb{H} \circ \partial = \hat{\Theta}. \tag{0.4.200}$$

We notice that for a fixed curvature R of $\Omega^2_{c,\mathbb{Z}}$, the same holonomy morphism $\mathbb{H}$ is defined for any other unitary R-ray. Thus, it provides a characterization of the whole gauge equivalence class of unitary R-rays classified by the differential invariant $[R]$. Equivalently stated, it provides a characterization of a spectral $[R]$-beam, and therefore we obtain a *holonomy cohomology class* in $H^1(X, U(1))$, defined by:

$$Hol(\partial S) = exp\left(i \int_{S} \Theta \right). \tag{0.4.201}$$

From the above, it is straightforward to conclude that given the *Chern morphism*:

$$\delta_c : H^1(X, U(1)) \to H^2(X, \mathbb{Z}), \tag{0.4.202}$$

and the natural morphism:

$$\iota : H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{R}), \tag{0.4.203}$$

the holonomy cohomology class in $H^1(X, U(1))$ of a spectral beam for fixed Θ, is in the inverse image of the Chern characteristic class:

$$c_1 = -\frac{1}{2\pi i}[\Theta] \in H^2(X, \mathbb{Z}), \tag{0.4.204}$$

under δ_c, or equivalently in the inverse image of the cohomology class $[\Theta]$ in $H^2(X, \mathbb{R})$, such that:

$$(\iota \circ \delta_c)(Hol) = [\Theta]. \tag{0.4.205}$$

Thus, we conclude that an $\mathbb{R}/\mathbb{Z} \cong U(1)$-observable holonomy morphism value, which formalizes the notion of a *non-integrable geometric phase factor*, is a global observable gauge-invariant characteristic of a spectral beam, that is it characterizes the whole gauge equivalence class of quantum unitary rays having the same curvature, such that in cohomology the above relation is satisfied.

0.5 The Functorial Imperative

0.5.1 *Representable Functors and Natural Transformations*

The definition of a category of algebraic objects of some kind with arrows being homomorphisms between them, constitutes an abstraction of the behavior of functions closed under the associative operation of *composition*. More precisely, the notion of a function is generalized to the notion of a homomorphism, whereas the associative operation of composition becomes an operation on sets of homomorphisms between algebraic objects of the same kind satisfying the same properties that functions and compositions satisfy. Notice that the composition of two functions f, g, denoted as $f \circ g$ is defined only in case that the codomain of g is the domain of f. Moreover, the composition of a function f with the identity of either its domain or codomain gives f again.

In a nutshell, the notion of a *category* of algebraic objects of some kind, for instance observables or events, is a conception based on the behavior of functions *closed under the associative operation of composition*, abstracted in terms of homomorphisms, which in turn, have been idealized as algebraic "generalized elements" (or their —*duals*) determining completely the algebraic objects themselves. Because there exists an obvious *duality* between incoming and outgoing homomorphisms with respect to an algebraic object, this is built into the definition of a category so that the operation of *arrows reversal* leaves invariant the concept of a category, meaning that this operation gives again a category being in dual or opposite relation in comparison to the given one. A consequence of this fact is that all categorical constructions come in dual pairs corresponding to the dual viewpoints of considering incoming or outgoing arrows with respect to a constituted algebraic object, see Mac Lane [1, 3].

Now, the establishment of the notion of a category of algebraic objects naturally rises the problem of defining the notion of a function whose domain and codomain are categories. Obviously such a function should *preserve* the composition operation binding a category as an associatively closed universe of discourse. This is precisely the notion of *functor* between categories. A *covariant functor* is a functor which preserves the directionality of an arrow in the domain category, whereas a *contravariant functor* is a functor which reverses it.

Each object A of a category $\mathcal{A}$ determines a covariant functor of homomorphisms emanating from A, denoted by $\curvearrowright_A : \mathcal{A} \to \mathcal{S}ets$, called the covariant $Hom_{\mathcal{A}}$-functor *represented* by A, and defined as follows:

[1]. For all objects X in $\mathcal{A}$, $\curvearrowright_A(X) := Hom_{\mathcal{A}}(A, X)$.

[2]. For all homomorphisms $f : X \to Y$ in $\mathcal{A}$,

$$\curvearrowright_A(f) : Hom_{\mathcal{A}}(A, X) \to Hom_{\mathcal{A}}(A, Y) \tag{0.5.1}$$

is defined as post-composition with f, viz., $\curvearrowright_A(f)(g) := f \circ g$.

The covariant representable functor $\curvearrowright_A : \mathcal{A} \to \mathcal{S}ets$, can be thought of as constructing an image of $\mathcal{A}$ in the category of $\mathcal{S}ets$ in a covariant way.

Now, let us consider the *opposite category* $\mathcal{A}^{op}$, and let A be an object in this category. The object A determines a contravariant functor of homomorphisms targeting A. Then, the contravariant $Hom_{\mathcal{A}}$-functor *represented* by A is the contravariant functor $\curvearrowright^A : \mathcal{A}^{op} \to \mathcal{S}ets$, defined as follows:

[1]. For all objects B in $\mathcal{A}^{op}$, $\curvearrowright^A(B) := Hom_{\mathcal{A}^{op}}(B, A)$.

[2]. For all homomorphisms $f : C \to B$ in $\mathcal{A}^{op}$,

$$\curvearrowright^A(f) : Hom_{\mathcal{A}^{op}}(B, A) \to Hom_{\mathcal{A}^{op}}(C, A) \tag{0.5.2}$$

is defined as pre-composition with f, viz., $\curvearrowright^A(f)(g) := g \circ f$.

The contravariant $Hom_{\mathcal{A}}$-functor represented by A, viz. $\curvearrowright^A : \mathcal{A}^{op} \to \mathcal{S}ets$, is called the *functor of generalized elements* (incoming homomorphisms) of A. Moreover, the information contained in A is *classified* completely by its functor of generalized elements $\curvearrowright^A$. Dually the covariant $Hom_{\mathcal{A}}$-functor represented by A, viz. $\curvearrowright_A : \mathcal{A} \to \mathcal{S}ets$ is called the *functor of generalized co-elements* (outgoing homomorphisms) of A. Similarly, the information contained in A is classified completely by its functor of generalized co-elements $\curvearrowright_A$.

Now, given a small category $\mathcal{A}$, viz. a category such that for all objects B, A, the Hom-class (class of homomorphisms) $Hom_{\mathcal{A}}(B, A)$ is a set, we may consider the $Hom_{\mathcal{A}}$-*bifunctor*, see Mac Lane [1]:

$$Hom_{\mathcal{A}} := \curvearrowright_B^A = \mathcal{A}^{op} \times \mathcal{A} \to \mathcal{S}ets \tag{0.5.3}$$

from the *product category* $\mathcal{A}^{op} \times \mathcal{A}$ to the category of sets, such that for objects B, A of $\mathcal{A}$, $\curvearrowright_B : \mathcal{A} \to \mathcal{S}ets$ is the covariant representable functor (represented by B), and $\curvearrowright^A : \mathcal{A}^{op} \to \mathcal{S}ets$ is the contravariant representable functor (represented by A).

Continuing in the same frame of thought, the next question arising is the following: Which is the proper notion of morphism which captures the notion of a *transformation from some functor to another functor* having both the same domain and the same codomain categories? Defining the proper notion of morphism between such functors is important because it would give us the possibility to consider legitimately the notion of a *functor category* $\mathcal{C}^{\mathcal{A}}$, where the algebraic objects would be functors $\mathbb{F} : \mathcal{A} \to \mathcal{C}$ and the morphisms would be the sought transformations between such functors. The leading idea has to do with the requirement that a transformation of the sought form should *compare two functorial processes* having the same domain and the same codomain in a way that is *not dependent* on the specific objects and arrows involved, that is it should relate the processes themselves without the intervention of ad hoc choices. This is precisely the notion required for the formalization of the concept of *naturality* referring to the relation or comparison of two functorial processes sharing the same source and the same target categories. Concomitantly the corresponding notion of morphism between functors of the above form is captured in the notion of a *natural transformation*.

More precisely, if $\mathbb{F}$, $\mathbb{G}$, are functors from the category $\mathcal{A}$ to the category $\mathcal{C}$, a *natural transformation* τ from $\mathbb{F}$ to $\mathbb{G}$ is a function assigning to each object A in $\mathcal{A}$ a morphism τ_A from $\mathbb{F}(A)$ to $\mathbb{G}(A)$ in $\mathcal{C}$, such that for every arrow $f : A \to B$ in $\mathcal{A}$ the following diagram in $\mathcal{C}$ commutes:

$$\begin{tikzcd}
\mathbb{F}(A) \arrow[r, "\tau_A"] \arrow[d, "\mathbb{F}(f)"'] & \mathbb{G}(A) \arrow[d, "\mathbb{G}(f)"] \\
\mathbb{F}(B) \arrow[r, "\tau_B"] & \mathbb{G}(B)
\end{tikzcd}$$

That is, for every arrow $f : A \to B$ in $\mathcal{A}$ we have:

$$\mathbb{G}(f) \circ \tau_A = \tau_B \circ \mathbb{F}(f). \tag{0.5.4}$$

A natural transformation $\tau : \mathbb{F} \to \mathbb{G}$ is called a *natural isomorphism (or natural equivalence)* if every component τ_A is *invertible*. Obviously, a natural isomorphism is an invertible natural transformation in the functor category $\mathcal{C}^{\mathcal{A}}$.

This is a key concept and a posteriori justifies the whole categorical framework, since it captures precisely the *criterion of naturality* referring to the comparison of any two functorial processes of the same kind.

0.5.2 *Adjoint Functors: Universals and Equivalence*

The fundamental fact about representable functors is that an object A in $\mathcal{A}$ is *entirely retrievable up to equivalence* (canonical isomorphism) from knowledge of the representable functor $\curvearrowright^A : \mathcal{A}^{op} \to \mathcal{S}ets$ (or equivalently from $\curvearrowright_A : \mathcal{A} \to \mathcal{S}ets$).

This fact, a consequence of a categorical proposition known as the *Yoneda Lemma*, see Mac Lane-Moerdijk [1], can be expressed this way:

Theorem 0.23. *Let A, B be objects in a category $\mathcal{A}$. Suppose we are given an isomorphism of their associated functors: $\curvearrowright^{A} \cong \curvearrowright^{B}$. Then there exists a* unique isomorphism *of the objects themselves, that is $A \cong B$, which gives rise to this isomorphism of functors.*

The conceptual significance of this result is that it bears the following relational interpretation:

An object A of a category $\mathcal{A}$ is *determined uniquely up to equivalence* by the *network of all internal relations* that the object A has with all the other objects in $\mathcal{A}$.

A natural further important question arising in this categorical setting is the following:

Is it possible for an object of a category to be determined uniquely up to equivalence by networks of relations among *partially congruent objects* targeting it that belong to another category functorially correlated with the former one?

It turns out that from a physical viewpoint this is the most *significant function* of the *whole categorical imperative*. This kind of *heteronymous determination* becomes possible in a functorial way if the network of these relations in the determining category becomes appropriately *internalized* in the former one, such that the determination factors uniquely via *a universal*, which exemplifies this determination in a paradigmatic or archetypal form. In technical terms, there should exist a *pair of adjoint functors* between these categories, which should be interpreted as *conceptual inverses* to each other, associated with a bidirectional process of *encoding/decoding* information associated with their structural form. It is instructive to explain this process by starting from the notion of *representation* of an arbitrary $\mathcal{S}ets$-valued functor as follows:

A *representation of a contravariant $\mathcal{S}ets$-valued functor* of the form $\mathbb{X} : \mathcal{A}^{op} \to \mathcal{S}ets$, where $\mathcal{A}$ is a (locally) small category consists of an object K in $\mathcal{A}$ and a natural isomorphism:

$$\mathbb{X} \cong Hom_{\mathcal{A}^{op}}(-, K), \tag{0.5.5}$$

or equivalently:

$$\mathbb{X} \cong \curvearrowright^{K}, \tag{0.5.6}$$

where K is called the *representing object* of the functor $\mathbb{X} : \mathcal{A}^{op} \to \mathcal{S}ets$. Thus, $\mathbb{X}$ is representable, if and only if such a representing object exists. Notice that, representations of contravariant $\mathcal{S}ets$-valued functors are unique up to a unique isomorphism. Evidently, there exists a dual formulation referring to a representation of a covariant $\mathcal{S}ets$-valued functor of the form $\mathbb{Z} : \mathcal{A} \to \mathcal{S}ets$. Namely, the covariant functor $\mathbb{Z}$ is representable if and only if there exists an object N in $\mathcal{A}$ and a natural isomorphism:

$$\mathbb{Z} \cong Hom_{\mathcal{A}}(N, -), \tag{0.5.7}$$

or equivalently:

$$\mathbb{Z} \cong \curvearrowright_N . \tag{0.5.8}$$

By analogy, we may consider the general case of a *bifunctor*:

$$\bigwedge(-,-) : \mathcal{C}^{op} \times \mathcal{L} \to \mathcal{S}ets \tag{0.5.9}$$

from the product category $\mathcal{C}^{op} \times \mathcal{L}$ to the category of sets. Using the conceptual approach of birepresentability, see Ellerman [1], we formulate the following proposition:

Theorem 0.24. *The bifunctor* $\bigwedge(-,-) : \mathcal{C}^{op} \times \mathcal{L} \to \mathcal{S}ets$ *is* birepresentable*, viz. it is representable within both the categories* $\mathcal{C}^{op}$*,* $\mathcal{L}$*, if and only if there exist two functors pointing into opposite directions, that is,* $\overrightarrow{\mathbb{A}} : \mathcal{C}^{op} \to \mathcal{L}$ *and* $\overleftarrow{\mathbb{A}} : \mathcal{L} \to \mathcal{C}^{op}$ *and a series of isomorphisms:*

$$Hom_{\mathcal{L}}(\overrightarrow{\mathbb{A}}(C), L) \cong \bigwedge(C, L) \cong Hom_{\mathcal{C}^{op}}(C, \overleftarrow{\mathbb{A}}(L)), \tag{0.5.10}$$

which is natural in both the arguments C *in* $\mathcal{C}^{op}$ *and* L *in* $\mathcal{L}$*.*

Equivalently, we state that the bifunctor: $\bigwedge(-,-) : \mathcal{C}^{op} \times \mathcal{L} \to \mathcal{S}ets$ is birepresentable, if and only if there exist two functors $\overrightarrow{\mathbb{A}} : \mathcal{C}^{op} \to \mathcal{L}$, $\overleftarrow{\mathbb{A}} : \mathcal{L} \to \mathcal{C}^{op}$ together with *natural isomorphisms of bifunctors*:

$$Hom_{\mathcal{L}}(\overrightarrow{\mathbb{A}}(-), -) \cong \bigwedge(-,-) \cong Hom_{\mathcal{C}^{op}}(-, \overleftarrow{\mathbb{A}}(-)). \tag{0.5.11}$$

In this case, we say that the two oppositely pointing functors $\overrightarrow{\mathbb{A}}$, $\overleftarrow{\mathbb{A}}$ form a *categorical adjunction*, induced by the requirement of birepresentability of the bifunctor: $\bigwedge(-,-) : \mathcal{C}^{op} \times \mathcal{L} \to \mathcal{S}ets$, where $\overrightarrow{\mathbb{A}} : \mathcal{C}^{op} \to \mathcal{L}$ is the *left adjoint functor* of the adjunction, and symmetrically, $\overleftarrow{\mathbb{A}} : \mathcal{L} \to \mathcal{C}^{op}$ is the *right adjoint functor* of the adjunction, denoted usually by $\mathbb{L}$ and $\mathbb{R}$ respectively. By ignoring the middle term we obtain the natural isomorphism of the Hom-bifunctors:

$$\Psi : Hom_{\mathcal{L}}(\overrightarrow{\mathbb{A}}(-), -) \cong Hom_{\mathcal{C}^{op}}(-, \overleftarrow{\mathbb{A}}(-)). \tag{0.5.12}$$

Equivalently, we obtain the isomorphism:

$$\Psi_{C,L} : Hom_{\mathcal{L}}(\overrightarrow{\mathbb{A}}(C), L) \cong Hom_{\mathcal{C}^{op}}(C, \overleftarrow{\mathbb{A}}(L)), \tag{0.5.13}$$

which is natural in both the arguments C in $\mathcal{C}^{op}$ and L in $\mathcal{L}$.

The requirement of birepresentability of a bifunctor of the form: $\bigwedge(-,-) : \mathcal{C}^{op} \times \mathcal{L} \to \mathcal{S}ets$, satisfied by the existence of a categorical adjunction, consisting of the functors $\overrightarrow{\mathbb{A}} : \mathcal{C}^{op} \to \mathcal{L}$ (left adjoint functor of the adjunction), and $\overleftarrow{\mathbb{A}} : \mathcal{L} \to \mathcal{C}^{op}$ (right adjoint functor of the adjunction), is tantamount to the establishment of a bidirectional functorial process of *information encoding/decoding* between the categories $\mathcal{L}$, $\mathcal{C}^{op}$. It is important to notice the directionality build in the process of information circulation by means of functorial encoding/decoding, viz. it is understood as an information-encoding of an object of $\mathcal{L}$ by an object of $\mathcal{C}^{op}$ via the functor $\overleftarrow{\mathbb{A}} : \mathcal{L} \to \mathcal{C}^{op}$, and inversely, as an information-decoding from an object of $\mathcal{C}^{op}$ back to an object of $\mathcal{L}$ via the functor $\overrightarrow{\mathbb{A}} : \mathcal{C}^{op} \rightarrow \mathcal{L}$, such that the structural relations of the respective categories are being preserved.

It is also important to emphasize that the bidirectional process of information encoding/decoding between the categories $\mathcal{L}$, $\mathcal{C}^{op}$ should be *functorial*. This is due to the fact that the appropriate information-encoders/decoders between objects are only those that preserve their structures; in this case, the objects, arrows, domains, codomains, compositions, and identities of the respective categories. Consequently, an information-encoding of an object of $\mathcal{L}$ by an object of $\mathcal{C}^{op}$ takes place via the functor $\overleftarrow{\mathbb{A}} : \mathcal{L} \rightarrow \mathcal{C}^{op}$, and inversely, an information-decoding from an object of $\mathcal{C}^{op}$ back to an object of $\mathcal{L}$ takes place via the functor $\overrightarrow{\mathbb{A}} : \mathcal{C}^{op} \rightarrow \mathcal{L}$. The functorial character of this bidirectional process of information encoding/decoding means precisely that the structural relations of the respective categories are being preserved.

The morphisms in $\mathcal{L}$, $\mathcal{C}^{op}$ related to each other by the above isomorphism are called *adjoint transposes*. Hence, if we consider a morphism:

$$h : \overrightarrow{\mathbb{A}}(C) \rightarrow L \tag{0.5.14}$$

in $\mathcal{L}$, then by virtue of the adjunction isomorphism, it has an adjoint transpose morphism in $\mathcal{C}^{op}$, namely:

$$h^* : C \rightarrow \overleftarrow{\mathbb{A}}(L). \tag{0.5.15}$$

Dually, if we consider a morphism:

$$g : C \rightarrow \overleftarrow{\mathbb{A}}(L) \tag{0.5.16}$$

in $\mathcal{C}^{op}$, then by virtue of the adjunction isomorphism, it has an adjoint transpose morphism in $\mathcal{L}$, namely:

$$g^* : \overrightarrow{\mathbb{A}}(C) \rightarrow L. \tag{0.5.17}$$

Now, we may consider the *identity* morphism at $\overrightarrow{\mathbb{A}}(C)$ in $\mathcal{L}$, that is $id_{\overrightarrow{\mathbb{A}}(C)}$ belonging to the set $Hom_{\mathcal{L}}(\overrightarrow{\mathbb{A}}(C), \overrightarrow{\mathbb{A}}(C))$. The adjoint transpose of $id_{\overrightarrow{\mathbb{A}}(C)}$ is called the *unit* morphism at C, that is:

$$(id_{\overrightarrow{\mathbb{A}}(C)})^* = \eta_C : C \rightarrow \overleftarrow{\mathbb{A}}(\overrightarrow{\mathbb{A}}(C)) \tag{0.5.18}$$

belonging to the set $Hom_{\mathcal{C}^{op}}(C, \overleftarrow{\mathbb{A}}(\overrightarrow{\mathbb{A}}(C)))$. Since the above is natural on the argument C in $\mathcal{C}^{op}$, we obtain a *natural transformation of the identity functor* on $\mathcal{C}^{op}$, via *self-referential information circulation* realized by functorially decoding and then encoding information, called the *unit natural transformation* of the adjunction:

$$\eta : id_{\mathcal{C}^{op}} \rightarrow \overleftarrow{\mathbb{A}}\overrightarrow{\mathbb{A}}. \tag{0.5.19}$$

Dually, we may consider the identity morphism at $\overleftarrow{\mathbb{A}}(L)$ in $\mathcal{C}^{op}$, that is $id_{\overleftarrow{\mathbb{A}}(L)}$ belonging to the set $Hom_{\mathcal{C}^{op}}(\overleftarrow{\mathbb{A}}(L), \overleftarrow{\mathbb{A}}(L))$. The adjoint transpose of $id_{\overleftarrow{\mathbb{A}}(L)}$ is called the *counit* morphism at L, that is:

$$(id_{\overleftarrow{\mathbb{A}}(L)})^* = \varepsilon_L : \overrightarrow{\mathbb{A}}(\overleftarrow{\mathbb{A}}(L)) \rightarrow L \tag{0.5.20}$$

belonging to the set $Hom_{\mathcal{L}}(\overrightarrow{\mathbb{A}}(\overleftarrow{\mathbb{A}}(L)), L)$. Since the above is natural on the argument L in $\mathcal{L}$, we obtain a natural transformation of the identity functor on $\mathcal{L}$, via self-referential information circulation realized by functorially encoding and then decoding information, called the *counit natural transformation* of the adjunction:

$$\varepsilon : \overrightarrow{\mathbb{A}}\overleftarrow{\mathbb{A}} \to id_{\mathcal{L}}. \tag{0.5.21}$$

The categorical adjunction being formed by $\overrightarrow{\mathbb{A}}$, $\overleftarrow{\mathbb{A}}$ can be equivalently represented in terms of the unit and counit natural transformations.

Furthermore, we notice that if we consider a morphism $h : \overrightarrow{\mathbb{A}}(C) \to L$ in $\mathcal{L}$, then by virtue of the unit natural transformation, its adjoint transpose morphism in $\mathcal{C}^{op}$ *factors uniquely* via the unit morphism at C as follows:

$$h^* = \overleftarrow{\mathbb{A}}(h) \circ \eta_C : C \to \overleftarrow{\mathbb{A}}(\overrightarrow{\mathbb{A}}(C)) \to \overleftarrow{\mathbb{A}}(L). \tag{0.5.22}$$

Since the above holds for every object L in $\mathcal{L}$, the pair $(\overrightarrow{\mathbb{A}}(C), \eta_C)$ constitutes a *universal* for the functor:

$$Hom_{\mathcal{C}^{op}}(C, \overleftarrow{\mathbb{A}}(-)) = \curvearrowright_C \circ \overleftarrow{\mathbb{A}} : \mathcal{L} \to \mathcal{S}ets. \tag{0.5.23}$$

Thus, for every object C in $\mathcal{C}^{op}$, the above composite encoding $\mathcal{S}ets$-valued functor becomes representable by the decoding of C in $\mathcal{L}$, viz. by the object $\overrightarrow{\mathbb{A}}(C)$ in $\mathcal{L}$:

$$(\curvearrowright_C \circ \overleftarrow{\mathbb{A}}) \cong Hom_{\mathcal{L}}(\overrightarrow{\mathbb{A}}(C), -). \tag{0.5.24}$$

Dually, if we consider a morphism $g : C \to \overleftarrow{\mathbb{A}}(L)$ in $\mathcal{C}^{op}$, then by virtue of the counit natural transformation, its adjoint transpose morphism in $\mathcal{L}$ factors uniquely via the counit morphism at L as follows:

$$\varepsilon_L \circ \overrightarrow{\mathbb{A}}(g) : \overrightarrow{\mathbb{A}}(C) \to \overrightarrow{\mathbb{A}}(\overleftarrow{\mathbb{A}}(L)) \to L. \tag{0.5.25}$$

Since the above holds for every object C in $\mathcal{C}^{op}$, the pair $(\overleftarrow{\mathbb{A}}(L), \varepsilon_L)$ constitutes a universal for the functor:

$$Hom_{\mathcal{L}}(\overrightarrow{\mathbb{A}}(-), L) = \curvearrowright^L \circ \overrightarrow{\mathbb{A}} : \mathcal{C}^{op} \to \mathcal{S}ets. \tag{0.5.26}$$

Thus, for every object L in $\mathcal{L}$, the above composite decoding $\mathcal{S}ets$-valued functor becomes representable by the encoding of L in $\mathcal{C}^{op}$, viz. by the object $\overleftarrow{\mathbb{A}}(L)$ in $\mathcal{C}^{op}$:

$$(\curvearrowright^L \circ \overrightarrow{\mathbb{A}}) \cong Hom_{\mathcal{C}^{op}}(-, \overleftarrow{\mathbb{A}}(L)). \tag{0.5.27}$$

Now, if both the unit and the counit natural transformations of the adjunction formed by the functors $\overrightarrow{\mathbb{A}}$, $\overleftarrow{\mathbb{A}}$, are natural isomorphisms then we obtain an *equivalence* of the categories $\mathcal{C}^{op}$, $\mathcal{L}$, that is we obtain a *duality*, $\mathcal{C}^{op} \simeq \mathcal{L}$. Thus, the notion of *functorial duality* is a consequence of the existence of a pair of adjoint functors between two categories when both the unit and counit natural transformations of the formed adjunction are natural isomorphisms.

We conclude that the three most fundamental concepts *epitomizing* the function of the whole categorical framework are the concepts of *naturality, universality and equivalence.* More concretely, category theory provides the precise formal means of identifying a universal in mathematical terms, that is the *archetypic form of instantiation* of some property which exemplify the property in such a paradigmatic way that all other instances have this property by virtue of *factorizing uniquely* through the corresponding universal. In this sense, objects are specified functorially via some universality property and only up to equivalence (canonical isomorphism) in a natural manner. All categorical universals are defined up to canonical isomorphism by the existence of pairs of adjoint functors between appropriate categories. An adjunction between two categories gives rise to a family of universal morphisms, one for each object in the first category and one for each object in the second. In this way, each object in the first category induces a certain property in the second category and the universal morphism caries the object to the universal for that property, that is it transports it to an agent of objectification in the second category exemplifying this property in a paradigmatic way. Most important, every adjunction extends to an *adjoint equivalence of certain subcategories* of the initially correlated categories. As we have explained previously this is described by the notion of *functorial duality*, when both the unit and counit natural transformations of the formed adjunction become natural isomorphisms.

0.5.3 *Probes and Adjoints to Realization Functors*

A pertinent problem that arises frequently is the determination of *not directly accessible* objects of some kind L by means of *controllable probes* Y. The categorical formulation of this problem involves two categories, namely the *small category of probes* $\mathcal{A}$ and the category of the not directly accessible objects $\mathcal{E}$, which is considered to be a *cocomplete* category. Moreover, we assume the existence of a *shaping or coordinatization functor* $\mathbb{M} : \mathcal{A} \to \mathcal{E}$, viz. a functor assigning to each probe Y in $\mathcal{A}$ the underlying object L in $\mathcal{E}$ and analogously for the arrows. In some cases the category of probes $\mathcal{A}$ is a subcategory, or even a generating subcategory of $\mathcal{E}$, such that a complete localization of $\mathcal{E}$ in terms of $\mathcal{A}$ is feasible. In the general case, the aim is to approximate an indirectly accessible object L in $\mathcal{E}$ inductively by means of appropriate structured families of probes Y. The important issue is if there exists a universal solution to this problem, which can be thus obtained in terms of a categorical adjunction.

It is rarely the case that the shaping functor $\mathbb{M} : \mathcal{A} \to \mathcal{E}$ has an adjoint such that an adjunction can be formed in a straightforward manner. For this reason, it is necessary to extend the probes from the categorical level of $\mathcal{A}$ to the categorical level of diagrams in $\mathcal{A}$, such that the initial probes can be embedded in the latter extended category. This process is accomplished by means of the Yoneda embedding $\curvearrowright$: $\mathcal{A} \longrightarrow \mathcal{S}ets^{\mathcal{A}^{op}}$, which constitutes the free completion of $\mathcal{A}$ under taking colimits of diagrams of probes.

The *functor category of presheaves of sets on probes*, denoted by $\mathcal{S}ets^{\mathcal{A}^{op}}$, has objects all functors $\mathbb{P} : \mathcal{A}^{op} \to \mathcal{S}ets$, and morphisms all natural transformations between such functors, where $\mathcal{A}^{op}$ is the opposite category of $\mathcal{A}$. Each object $\mathbb{P}$ in the category of presheaves $\mathcal{S}ets^{\mathcal{A}^{op}}$ is a contravariant set-valued functor on $\mathcal{A}$, called a *presheaf* on $\mathcal{A}$, defined as follows: For each probe Y of $\mathcal{A}$, $\mathbb{P}(\mathrm{Y})$ is a set, and for each arrow $f : C \to Y$, $\mathbb{P}(f) : \mathbb{P}(Y) \to \mathbb{P}(\mathrm{C})$ is a set-theoretic function such that if $p \in \mathbb{P}(\mathrm{Y})$, the value $\mathbb{P}(f)(p)$ for an arrow $f : C \to Y$ in $\mathcal{A}$ is called the restriction of p along f and is denoted by $\mathbb{P}(f)(p) = p \cdot f$.

We notice that each probe Y of $\mathcal{A}$ gives rise to a contravariant Hom-functor $\curvearrowright[Y] := Hom_{\mathcal{A}}(-, Y)$. This functor defines a presheaf on $\mathcal{A}$ for each Y in $\mathcal{A}$. Concomitantly, the functor $\curvearrowright$ is a full and faithful functor from $\mathcal{A}$ to the contravariant functors on $\mathcal{A}$, viz.:

$$\curvearrowright : \mathcal{A} \longrightarrow \mathcal{S}ets^{\mathcal{A}^{op}}, \tag{0.5.28}$$

defining the Yoneda embedding $\mathcal{A} \hookrightarrow \mathcal{S}ets^{\mathcal{A}^{op}}$.

According to the *Yoneda Lemma*, there exists an injective correspondence between elements of the set $\mathbb{P}(Y)$ and natural transformations in $\mathcal{S}ets^{\mathcal{A}^{op}}$ from $\curvearrowright[Y]$ to $\mathbb{P}$ and this correspondence is natural in both $\mathbb{P}$ and Y, for every presheaf of sets $\mathbb{P}$ in $\mathcal{S}ets^{\mathcal{A}^{op}}$ and probe Y in $\mathcal{A}$. The functor category of presheaves of sets on probes $\mathcal{S}ets^{\mathcal{A}^{op}}$ is a complete and cocomplete category. Thus, the *Yoneda embedding* $\curvearrowright : \mathcal{A} \longrightarrow \mathcal{S}ets^{\mathcal{A}^{op}}$ constitutes the sought *free completion of $\mathcal{A}$ under colimits of diagrams of probes.*

The deep meaning of this fact is that if we consider a shaping or coordinatization functor $\mathbb{M} : \mathcal{A} \to \mathcal{E}$ there exists *precisely one* corresponding uniquely defined, up to isomorphism, *colimit-preserving* functor $\widehat{\mathbb{M}} : \mathcal{S}ets^{\mathcal{A}^{op}} \to \mathcal{E}$, such that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{A} \arrow[d, tail, "\curvearrowright"'] \arrow[dr, "\mathbb{M}"] & \\
\mathcal{S}ets^{\mathcal{A}^{op}} \arrow[r, "\widehat{\mathbb{M}}"] & \mathcal{E}
\end{tikzcd}$$

Consequently, every morphism from a probe Y in $\mathcal{A}$ to an indirectly accessible object L in $\mathcal{E}$ factors *uniquely* through the functor category $\mathcal{S}ets^{\mathcal{A}^{op}}$ and the computation of the colimit-preserving functor $\widehat{\mathbb{M}} : \mathcal{S}ets^{\mathcal{A}^{op}} \to \mathcal{E}$ is crucial for understanding the underlying structure of an object L in $\mathcal{E}$ in terms of diagrams of probes.

The physical significance of the functor $\widehat{\mathbb{M}}$ is unraveled by the fact that it plays the role of a left adjoint $\mathbb{L}$, and thus colimit-preserving functor, from $\mathcal{S}ets^{\mathcal{A}^{op}}$ to $\mathcal{E}$. More precisely, the functor $\widehat{\mathbb{M}} := \mathbb{L}$ is the left adjoint of the *categorical adjunction* between the categories $\mathcal{S}ets^{\mathcal{A}^{op}}$ and $\mathcal{E}$, where the right adjoint $\mathbb{R} : \mathcal{E} \to \mathbf{Sets}^{\mathcal{A}^{op}}$, is physically interpreted as the realization functor of $\mathcal{E}$ in terms of probes.

More precisely, the *probes-induced realization functor of $\mathcal{E}$ in $\mathcal{S}ets^{\mathcal{A}^{op}}$* is defined as follows:

$$\mathbb{R} : \mathcal{E} \to \mathcal{S}ets^{\mathcal{A}^{op}}, \tag{0.5.29}$$

where the action on a probe Y in $\mathcal{A}$ is given by:

$$\mathbb{R}(L)(Y) := \mathbb{R}_L(Y) = Hom_{\mathcal{E}}(\mathbb{M}(Y), L). \tag{0.5.30}$$

The functor $\mathbb{R}(L)(-) := \mathbb{R}_L(-) = Hom_{\mathcal{E}}(\mathbb{M}(-), L)$ is called the *functor of probing frames* of L, where $\mathbb{M} : \mathcal{A} \to \mathcal{E}$ is a *shaping or coordinatization functor* of $\mathcal{E}$. Since the physical interpretation of the functor $\mathbb{R}(L)(-)$ refers to the functorial realization of the indirectly accessible object L in $\mathcal{E}$ in terms of probes Y in $\mathcal{A}$, we think of $\mathbb{R}(L)(-)$ as the *spectrum functor* of L through the probes Y.

We notice that the probing frames of L, denoted by $\psi_Y : \mathbb{M}(Y) \to L$, instantiated by the evaluation of the functor $\mathbb{R}(L)(-)$ at each probe Y in $\mathcal{A}$, are not ad hoc but they are inter-related by the operation of presheaf restriction. Explicitly, this means that for each arrow $f : C \to Y$, $\mathbb{R}(L)(f) : \mathbb{R}(L)(Y) \to \mathbb{R}(L)(C)$ is a function between sets of probing frames of L in the opposite direction, such that if $\psi_Y \in \mathbb{R}(L)(Y)$ is a probing frame, the value of $\mathbb{R}(L)(f)(\psi_Y)$, or equivalently the corresponding probing frame $\psi_C : \mathbb{M}(C) \to L$ is given by the restriction or pullback of ψ_Y along f, denoted by $\mathbb{R}(L)(f)(\psi_Y) = \psi_Y \cdot f = \psi_C$.

In this setting, the problem of *approximating* an indirectly accessible object L in $\mathcal{E}$ by means of diagrams of probes Y has a *universal solution*, which is provided by the left adjoint functor $\mathbb{L} : \mathcal{S}ets^{\mathcal{A}^{op}} \to \mathcal{E}$ to the realization functor $\mathbb{R} : \mathcal{E} \to \mathcal{S}ets^{\mathcal{A}^{op}}$. In other words, the existence of the left adjoint functor $\mathbb{L}$ paves the way for an explicit inductive synthesis of an object L in $\mathcal{E}$ by means of appropriate diagrams of probes in a functorial manner.

Theorem 0.25. *There exists a* categorical adjunction *between the categories $\mathcal{S}ets^{\mathcal{A}^{op}}$ and $\mathcal{E}$. More precisely, there exists* a pair of adjoint functors $\mathbb{L} \dashv \mathbb{R}$ *as follows:*

$$\mathbb{L} : \mathcal{S}ets^{\mathcal{A}^{op}} \underset{\longrightarrow}{\longleftarrow} \mathcal{E} : \mathbb{R}. \tag{0.5.31}$$

First, it is crucial to notice that every presheaf $\mathbb{P}$ in $\mathcal{S}ets^{\mathcal{A}^{op}}$ gives rise to a category. *The category of elements of a presheaf* $\mathbb{P}$, denoted by $\int(\mathbb{P}, \mathcal{A})$, has objects all pairs (Y, p), and morphisms $(\acute{Y}, \acute{p}) \to (Y, p)$ are those morphisms $u : \acute{Y} \to Y$ of $\mathcal{A}$ for which $p \cdot u = \acute{p}$, that is the restriction or pullback of p along u is $\acute{p}$. Projection on the second coordinate of $\int(\mathbb{P}, \mathcal{A})$, defines a functor $\int_{\mathbb{P}} : \int(\mathbb{P}, \mathcal{A}) \to \mathcal{Y}$ called the *split discrete uniform fibration* induced by $\mathbb{P}$, where $\mathcal{A}$ is the base category of the fibration as in the diagram below. We note that the fibers are categories in which the only arrows are identity arrows. If Y is a probe in $\mathcal{A}$, the inverse image under $\int_{\mathbb{P}}$ of Y is simply the set $\mathbb{P}$(Y), although its elements are written as pairs so as to form a disjoint union.

$$
\begin{tikzcd}
\int(\mathbb{P},\mathcal{A}) \arrow[d, "\int_{\mathbb{P}}"'] & \\
\mathcal{A} \arrow[r, "\mathbb{P}"] & \mathcal{S}ets
\end{tikzcd}
$$

Second, a natural transformation τ between the presheaves $\mathbb{P}$ and $\mathbb{R}(L)$ on the category of probes $\mathcal{A}$, $\tau : \mathbb{P} \longrightarrow \mathbb{R}(L)$ is a family τ_Y indexed by probes Y of $\mathcal{A}$ for which each τ_Y is a map of sets,

$$\tau_Y : \mathbb{P}(Y) \rightarrow Hom_{\mathcal{E}}(\mathbb{M}(Y), L), \tag{0.5.32}$$

such that the diagram of sets below commutes for each arrow $u : \acute{Y} \rightarrow Y$ of $\mathcal{A}$.

$$
\begin{tikzcd}
\mathbb{P}(Y) \arrow[r, "\tau_Y"] \arrow[d, "\mathbb{P}(u)"'] & Hom_{\mathcal{E}}(\mathbb{M}(Y), L) \arrow[d, "{}^{*}\mathbb{M}(u)"] \\
\mathbb{P}(\acute{Y}) \arrow[r, "\tau_Y"] & Hom_{\mathcal{E}}(\mathbb{M}(\acute{Y}), L)
\end{tikzcd}
$$

Third, from the perspective of the category of elements of the presheaf P the map τ_Y, defined above, is identical with the map:

$$\tau_Y : (Y, p) \rightarrow Hom_{\mathcal{L}}\left(\mathbb{M} \circ \int_{\mathbb{P}} (Y, p), L\right). \tag{0.5.33}$$

Therefore, such a τ may be represented as a family of arrows of $\mathcal{E}$ which is being indexed by objects (Y, p) of the category of elements of the presheaf $\mathbb{P}$, namely

$$\{\tau_Y(p) : \mathbb{M}(Y) \rightarrow L\}_{(Y,p)}. \tag{0.5.34}$$

Thus, the condition of the commutativity of the above diagram, is equivalent to the condition that for each arrow u the following diagram commutes:

$$\begin{tikzcd}
\mathbb{M}(Y) \arrow[r, equal] & \mathbb{M}\circ\int_{\mathbb{P}}(Y,p) \arrow[dr, "\tau_Y(p)"] & \\
& & L \\
\mathbb{M}(\acute{Y}) \arrow[uu, "\mathbb{M}(u)"] \arrow[r, equal] & \mathbb{M}\circ\int_{\mathbb{P}}(\acute{Y},\acute{p}) \arrow[uu, "u_*"'] \arrow[ur, "\tau_{\acute{Y}}(\acute{p})"'] &
\end{tikzcd}$$

Consequently, according to the above diagram, the arrows $\tau_Y(p)$ form a cocone from the functor $\mathbb{M}\circ\int_{\mathbb{P}}$ to L. The categorical definition of a colimit, points to the conclusion that each such cocone emerges by the composition of the colimiting cocone with a unique arrow from the colimit $\mathbb{LP}$ to L. Equivalently, we conclude that there exists a bijection, which is natural in $\mathbb{P}$ and L:

(0.5.35) $$Hom_{\mathbb{S}ets^{\mathcal{A}^{op}}}(\mathbb{P}, Hom_{\mathcal{E}}(\mathbb{M}(-),L)) \cong Hom_{\mathcal{L}}(\mathbb{LP},L),$$

(0.5.36) $$Hom_{\mathbb{S}ets^{\mathcal{A}^{op}}}(\mathbb{P},\mathbb{R}(L)) \cong Hom_{\mathcal{L}}(\mathbb{LP},L)$$

abbreviated as follows:

(0.5.37) $$Nat(\mathbb{P},\mathbb{R}(L)) \cong Hom_{\mathcal{L}}(\mathbb{LP},L).$$

Hence, the probes-induced realization functor of $\mathcal{E}$, realized for each L in $\mathcal{E}$ by the presheaf of probing frames $\mathbb{R}(L) = Hom_{\mathcal{E}}(\mathbb{M}(-),L)$ in $\mathcal{S}ets^{\mathcal{A}^{op}}$, has a left adjoint functor $\mathbb{L} : \mathcal{S}ets^{\mathcal{A}^{op}} \to \mathcal{E}$, which is defined for each presheaf $\mathbb{P}$ in $\mathcal{S}ets^{\mathcal{A}^{op}}$ as the colimit $\mathbb{L}(\mathbb{P})$. Thus, the following diagram, where the Yoneda embedding is denoted by $\curvearrowright$, commutes:

$$\begin{tikzcd}
\mathcal{A} \arrow[d, tail, "\curvearrowright"'] \arrow[dr, "\mathbb{M}"] & \\
\mathbb{S}ets^{\mathcal{A}^{op}} \arrow[r, dotted, shift left, "\mathbb{L}"] & \mathcal{E} \arrow[l, dotted, shift left, "\mathbb{R}"]
\end{tikzcd}$$

The pair of adjoint functors $\mathbb{L} \dashv \mathbb{R}$ formalizes category-theoretically the functorial process of *encoding and decoding* information between diagrams of probes Y and not directly accessible objects L via the action of probing frames $\psi_Y : \mathbb{M}(Y) \to L$.

Furthermore, the existence of an adjunction between two categories always gives rise to a family of *universal morphisms*, called unit and counit of the adjunction, one for each object in the first category and one for each object in the second. Most importantly, every adjunction gives rise to an *adjoint equivalence* of certain subcategories of the initial functorially correlated categories. It is precisely this category-theoretic fact which determines the necessary and sufficient conditions for the isomorphic representation of an object L in $\mathcal{E}$ by means of suitably restricted functors of probing frames.

For any presheaf $\mathbb{P}$ in the functor category $\mathcal{S}ets^{Y^{op}}$, the *unit natural transformation* of the adjunction is defined as follows:

(0.5.38) $$\delta_{\mathbb{P}} : \mathbb{P} \longrightarrow \mathbb{RLP}.$$

On the other side, for each object L in $\mathcal{E}$ the *counit natural transformation* of the adjunction is defined as follows:

(0.5.39) $$\epsilon_L : \mathbb{LR}(L) \longrightarrow L.$$

The representation of an object L in $\mathcal{E}$, in terms of the functor of probing frames $\mathbb{R}(L)$ of L, is *full and faithful*, if and only if the counit of the adjunction is an isomorphism, that is structure-preserving, injective and surjective. In turn, the counit of the adjunction is an isomorphism, if and only if the right adjoint functor is full and faithful. In the latter case we characterize the shaping or coordinatization functor $\mathbb{M} : \mathcal{A} \to \mathcal{E}$ as a *proper* or *dense* functor.

0.5.4 ***Hom-Tensor Adjunction***

We have shown that formally the left adjoint functor of the adjunction, viz. $\mathbb{L} : \mathcal{S}ets^{\mathcal{A}^{op}} \to \mathcal{E}$, is defined for each presheaf $\mathbb{P}$ in $\mathcal{S}ets^{\mathcal{A}^{op}}$ as the colimit $\mathbb{L}(\mathbb{P})$. The functorial representation of an object L in $\mathcal{E}$ through the category $\mathcal{S}ets^{\mathcal{A}^{op}}$ requires an explicit calculation of this colimit.

For this purpose, it is instructive to explain the general categorical method of calculating the colimit of any functor $\mathbb{X} : \mathcal{I} \to \mathcal{E}$ from some index category $\mathcal{I}$ to $\mathcal{E}$. Let $\mu_i : \mathbb{X}(i) \to \sum_i \mathbb{X}(i)$, i in $\mathcal{I}$, be the injections of $\mathbb{X}(i)$ into their coproduct $\sum_i \mathbb{X}(i)$. A morphism from this coproduct, $\chi : \sum_i \mathbb{X}(i) \to \mathcal{E}$, is determined uniquely by the set of its components $\chi_i = \chi\mu_i$. These components χ_i are going to form a cocone over $\mathbb{X}$ to the vertex L only when for all arrows $v : i \to j$ of the index category $\mathcal{I}$ the following conditions are satisfied:

(0.5.40) $$(\chi\mu_j)\mathbb{X}(v) = \chi\mu_i,$$

$$\begin{tikzcd}
\mathbb{X}(i) \arrow[d, "\mu_i"'] \arrow[dr, "\chi\mu_i"] & \\
\sum \mathbb{X}(i) \arrow[r, dotted, "\chi"] & L \\
\mathbb{X}(j) \arrow[u, "\mu_j"] \arrow[ur, "\chi\mu_j"] &
\end{tikzcd}$$

So we consider all $\mathbb{X}(dom\, v)$ for all arrows v with its injections μ_v and obtain their coproduct $\sum_{v:i\to j}\mathbb{X}(dom\, v)$. Next we construct two arrows ζ and η, defined in terms of the injections μ_v and μ_i, for each $v : i \to j$, by the conditions:

$$\zeta\mu_v = \mu_i, \tag{0.5.41}$$

$$\eta\mu_v = \mu_j\mathbb{X}(v), \tag{0.5.42}$$

together with their coequalizer χ:

$$\begin{tikzcd}
\mathbb{X}(dom\, v) \arrow[d, "\mu_v"'] & \mathbb{X}(i) \arrow[d, "\mu_i"'] \arrow[dr, dotted, "\chi\mu_i"] & \\
\sum_{v:i\to j}\mathbb{X}(dom\, v) \arrow[r, shift left, "\zeta"] \arrow[r, shift right, "\eta"'] & \sum \mathbb{X}(i) \arrow[r, dotted, "\chi"] & L
\end{tikzcd}$$

The coequalizer condition $\chi\zeta = \chi\eta$ means that the arrows $\chi\mu_i$ form a cocone over $\mathbb{X}$ to the vertex L. We further note that since χ is the coequalizer of the arrows ζ and η this cocone is the colimiting cocone for the functor $\mathbb{X} : \mathcal{I} \to \mathcal{E}$ from some index category $\mathcal{I}$ to $\mathcal{E}$. Hence, the colimit of the functor $\mathbb{X}$ can be constructed as a coequalizer of coproduct according to the following diagram:

$$\begin{tikzcd}
\sum_{v:i\to j}\mathbb{X}(dom\, v) \arrow[r, shift left, "\zeta"] \arrow[r, shift right, "\eta"'] & \sum \mathbb{X}(i) \arrow[r, "\chi"] & Colim\, \mathbb{X}
\end{tikzcd}$$

In our case, we need to calculate the colimit $\mathbb{L}(\mathbb{P})$ for a functor $\mathbb{P}$ in $\mathcal{S}ets^{\mathcal{A}^{op}}$. According to the above general scheme, it is necessary to specify the index category corresponding to the presheaf functor $\mathbb{P}$ defined on the base category of probes $\mathcal{A}$.

The index category corresponding to the presheaf functor $\mathbb{P}$ is the category of its elements $\mathcal{I} \equiv \int(\mathbb{P}, \mathcal{A})$, whence the functor $[\mathbb{M} \circ \int_{\mathbb{P}}]$ plays the role of the functor $\mathbb{X} : \mathcal{I} \to \mathcal{E}$. Hence, we obtain:

$$\mathbb{L}(\mathbb{P}) = \mathbb{L}_{\mathbb{M}}(\mathbb{P}) = Colim \left\{ \int (\mathbb{P}, \mathcal{A}) \xrightarrow{\int_{\mathbb{P}}} \mathcal{A} \xrightarrow{\mathbb{M}} \mathcal{E} \right\}. \tag{0.5.43}$$

In this manner, referring to the coequalizer of coproduct diagram of the general scheme, in the present case, the second coproduct is over all the objects (Y, p) with $p \in \mathbb{P}(Y)$ of the category of elements, while the first coproduct is over all the arrows $v : (\acute{Y}, \acute{p}) \to (Y, p)$ of that category, so that $v : \acute{Y} \to Y$ and the condition $p \cdot v = \acute{p}$ is satisfied. We conclude that the colimit $\mathbb{L}_{\mathbb{M}}(\mathbb{P})$ can be equivalently presented as the following coequalizer of coproduct:

$$\textstyle\sum_{v:\acute{Y}\to Y} \mathbb{M}(\acute{Y}) \quad \overset{\zeta}{\underset{\eta}{\rightrightarrows}} \quad \sum_{(Y,p)} \mathbb{M}(Y) \xrightarrow{\chi} \mathbb{L}_{\mathbb{M}}(\mathbb{P}).$$

In order to analyze in more detail the colimit in the category of elements of $\mathbb{P}$ induced by the shaping functor $\mathbb{M}$, we consider the standard case where there exists a colimit-preserving functor from the category $\mathcal{E}$ to $\mathcal{S}ets$, such that the calculation can be performed set-theoretically. Then the coproduct $\sum_{(Y,p)} \mathbb{M}(Y)$ is a coproduct of sets, which is equivalent to the product $\mathbb{P}(Y) \times \mathbb{M}(Y)$ for $Y \in \mathcal{A}$. The coequalizer is thus equivalent to the definition of the tensor product $\mathbb{P} \otimes_{\mathcal{A}} \mathbb{M}$ of the set valued factors:

$$\mathbb{P} : \mathcal{A}^{op} \to \mathcal{S}ets, \qquad \mathbb{M} : \mathcal{A} \to \mathcal{S}ets, \tag{0.5.44}$$

where the contravariant functor $\mathbb{P}$ is considered as a right $\mathcal{A}$-module and the covariant functor $\mathbb{M}$ as a left $\mathcal{A}$-module, in complete analogy with the algebraic definition of the tensor product of a right $\mathcal{A}$-module with a left $\mathcal{A}$-module over a ring of coefficients $\mathcal{A}$. We call this construction the *functorial tensor product decomposition of the colimit in the category of elements of* $\mathbb{P}$ *induced by the shaping functor* $\mathbb{M} : \mathcal{A} \to \mathcal{E}$ of $\mathcal{E}$:

$$\textstyle\sum_{Y,\acute{Y}} \mathbb{P}(Y) \times Hom(\acute{Y}, Y) \times \mathbb{M}(\acute{Y}) \overset{\zeta}{\underset{\eta}{\rightrightarrows}} \sum_{Y} \mathbb{P}(Y) \times \mathbb{M}(Y) \xrightarrow{\chi} \mathbb{P} \otimes_{\mathcal{A}} \mathbb{M}.$$

Therefore, we formulate the following theorem:

Theorem 0.26. *The pair of adjoint functors* $\mathbb{L} \dashv \mathbb{R}$ *constitute a* Hom-Tensor adjunction *defined by the natural bijection:*

$$Hom_{\mathbb{S}ets^{\mathcal{A}^{op}}}(\mathbb{P}, Hom_{\mathcal{E}}(\mathbb{M}(-), L)) \cong Hom_{\mathcal{E}}(\mathbb{P} \otimes_{\mathcal{A}} \mathbb{M}, L) \tag{0.5.45}$$

abbreviated as follows:

$$Nat(\mathbb{P}, \mathbb{R}(L)) \cong Hom_{\mathcal{E}}(\mathbb{P} \otimes_{\mathcal{A}} \mathbb{M}, L). \tag{0.5.46}$$

According to the above diagram for elements $p \in \mathbb{P}(Y)$, $v : \acute{Y} \to Y$ and $\acute{q} \in \mathbb{M}(\acute{Y})$ the following equations hold:

$$\zeta(p, v, \acute{q}) = (p \cdot v, \acute{q}), \qquad \eta(p, v, \acute{q}) = (p, v(\acute{q})) \tag{0.5.47}$$

symmetric in $\mathbb{P}$ and $\mathbb{M}$. Hence the elements of the set $\mathbb{P} \otimes_{\mathcal{A}} \mathbb{M}$ are all of the form $\chi(p, q)$. This element can be written as:

$$\chi(p, q) = p \otimes q, \quad p \in \mathbb{P}(Y), q \in \mathbb{M}(Y). \tag{0.5.48}$$

Thus, if we take into account the definitions of ζ and η above, we obtain:

$$p \cdot v \otimes \acute{q} = p \otimes v(\acute{q}), \quad p \in \mathbb{P}(Y), \acute{q} \in \mathbb{M}(\acute{Y}), v : \acute{Y} \to Y. \tag{0.5.49}$$

We conclude that the set $\mathbb{P} \otimes_{\mathcal{A}} \mathbb{M}$ is actually the quotient of the set $\sum_Y \mathbb{P}(Y) \times \mathbb{M}(Y)$ by the smallest equivalence relation generated by the above equations, whence the elements $p \otimes q$ of the quotient set $\mathbb{P} \otimes_{\mathcal{A}} \mathbb{M}$ are the equivalence classes of this relation.

Furthermore, if we define the arrows:

$$k_Y : \mathbb{P} \otimes_{\mathcal{A}} \mathbb{M} \to L, \qquad l_Y : \mathbb{P}(Y) \to Hom_{\mathcal{E}}(\mathbb{M}(Y), L), \tag{0.5.50}$$

they are related under the adjunction by:

$$k_Y(p, q) = l_Y(p)(q), \qquad Y \in \mathcal{A}, p \in \mathbb{P}(Y), q \in \mathbb{M}(Y). \tag{0.5.51}$$

Here we consider k as a function on $\sum_Y \mathbb{P}(Y) \times \mathbb{M}(Y)$ with components $k_Y : \mathbb{P}(Y) \times \mathbb{M}(Y) \to L$ satisfying:

$$k_{\acute{Y}}(p \cdot v, \acute{q}) = k_Y(p, v(\acute{q})), \tag{0.5.52}$$

in agreement with the equivalence relation defined above.

Now, we replace back the category $\mathcal{S}ets$ by the category $\mathcal{E}$ under study. The element q in the set $\mathbb{M}(Y)$ is replaced by a generalized element $q : \mathbb{M}(C) \to \mathbb{M}(Y)$ from some object $\mathbb{M}(C)$ of $\mathcal{E}$. Then we consider k as a function $\sum_{(Y,p)} \mathbb{M}(Y) \to L$ with components $k_{(Y,p)} : \mathbb{M}(Y) \to L$ for each $p \in \mathbb{P}(Y)$, which for all arrows $v : \acute{Y} \to Y$ satisfy:

$$k_{(\acute{Y}, p \cdot v)} = k_{(Y,p)} \circ \mathbb{M}(v). \tag{0.5.53}$$

Then the condition defining the bijection holding by virtue of the Hom-Tensor adjunction is given by:

$$k_{(Y,p)} \circ q = l_Y(p) \circ q : \mathbb{M}(C) \to L. \tag{0.5.54}$$

This argument, being natural in the object $\mathbb{M}(C)$, is determined by setting $\mathbb{M}(C) = \mathbb{M}(Y)$ with q being the identity map. Hence the bijection takes the simple form:

$$k_{(Y,p)} = l_Y(p). \tag{0.5.55}$$

As a first application of the above method, we consider the case $\mathbb{P} = \curvearrowright[Y]$, viz. the representable Hom-functor $Hom_{\mathcal{A}}(-, Y)$, represented by the probe Y in $\mathcal{A}$. Then, we have:

$$\mathbb{L}(\curvearrowright[Y]) = Colim \left\{ \int (\curvearrowright[Y], \mathcal{A}) \xrightarrow{\int_{\curvearrowright[Y]}} \mathcal{A} \xrightarrow{\mathbb{M}} \mathcal{E} \right\} = \curvearrowright[Y] \otimes_{\mathcal{A}} \mathbb{M} \cong \mathbb{M}(Y). \tag{0.5.56}$$

Thus, we express the evaluation of the shaping functor $\mathbb{M}$ at each probe Y in $\mathcal{A}$ as the colimit taken in the category of elements of the representable Hom-functor $\curvearrowright[Y]$ according to the above.

The next case of interest is the one where $\mathbb{P} = \mathbb{R}(L)$, viz. the functor of probing frames of L, $Hom_{\mathcal{E}}(\mathbb{M}(-), L)$. The significance of the calculation of the colimit $\mathbb{L}(\mathbb{R}(L))$:

$$\mathbb{L}(\mathbb{R}(L)) = Colim\{\int(\mathbb{R}(L), \mathcal{A}) \xrightarrow{\int_{\mathbb{R}(L)}} \mathcal{A} \xrightarrow{\mathbb{M}} \mathcal{E}\} = \mathbb{R}(L) \otimes_{\mathcal{A}} \mathbb{M}, \tag{0.5.57}$$

stems from the fact that if the counit of the Hom-Tensor adjunction:

$$\epsilon_L : \mathbb{L}\mathbb{R}(L) \longrightarrow L, \tag{0.5.58}$$

is an isomorphism, that is structure-preserving, injective and surjective, then we obtain a full and faithful isomorphic representation of an object L in $\mathcal{E}$, in terms of the functor of probing frames $\mathbb{R}(L)$ of L. Thus, it is important to specify the conditions which force the counit ϵ_L to be an isomorphism.

The counit natural transformation ϵ_L elucidates the notion of *spectral observation via probes* of an indirectly accessible object L in $\mathcal{E}$ with a precise operational interpretation. More concretely, spectral observation of L constitutes a natural transformation of the identity of L. This natural transformation is expressed concretely by the counit ϵ_L for every L in $\mathcal{E}$. In this sense, spectral observation of an object L, via the functor of probing frames $\mathbb{R}(L)$, where the probes Y in $\mathcal{A}$ play the role of partial or local information carriers classifying the global information content of L, is interpreted as the operational implementation of the counit natural transformation ϵ_L. Thus, the composite endofunctor:

$$\mathbb{G} := \mathbb{L}\mathbb{R}(L) : \mathcal{E} \to \mathcal{S}ets^{\mathcal{A}^{op}} \to \mathcal{E}, \tag{0.5.59}$$

may be called the *global spectral observation functor* of L via the functor of probing frames $\mathbb{R}(L)$. Notice, that if the counit is an isomorphism, then L can be considered as a fixed point of the corresponding global spectral observation functor $\mathbb{G}$. The counit natural transformation ϵ_L is a natural isomorphism, if and only if the right adjoint functor is full and faithful, or equivalently if and only if the cocone from the functor $\mathbb{M} \circ \int_{\mathbb{R}(L)}$ to L is universal for each L in $\mathcal{E}$. In the latter case, we characterize the functor $\mathbb{M} : \mathcal{A} \to \mathcal{E}$ as a dense shaping or coordinatization functor. In this way, it is important to specify the necessary and sufficient conditions which force the counit ϵ_L to be an isomorphism. This requirement leads to the notion of sheaf-theoretic localization of L through the probing frames $\psi_Y : \mathbb{M}(Y) \to L$.

0.6 Grothendieck Topos Interpretation of the Hom-Tensor Adjunction

We introduce the notion of a *functor of prelocalizations* $\mathbb{T}(L)$ for an object L in $\mathcal{E}$ as follows: $\mathbb{T}(L)$ is defined as a subfunctor of the functor of probing frames $\mathbb{R}(L)$ of L, viz. $\mathbb{T}(L) \hookrightarrow \mathbb{R}(L)$, or equivalently $\mathbb{T}(L)(Y) \subseteq [\mathbb{R}(L)](Y)$ for each probe Y in $\mathcal{A}$. A functor of prelocalizations for an object L evaluated at a probe Y can be expressed in the form of a right ideal $\mathbb{R}(L)(Y) \triangleleft T(L)(Y)$ consisting of prelocalizing probing frames $\psi_Y : \mathbb{M}(Y) \to L$. In more detail, this means that *prelocalizing probing frames* of L are characterized by the following property fitting them into *right ideals*: If $[\psi_Y : \mathbb{M}(Y) \to L] \in \mathbb{T}(L)(Y)$, and $\mathbb{M}(v) : \mathbb{M}(\acute{Y}) \to \mathbb{M}(Y)$ in $\mathcal{E}$, for $v : \acute{Y} \to Y$ in $\mathcal{A}$, then $[\psi_Y \circ \mathbb{M}(v) : \mathbb{M}(\acute{Y}) \to \mathcal{E}] \in \mathbb{T}(L)(Y)$. A functor of prelocalizations $\mathbb{T}(L)$ of L is equivalently called a *(spectral) sieve* of L.

A family of probing frames $\psi_Y : \mathbb{M}(Y) \longrightarrow L$, Y in $\mathcal{A}$, is the *generator of a right ideal of prelocalizations* $\mathbb{T}(L)(Y)$, if and only if, this ideal is the smallest among all that contains that family. The right ideals of prelocalizations for an L in $\mathcal{E}$ constitute a partially ordered set under inclusion. The minimal right ideal is the empty one, namely $\mathbb{T}(L)(Y) = \emptyset$ for all Y in $\mathcal{A}$, whereas the maximal right ideal is the functor of probing frames $\mathbb{R}(L)$ of L itself.

In order to demonstrate explicitly the implications of the notion of a sieve $\mathbb{T}(L)$, it is essential to calculate explicitly the colimit $\mathbb{L}(\mathbb{T}(L))$:

$$\mathbb{L}(\mathbb{T}(L)) = Colim \left\{ \int (\mathbb{T}(L), \mathcal{A}) \xrightarrow{\int_{\mathbb{T}(L)}} \mathcal{A} \xrightarrow{\mathbb{M}} \mathcal{E} \right\} = \mathbb{T}(L) \otimes_{\mathcal{A}} \mathbb{M}, \tag{0.6.1}$$

for any sieve $\mathbb{T}(L)$. According to the preceding, the sought colimit is expressed as the tensor product:

$$\sum\nolimits_{Y,\acute{Y}} \mathbb{T}(L)(Y) \times Hom(\acute{Y}, Y) \times \mathbb{M}(\acute{Y}) \ \underset{\eta}{\overset{\zeta}{\rightrightarrows}} \ \sum\nolimits_{Y} \mathbb{T}(L)(Y) \times \mathbb{M}(Y) \xrightarrow{\chi}$$

$$\xrightarrow{\chi} \mathbb{T}(L) \otimes_{\mathcal{A}} \mathbb{M}.$$

Thus, for prelocalizing probes $\psi_Y \in \mathbb{T}(L)(Y)$, $v : \acute{Y} \to Y$ and $\acute{q} \in \mathbb{M}(\acute{Y})$ we obtain the following identification equations:

$$\psi_Y \cdot v \otimes \acute{q} = \psi_Y \otimes v(\acute{q}). \tag{0.6.2}$$

We conclude that the set $\mathbb{T}(L) \otimes_{\mathcal{A}} \mathbb{M}$ is the quotient of the set $\sum_Y \mathbb{T}(L)(Y) \times \mathbb{M}(Y)$ by the smallest equivalence relation generated by the above equations. Clearly, this equivalence relation generates a right ideal of prelocalizations, where the elements $\psi_Y \otimes q$ of the $\mathbb{T}(L) \otimes_{\mathcal{A}} \mathbb{M}$ are the equivalence classes of this relation. An equivalence class of the form $\psi_Y \otimes q$ demarcates a *maximal connected component* in the category of elements $\int (\mathbb{T}(L), \mathcal{A})$ of the sieve $\mathbb{T}(L)$, where the identification is forced by means of admissible underlying transition arrows according to the above description. Notice that the elements in $\int (\mathbb{T}(L), \mathcal{A})$ are expressed in the form of pairs (ψ_Y, q), and thus, they should be thought of as *pointed probing frames* of L.

The *pullback of the prelocalizing probing frames* $\psi_Y : \mathbb{M}(Y) \longrightarrow L$, Y in $\mathcal{A}$, and $\psi_{\acute{Y}} : \mathbb{M}(\acute{Y}) \longrightarrow L$, $\acute{Y}$ in $\mathcal{A}$, with common codomain L, consists of the common refinement $\mathbb{M}(Y) \times_L \mathbb{M}(\acute{Y})$ and two arrows $\psi_{Y\acute{Y}}$ and $\psi_{\acute{Y}Y}$, called projections, as shown in the following diagram. The square commutes and for any K and arrows h and g that make the outer square commute, there exists a unique $u : \mathbb{M}(\acute{\acute{Y}}) \longrightarrow \mathbb{M}(Y) \times_L \mathbb{M}(\acute{Y})$ that makes the whole diagram commute.

$$\begin{tikzcd}
\mathbb{M}(\acute{\acute{Y}}) \arrow[dr, "u"] \arrow[drr, bend left, "h"] \arrow[ddr, bend right, "g"'] & & \\
& \mathbb{M}(Y) \times_L \mathbb{M}(\acute{Y}) \arrow[r, "\psi_{Y,\acute{Y}}"] \arrow[d, "\psi_{\acute{Y},Y}"] & \mathbb{M}(Y) \arrow[d, "\psi_Y"] \\
& \mathbb{M}(\acute{Y}) \arrow[r, "\psi_{\acute{Y}}"] & L
\end{tikzcd}$$

We notice that if the prelocalizing probing frames ψ_Y and $\psi_{\acute{Y}}$ are injective, then their pullback is given by their intersection. Consequently, we define the *gluing isomorphism* of the prelocalizing probing frames ψ_Y and $\psi_{\acute{Y}}$, as follows:

$$\Omega_{Y,\acute{Y}} : \psi_{\acute{Y}Y}(\mathbb{M}(Y) \times_L \mathbb{M}(\acute{Y})) \longrightarrow \psi_{Y\acute{Y}}(\mathbb{M}(Y) \times_L \mathbb{M}(\acute{Y})), \tag{0.6.3}$$

$$\Omega_{Y,\acute{Y}} = \psi_{Y\acute{Y}} \circ {\psi_{\acute{Y}Y}}^{-1}. \tag{0.6.4}$$

From the previous definition, we deduce the satisfaction of the following *cocycle conditions*:

$$\Omega_{Y,Y} = id_Y, \tag{0.6.5}$$

$$\Omega_{Y,\acute{Y}} \circ \Omega_{\acute{Y},\acute{\acute{Y}}} = \Omega_{Y,\acute{\acute{Y}}}, \tag{0.6.6}$$

$$\Omega_{Y,\acute{Y}} = \Omega^{-1}{}_{\acute{Y},Y} \tag{0.6.7}$$

where, in the first condition id_Y denotes the identity of $\mathbb{M}(Y)$, in the second $\psi_Y \times_L \psi_{\acute{Y}} \times_L \psi_{\acute{\acute{Y}}} \neq 0$, and in the third $\psi_Y \times_L \psi_{\acute{Y}} \neq 0$.

Thus, the *gluing isomorphism* between any two prelocalizing frames of a sieve $\mathbb{T}(L)$ assures that $\psi_{\acute{Y}Y}(\mathbb{M}(Y) \times_L \mathbb{M}(\acute{Y}))$ and $\psi_{Y\acute{Y}}(\mathbb{M}(Y) \times_L \mathbb{M}(\acute{Y}))$ probe L on their common refinement in a compatible way. This provides the sought criterion of qualifying prelocalizing frames of $\mathbb{T}(L)$ to localizing ones.

Before we formulate the notion of localizing sieves of L, based on the above criterion characterizing localizing probing frames of L, we need to introduce the notion of *covering sieves* of L. In other words, it is necessary to distinguish among the set of all sieves of L the ones that we will think of as covering sieves of L by means of the following constitutive properties:

[i]. The maximal L-sieve $\mathbb{S}_m(L)$ is a covering L-sieve for any L in $\mathcal{E}$.

[ii]. The covering L-sieves are stable under pullback operations in $\mathcal{E}$, and in particular, the intersection of L-covering sieves is also a L-covering sieve.

[iii]. The covering L-sieves are transitive, such that any two covering L-sieves in $\mathcal{E}$ admit a common refinement.

The above conditions characterizing the properties of covering L-sieves may be equivalently formulated in terms of their *generating families* of probing frames $\{\psi_{Y_j} : \mathbb{M}(Y_j) \to L\}$, Y_j in $\mathcal{A}$, as follows:

[i]. The family of all probing frames of L is a L-covering family.

[ii]. If $\{\mathbb{M}(Y_j) \to L\}$ is a L-covering family and $D \to L$ is any morphism in $\mathcal{E}$, then for every index j in this family, the following diagram is a pullback diagram over L:

$$\begin{tikzcd}
\mathbb{M}(Y_j)\times_L D \arrow[r] \arrow[d] & \mathbb{M}(Y_j) \arrow[d] \\
D \arrow[r] & L
\end{tikzcd}$$

such that $\{\mathbb{M}(Y_j)\times_L D \to D\}$ is a D-covering family.

[iii]. If $\{\mathbb{M}(Y_j) \to L\}$ is a L-covering family and for every index k such that $\{\mathbb{M}(Y_{jk}) \to \mathbb{M}(Y_j)\}$ is an $\mathbb{M}(Y_j)$-covering family, then the collection of composite probing frames $\{\mathbb{M}(Y_{jk}) \to \mathbb{M}(Y_j) \to L\}$ is a L-covering family.

We call any probing frame $\{\psi_{Y_j} : \mathbb{M}(Y_j) \to L\}$, Y_j in $\mathcal{A}$, of a L-covering sieve $\mathbb{S}(L) := \mathbb{S}L$, an $\mathbb{M}(Y_j)$-spectrum *cover* of L belonging to some L-covering generating family of $\mathbb{S}L$. As a consequence of conditions [ii] and [iii], we notice that if $\{\mathbb{M}(Y_j) \to L\}$ is a L-covering family and $\{\mathbb{M}(Y_k) \to L\}$ is another L-covering family, then for each index j in the first family and for each index k in the second, the following diagram is a *pullback diagram* over L:

$$\begin{tikzcd}
\mathbb{M}(Y_j)\times_L \mathbb{M}(Y_k) \arrow[r, "\psi_{jk}"] \arrow[d, "\psi_{kj}"'] & \mathbb{M}(Y_j) \arrow[d, "\psi_j"'] \\
\mathbb{M}(Y_k) \arrow[r, "\psi_k"] & L
\end{tikzcd}$$

and $\{[\mathbb{M}(Y_j)\times_L\mathbb{M}(Y_k)] \to \mathbb{M}(Y_j) \to L\}$, $\{[\mathbb{M}(Y_j)\times_L\mathbb{M}(Y_k)] \to \mathbb{M}(Y_k) \to L\}$ are both L-covering families. Moreover, it is clear that the L-covering family $\{[\mathbb{M}(Y_j)\times_L\mathbb{M}(Y_k)] \to L\}$ is a common refinement of the L-covering families $C_j = \{\mathbb{M}(Y_j) \to L\}$ and $C_k = \{\mathbb{M}(Y_k) \to L\}$. In particular, if both of the L-covering families C_j and C_k consist of injective covers of L, then the L-covering family $C_{jk} = \{[\mathbb{M}(Y_j)\times_L\mathbb{M}(Y_k)] \to L\}$, viz. their common refinement is actually the same as their intersection. Then for each index j in the first family and for each index k in the second one, we define the following morphism:

$$\Omega_{jk} : \psi_{kj}(\mathbb{M}(Y_j)\times_L\mathbb{M}(Y_k)) \to \psi_{jk}(\mathbb{M}(Y_j)\times_L\mathbb{M}(Y_k)) \tag{0.6.8}$$

by the assignment:

$$\Omega_{jk} = \psi_{jk} \circ {\psi_{kj}}^{-1}. \tag{0.6.9}$$

If for each index j in the first family and for each index k in the second one the morphism Ω_{jk} is an isomorphism, then we say that the $\mathbb{M}(Y_j)$-covering family $\{[\mathbb{M}(Y_j)\times_L\mathbb{M}(Y_k)] \to \mathbb{M}(Y_j)\}$ is compatible with the $\mathbb{M}(Y_k)$-covering family $\{[\mathbb{M}(Y_j)\times_L\mathbb{M}(Y_k)] \to \mathbb{M}(Y_k)\}$ with respect to L. Now, since the L-covering family $C_{jk} = \{[\mathbb{M}(Y_j)\times_L\mathbb{M}(Y_k)] \to L\}$ is a common refinement of the L-covering families $C_j = \{\mathbb{M}(Y_j) \to L\}$ and $C_k = \{\mathbb{M}(Y_k) \to L\}$, if Ω_{jk} is an isomorphism for each j, k, it is called a *gluing isomorphism*, because $\psi_{kj}(\mathbb{M}(Y_j)\times_L\mathbb{M}(Y_k))$ and $\psi_{jk}(\mathbb{M}(Y_j)\times_L\mathbb{M}(Y_k))$ cover the same part of L in a compatible way. Equivalently, L has compatible $[\mathbb{M}(Y_j)\times_L\mathbb{M}(Y_k)]$-Spectrum covers belonging to the common refinement L-covering family C_{jk} of the L-covering families C_j and C_k.

It is straightforward to show that the gluing isomorphism Ω_{jk} satisfies the following *cocycle compatibility conditions* (whenever they are defined), where j, k refer to the common refinement L-covering family C_{jk}:

$$\Omega_{jj} = id_{\mathbb{M}(Y_j)}, \tag{0.6.10}$$

$$\Omega_{jk} \circ \Omega_{km} = \Omega_{jm}, \tag{0.6.11}$$

$$\Omega_{jk} = {\Omega^{-1}}_{kj}. \tag{0.6.12}$$

Consequently, we conclude as follows:

Theorem 0.27. *A $\mathbb{M}(-)$-spectrum L-sieve $\mathbb{S}(L)$ is a localizing L-sieve, or equivalently a functor of localizations of L, if and only if it is closed with respect to covers of L and the above cocycle compatibility conditions are satisfied.*

For our purposes, the conceptual significance of a localizing $\mathbb{M}(-)$-spectrum L-sieve, lies on the fact that the functor $\mathbb{R}(L)$ becomes a *sheaf* when restricted to it.

Technically, in order to be able to transform the functor of probing frames $\mathbb{R}(L)$ into a sheaf it is necessary to define a notion of topology on the base category of probes $\mathcal{A}$. This is made possible by restricting the description of covering sieves on the category $\mathcal{A}$ as follows:

We define that a Y-sieve $\mathbb{S}(Y)$ on a probe Y in $\mathcal{A}$ is a covering Y-sieve, if the images of all the arrows $s_j : Y_j \to Y$ belonging to the Y-sieve $\mathbb{S}(Y)$, under the action of the shaping functor $\mathbb{M} : \mathcal{A} \to \mathcal{E}$, together form a covering $\mathbb{M}(Y)$-sieve in $\mathcal{E}$, generated by the $\mathbb{M}(Y)$-covering family $\{\mathbb{M}(Y_j) \to \mathbb{M}(Y)\}$ in $\mathcal{E}$.

The function $\mathbb{J}$ which assigns to each probe Y in $\mathcal{A}$ a set $\mathbb{J}(Y)$ of covering Y-sieves defines a *Grothendieck topology* on the base category $\mathcal{A}$. The category $\mathcal{A}$ together with a topology $\mathbb{J}$ is called a *site*, denoted by $(\mathcal{A}, \mathbb{J})$.

Then, the functor of probing frames $\mathbb{R}(L)$ on $\mathcal{A}$ is a sheaf for $\mathbb{J}$, if and only if for any Y-covering family $\{s_j : Y_j \to Y\}$ belonging to $\mathbb{J}(Y)$, any compatible family of probing frames $\{\psi_j : \mathbb{M}(Y_j) \to L\}$ can be glued together uniquely. Equivalently, any compatible family of frames $\{\psi_j\}$ has a unique amalgamation, in the sense that there exists a unique frame $\psi : \mathbb{M}(Y) \to L$ in the set $\mathbb{R}(L)(Y)$, such that the restriction of ψ at s_j gives ψ_j, that is $\psi \cdot s_j = \psi_j$. If the above holds for a particular covering sieve in the topology, we obtain that the functor of probing frames $\mathbb{R}(L)$ satisfies the sheaf condition, that is it becomes a sheaf with respect to this covering sieve.

A technically equivalent formulation of the sheaf condition, see Mac Lane - I. Moerdijk [1], referring in the present context of enquiry to the functor of probing frames $\mathbb{R}(L)$, is described in terms of an equalizer condition, expressed in terms of covering Y-sieves $\mathbb{S}(Y)$, as in the following diagram in $\mathcal{S}ets$:

$$\prod_{f\circ g\in\mathbb{S}(Y)} \mathbb{R}(L)(dom(g)) \;\underleftarrow{\overleftarrow{\qquad\quad}}\; \prod_{f\in\mathbb{S}(Y)}\mathbb{R}(L)(dom(f)) \xleftarrow{\;e\;} \mathbb{R}(L)(Y).$$

If the above diagram is an equalizer for a particular covering Y-sieve $\mathbb{S}(Y)$, we obtain that the functor of probing frames $\mathbb{R}(L)$ satisfies the sheaf condition with respect to the covering Y-sieve $\mathbb{S}(Y)$.

A *Grothendieck topos* over the small category of probes $\mathcal{A}$ is a category which is equivalent to the category of sheaves $\mathcal{S}h(\mathcal{A}, \mathbb{J})$ on a site $(\mathcal{A}, \mathbb{J})$. The site can be thought of as a system of generators and relations for the topos. We note that a category of sheaves $\mathcal{S}h(\mathcal{A}, \mathbb{J})$ on a site $(\mathcal{A}, \mathbb{J})$ is a full subcategory of the functor category of presheaves $\mathcal{S}ets^{\mathcal{A}^{op}}$.

The basic properties of a Grothendieck topos are the following:

(1). It admits finite projective limits; in particular, it has a terminal object, and it admits fibered products.

(2). If $(Y_i)_{i\in I}$ is a family of objects of the topos, then the sum $\coprod_{i\in I} Y_i$ exists and is disjoint.

(3). There exist quotients by equivalence relations and have the same good properties as in the category of sets.

0.7 The Grothendieck Topology of Epimorphic Families

As a concrete application of this framework, we define a covering $\mathbb{M}(-)$-spectrum L-sieve $\mathbb{S}(L)$ consisting of a generating family of coordinatizing probing frames $\{\psi_{Y_j} : \mathbb{M}(Y_j) \to L\}$, Y_j in $\mathcal{A}$, which are stable under pullback operations and jointly define an epimorphism in $\mathcal{E}$:

$$T : \coprod\nolimits_{(Y_j \in \mathcal{Y}, \psi_j : \mathbb{M}(Y_j) \to L)} \mathbb{M}(Y_j) \to L. \tag{0.7.1}$$

Concomitantly, we define that a Y-sieve $\mathbb{S}(Y)$ on a probe Y in $\mathcal{A}$ is a covering Y-sieve, if the images of all the arrows $s_j : Y_j \to Y$ belonging to the Y-sieve $\mathbb{S}Y$, under the action of the shaping functor $\mathbb{M} : \mathcal{A} \to \mathcal{E}$, together form an *epimorphic family* in $\mathcal{E}$.

Theorem 0.28. *The specification of covering sieves on the probes Y in $\mathcal{A}$, in terms of epimorphic families of coordinatizing probing frames in $\mathcal{E}$, expressed in terms of the morphism*

$$G : \coprod\nolimits_{(s_j : Y_j \to Y) \in \mathbb{S}Y} Y_j \to Y, \tag{0.7.2}$$

such that G is an epimorphism in $\mathcal{E}$, is a Grothendieck topology $\mathbb{J}$ on $\mathcal{A}$, called the Grothendieck topology of epimorphic families.

First of all, we notice that the maximal sieve on each probe Y in $\mathcal{A}$ includes the identity $Y \to Y$, and thus it is a covering sieve. The satisfaction of the transitivity property of the defined covering sieves is obvious. It remains to demonstrate that the covering sieves remain stable under pullback. For this purpose we consider the pullback of such a covering sieve $\mathbb{S}(Y)$ on Y along any morphism $h : Y' \to Y$ in $\mathcal{A}$, according to the following diagram:

$$\begin{tikzcd}
\coprod_{s \in \mathbb{S}(Y)} C \times_Y \acute{Y} \arrow[r] \arrow[d] & \acute{Y} \arrow[d, "h"] \\
\coprod_{s \in \mathbb{S}(Y)} C \arrow[r, "G"] & Y
\end{tikzcd}$$

For each morphism $s : C \to Y$ in $\mathbb{S}(Y)$ there exists a morphism $\coprod_{s \in \mathbb{S}Y} [D]^s \to C \times_Y \acute{Y}$ being an epimorphism in $\mathcal{E}$ under the action of the shaping functor $\mathbb{M}$. Equivalently, under $\mathbb{M}$ there exists a jointly epimorphic family of morphisms $\{[D]^s{}_j \to C \times_Y \acute{Y}\}_j$, where each domain $[D]^s$ is a probe. Consequently, the collection of all the composites:

$$[D]^s{}_j \to C \times_Y \acute{Y} \to \acute{Y} \tag{0.7.3}$$

for all $s : D \to Y$ in $\mathbb{S}(Y)$, and all indices j jointly form an epimorphic family in $\mathcal{E}$, which is contained in the sieve $h^*(\mathbb{S}(Y))$, being the pullback of $\mathbb{S}(Y)$ along the morphism $h : \acute{Y} \to Y$. Therefore, the sieve $h^*(\mathbb{S}(Y))$ is a $\acute{Y}$-covering sieve. Thus, the operation $\mathbb{J}$, which assigns to each probe Y in $\mathcal{A}$, a collection $\mathbb{J}(Y)$ of covering Y-sieves, being epimorphic families of arrows in $\mathcal{E}$ under $\mathbb{M}$, constitutes a Grothendieck topology $\mathbb{J}$ on $\mathcal{A}$.

Now, we may express a covering Y-sieve, that is

$$G : \coprod_{(s_j : Y_j \to Y) \in \mathbb{S}Y} Y_j \to Y \tag{0.7.4}$$

being an epimorphism in $\mathcal{E}$ (under the action of $\mathbb{M}$), equivalently as the pullback of G along itself:

$$\begin{tikzcd}
\coprod_{\acute{s}} \acute{C} \times_Y \coprod_s C \arrow[r] \arrow[d] & \coprod_s C \arrow[d, "G"] \\
\coprod_{\acute{s}} \acute{C} \arrow[r, "G"] & Y
\end{tikzcd}$$

Given that pullbacks in $\mathcal{E}$ preserve coproducts, the epimorphic family associated with a covering sieve on Y, admits the following coequalizer presentation:

$$\begin{tikzcd}
\coprod_{\acute{s},s} \acute{C} \times_Y C \arrow[r, shift left, "q_1"] \arrow[r, shift right, "q_2"'] & \coprod_s C \arrow[r, "G"] & Y
\end{tikzcd}$$

Further, we notice that for each pair of morphisms $s : C \to Y$ and $\acute{s} : \acute{C} \to Y$ in a covering Y-sieve $\mathbb{S}Y$, there exists (under the action of $\mathbb{M}$) an epimorphic family $\{D \to \acute{C} \times_Y C\}$, such that each domain D is a probe in $\mathcal{A}$. Consequently, each epimorphic family of morphisms associated with a covering Y-sieve $\mathbb{S}(Y)$ may be represented by a commutative diagram of the following form:

$$\begin{tikzcd}
D \arrow[r, "l"] \arrow[d, "k"] & C \arrow[d, "s"] \\
\acute{C} \arrow[r, "\acute{s}"] & Y
\end{tikzcd}$$

Now we may combine the representation of epimorphic families by commutative squares, together with the coequalizer presentation of the same epimorphic families as follows:

$$\coprod_D D \overset{y_1}{\underset{y_2}{\rightrightarrows}} \coprod_s C \xrightarrow{G} Y$$

where, the first coproduct is indexed by all D in the commutative squares representing epimorphic families.

Then, for each L in $\mathcal{E}$, we consider the functor of probing frames $\mathbb{R}(L)$ in $\mathcal{S}ets^{\mathcal{A}^{op}}$. If we apply the functor $\mathbb{R}(L)$ to the above coequalizer diagram we obtain an equalizer diagram in $\mathcal{S}ets$ as follows:

$$\prod_D \mathbb{R}(L)(D) \leftleftarrows \prod_{s \in \mathbb{S}Y} \mathbb{R}(L)(C) \longleftarrow \mathbb{R}(L)(Y)$$

where the first product is indexed by all D in the commutative squares representing epimorphic families. Consequently, since the above diagram is an equalizer in $\mathcal{S}ets$ we conclude that the functor of probing frames $\mathbb{R}(L)$ in $\mathcal{S}ets^{\mathcal{A}^{op}}$, satisfies the sheaf condition for the covering Y-sieve $\mathbb{S}(Y)$. Moreover, the equalizer condition holds for every covering sieve in the Grothendieck topology of epimorphic families. By rephrasing the above, we conclude as follows:

Theorem 0.29. *The functor of probing frames $\mathbb{R}(L)$ is a sheaf for the Grothendieck topology of epimorphic families on the base category $\mathcal{A}$ of probes.*

0.8 Unit and Counit of the Hom-Tensor Adjunction

We focus again our attention in the Hom-Tensor adjunction and investigate the unit and the counit of it. For any presheaf $\mathbb{P} \in \mathcal{S}ets^{\mathcal{A}^{op}}$, we deduce that the unit $\delta_{\mathbb{P}} : \mathbb{P} \longrightarrow Hom_{\mathcal{L}}(\mathbb{M}(\_), \mathbb{P} \otimes_{\mathcal{A}} \mathbb{M})$ has components:

(0.8.1) $$\delta_{\mathbb{P}}(Y) : \mathbb{P}(Y) \longrightarrow Hom_{\mathcal{E}}(\mathbb{M}(Y), \mathbb{P} \otimes_{\mathcal{A}} \mathbb{M})$$

for each probe Y of $\mathcal{A}$. If we make use of the representable presheaf $y[Y]$, we obtain:

(0.8.2) $$\delta_{\curvearrowright[Y]} : \curvearrowright[Y] \to Hom_{\mathcal{L}}(\mathbb{M}(\_), \curvearrowright[Y] \otimes_{\mathcal{A}} \mathbb{M}).$$

Hence, for each probe Y of $\mathcal{A}$ the unit, in the case considered, corresponds to a morphism:

(0.8.3) $$\mathbb{M}(Y) \to \curvearrowright[Y] \otimes_{\mathcal{A}} \mathbb{M}.$$

But, since

(0.8.4) $$\curvearrowright[Y] \otimes_{\mathcal{A}} \mathbb{M} \cong \mathbb{M}(Y),$$

the unit for the representable presheaf of probes, which is a sheaf for the Grothendieck topology of epimorphic families, is clearly an isomorphism. By the preceding discussion we can see that the diagram commutes:

$$\begin{tikzcd}
\mathcal{A} \arrow[d, tail, "\curvearrowright"'] \arrow[drr, "\mathbb{M}"] & & \\
\mathbb{S}ets^{\mathcal{A}^{op}} \arrow[rr, "{[-]\otimes_{\mathcal{E}}\mathbb{M}}"] & & \mathcal{E}
\end{tikzcd}$$

Theorem 0.30. *The unit of the Hom-Tensor adjunction, referring to the representable sheaf* $\curvearrowright[Y]$ *of the category of probes,* $\delta_{\curvearrowright[Y]} : \mathbb{M}(Y) \longrightarrow \curvearrowright[Y]\otimes_{\mathcal{A}}\mathbb{M}$*, is an isomorphism.*

Next, we show that the counit natural transformation

$$\epsilon_L : \mathbb{L}\mathbb{R}(L) \rightarrow L \tag{0.8.5}$$

of the Hom-Tensor adjunction is a natural isomorphism if the domain is restricted to the sheaf of probing frames $\mathbb{R}(L)$ (for the Grothendieck topology $\mathbb{J}$ of epimorphic families on $\mathcal{A}$, or equivalently for each covering Y-sieve $\mathbb{S}(Y)$ in $\mathbb{J}$).

For each L in $\mathcal{E}$, the counit is defined as follows:

$$\epsilon_L : Hom_{\mathcal{L}}(\mathbb{M}(\_), L)\otimes_{\mathcal{A}}\mathbb{M} \longrightarrow L. \tag{0.8.6}$$

The counit corresponds to the vertical map in the following coequalizer diagram:

$$\begin{tikzcd}
\coprod_{v:Y\rightarrow E}\mathbb{M}(Y) \arrow[r, shift left, "\zeta"] \arrow[r, shift right, "\eta"'] & \coprod_{(E,\psi)}\mathbb{M}(E) \arrow[r] \arrow[dr] & {[\mathbb{R}(L)](-)\otimes_{\mathcal{A}}\mathbb{M}} \arrow[d, dotted, "\epsilon_L"] \\
& & L
\end{tikzcd}$$

where the first coproduct is indexed by all arrows $v : Y \rightarrow E$, with Y, E probes in $\mathcal{A}$, whereas the second coproduct is indexed by all probes Y in $\mathcal{A}$ and probing frames $\psi : \mathbb{M}(E) \rightarrow L$, belonging to a covering sieve of L by objects of the category of probes $\mathcal{A}$.

We remind that a covering $\mathbb{M}(Y)$-spectrum L-sieve $\mathbb{S}(L)$ consists of a generating family of coordinatizing probing frames $\{\psi_{Y_j} : \mathbb{M}(Y_j) \rightarrow L\}$, Y_j in $\mathcal{A}$, which are stable under pullback operations and jointly define an epimorphism in $\mathcal{E}$:

$$T : \coprod_{(Y_j\in\mathcal{Y},\psi_j:\mathbb{M}(Y_j)\rightarrow L)}\mathbb{M}(Y_j) \rightarrow L. \tag{0.8.7}$$

Equivalently, a covering L-sieve $\mathbb{S}(L)$ admits the following coequalizer diagram presentation in $\mathcal{E}$:

$$\coprod_{\nu}\mathbb{M}(D) \underset{w_2}{\overset{w_1}{\rightrightarrows}} \coprod_{\psi}\mathbb{M}(C) \xrightarrow{T} L$$

where, the first coproduct is indexed by all ν, representing commutative squares in $\mathcal{E}$ corresponding to epimorphic families of coordinatizing probing frames:

$$\begin{tikzcd} \mathbb{M}(D) \arrow[r, "\mathbb{M}(l)"] \arrow[d, "\mathbb{M}(k)"] & \mathbb{M}(C) \arrow[d, "\psi_{\mathbb{M}(C)}"] \\ \mathbb{M}(\acute{C}) \arrow[r, "\psi_{\mathbb{M}(\acute{C})}"] & L \end{tikzcd}$$

where D, C, $\acute{C}$ are probes in $\mathcal{A}$.

Next, we show that a covering $\mathbb{M}(Y)$-spectrum L-sieve $\mathbb{S}(L)$ is the coequalizer of (w_1, w_2) if and only if it is the coequalizer of (ζ, η), under the proviso that $\mathbb{R}(L)$ is a sheaf of probing frames.

We notice that the coequalizer condition

$$T \circ w_1 = T \circ w_2, \tag{0.8.8}$$

is equivalent to the condition:

$$T_{(\mathbb{M}(C),\psi_{\mathbb{M}(C)})} \circ \mathbb{M}(l) = T_{(\mathbb{M}(\acute{C}),\psi_{\mathbb{M}(\acute{C})})} \circ \mathbb{M}(k) \tag{0.8.9}$$

for each commutative square ν of the form above.

Correspondingly, the coequalizer condition $T \circ \zeta = T \circ \eta$ is equivalent to the condition:

$$T_{(\mathbb{M}(C),\psi_{\mathbb{M}(C)})} \circ \mathbb{M}(u) = T_{(\mathbb{M}(\acute{C}),\psi_{\mathbb{M}(C)} \circ \mathbb{M}(u))} \tag{0.8.10}$$

for every morphism of probes $u : \acute{C} \to C$, with $\acute{C}$, C probes in $\mathcal{A}$ and $\psi : \mathbb{M}(C) \to L$, a coordinatizing probing belonging to a covering sieve of L. Therefore, our claim is proved if we show that the above conditions are equivalent.

On the one hand, $T \circ \zeta = T \circ \eta$, implies for every commutative diagram of the form ν:

$$\begin{tikzcd} \mathbb{M}(D) \arrow[r, "\mathbb{M}(l)"] \arrow[d, "\mathbb{M}(k)"] & \mathbb{M}(C) \arrow[d, "\psi_{\mathbb{M}(C)}"] \\ \mathbb{M}(\acute{C}) \arrow[r, "\psi_{\mathbb{M}(\acute{C})}"] & L \end{tikzcd}$$

where D, C, $\acute{C}$ are probes in $\mathcal{A}$, the following relations hold:

$$T_{(\mathbb{M}(C),\psi_{\mathbb{M}(C)})} \circ \mathbb{M}(l) = T_{(\mathbb{M}(\acute{C}),\psi_{\mathbb{M}(\acute{C})})} \circ \mathbb{M}(k). \tag{0.8.11}$$

On the other hand, the coequalizer condition $T \circ w_1 = T \circ w_2$, implies that for every morphism $\mathbb{M}(u) : \mathbb{M}(\acute{C}) \to \mathbb{M}(C)$, with C, $\acute{C}$ probes in $\mathcal{A}$ and $\psi_{\mathbb{M}(C)} : \mathbb{M}(C) \to L$, the diagram of the form ν below:

$$\begin{tikzcd}
\mathbb{M}(\acute{C}) \arrow[r, "\mathbb{M}(u)"] \arrow[d, "id"'] & \mathbb{M}(C) \arrow[d, "\psi_{\mathbb{M}(C)}"] \\
\mathbb{M}(\acute{C}) \arrow[r] & L
\end{tikzcd}$$

commutes and provides the condition:

$$T_{(\mathbb{M}(C),\psi_{\mathbb{M}(C)})} \circ \mathbb{M}(u) = T_{(\mathbb{M}(\acute{C}),\psi_{\mathbb{M}(C)} \circ \mathbb{M}(u))}. \tag{0.8.12}$$

Consequently, the pairs of arrows (ζ, η) and (w_1, w_2) have isomorphic coequalizers, proving that the counit of the Hom-Tensor adjunction restricted to sheaves for the Grothendieck topology of epimorphic families on $\mathcal{A}$ is an isomorphism.

Theorem 0.31. *The epimorphism $T : \coprod_{(Y_j \in \mathcal{Y}, \psi_j : \mathbb{M}(Y_j) \to L)} \mathbb{M}(Y_j) \to L$ that defines a covering L-sieve, induces an isomorphism in $\mathcal{E}$: $\mathbb{LR}(L) \cong L$ if the functor of probing frames $\mathbb{R}(L)$ is a sheaf for the Grothendieck topology $\mathbb{J}$ of epimorphic families. Equivalently, the counit natural transformation $\epsilon_L : \mathbb{LR}(L) \to L$ of the Hom-Tensor adjunction is a natural isomorphism if restricted to the sheaf $\mathbb{R}(L)$ for this topology $\mathbb{J}$.*

Hence, the global information content of an object L in $\mathcal{E}$ is completely described up to isomorphism in terms of the colimit $\mathbb{LR}(L)$ taken in the fibred category $\int(\mathbb{R}, \mathcal{A})$ of the sheaf of probing frames $\mathbb{R}(L)$, because the counit ϵ_L of the Hom-Tensor adjunction for every covering L-sieve $\mathbb{S}(L)$ is an isomorphism.

As a consequence of the unit and counit isomorphism, we obtain the following:

Theorem 0.32. *The Hom-Tensor adjunction can be restricted to a dual functorial equivalence of categories:*

$$\mathbb{L} : \mathcal{S}h(\mathcal{A}, \mathbb{J}) \rightleftarrows \mathcal{E} : \mathbb{R}, \tag{0.8.13}$$

$$\mathbb{L} : \mathcal{S}h(\mathcal{A}, \mathbb{J}) \cong \mathcal{E} : \mathbb{R}. \tag{0.8.14}$$

Thus, the category $\mathcal{E}$ can be made equivalent to the category of sheaves $\mathcal{S}h(\mathcal{A}, \mathbb{J})$ on the site $(\mathcal{Y}, \mathbb{J})$.

We note that the Grothendieck topos extension of the Hom-Tensor adjunction according to the above general functorial framework has been applied for the sheaf-theoretic localization and representation of quantum logical event algebras via local Boolean probing measurement frames, see Zafiris [1, 2, 5, 6, 10] and Zafiris-Karakostas [1], quantum measure algebras via local Boolean probabilistic frames, see Zafiris [9], and quantum observables algebras via local commutative observables frames, see Zafiris [3, 4, 7, 8] and Epperson-Zafiris [1].

It is instructive to summarize the previous discussion as follows: The problem of approximating an indirectly accessible object L in $\mathcal{E}$ by means of diagrams of probes Y has a *universal solution*, which is obtained by the Hom-Tensor adjunction. More precisely, the universal solution is given by the left adjoint functor $\mathbb{L} : \mathcal{S}ets^{\mathcal{A}^{op}} \to \mathcal{E}$ to the realization functor $\mathbb{R} : \mathcal{E} \to \mathcal{S}ets^{\mathcal{A}^{op}}$. If the functor of probing frames $\mathbb{R}(L)$ of L is restricted to a localization functor, or equivalently to a sheaf for the Grothendieck topology $\mathbb{J}$ of epimorphic families, then the counit natural transformation ϵ_L of the Hom-Tensor adjunction is a natural isomorphism if restricted to the sheaf $\mathbb{R}(L)$ for this topology $\mathbb{J}$ and the right adjoint is a full and faithful functor. It is useful to recapitulate the role of localization functors by means of the following diagram displaying the function of the counit of the Hom-Tensor adjunction:

$$\begin{tikzcd}
\mathbb{R}(L) \otimes_{\mathcal{A}} \mathbb{M} \arrow[dr, "\epsilon_L"] & \\
\mathbb{M}(Y) \arrow[u, "\psi_Y \otimes (-)"] \arrow[r, "\psi_Y"] & L
\end{tikzcd}$$

We conclude that the process of *spectral observation via probes* of an indirectly accessible object L in $\mathcal{E}$ by means of a localization functor $\mathbb{R}(L)$ constitutes a natural isomorphism of the identity of L. Therefore, under the hypothesis that there exists a colimit-preserving functor from the category $\mathcal{E}$ to $\mathcal{S}ets$, the elements of an object L can be represented isomorphically in terms of equivalence classes or germs $\psi_Y \otimes q$, $q \in \mathbb{M}(Y)$, of pointed probing frames qualified as local covers of L. Thus, the action of localization functors on L makes the category $\mathcal{E}$ a reflection of $\mathcal{S}ets^{Y^{op}}$, which can be further identified with the category of sheaves $\mathcal{S}h(\mathcal{A}, \mathbb{J})$ on the site $(\mathcal{Y}, \mathbb{J})$.