

**APPLIED VIRTUALITY
BOOK SERIES
CODING AS LITERACY
—METALITHIKUM IV**

BIRKHÄUSER
Basel

APPLIED VIRTUALITY
BOOK SERIES
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— **METALITHIKUM IV**
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V THE NATURE OF LOCAL/GLOBAL DISTINCTIONS, GROUP ACTIONS, AND PHASES: A SHEAF-THEORETIC APPROACH TO QUANTUM GEOMETRIC SPECTRA ELIAS ZAFIRIS

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The basic idea of a physical geometric spectrum implied by the principles of gauge theory is based on the notion of invariance under the action of a local symmetry group. In particular, the sheaf-theoretic interpretation of gauge geometry introduces a subtle distinction between local and global physical information carriers. A natural question arising in this context is how these aspects are reflected in the case of quantum mechanics. We explain that local information carriers are encoded as observables, whereas global ones are encoded either as “global memories” of states during processes of cyclic evolution, or as relative phase differences between distinct histories of events. We conclude that quantum geometric spectra

admit a sheaf-theoretic interpretation in combination with the non-spatiotemporal gauge structure of quantum mechanics.

I OBSERVABLES AND GEOMETRIC SPECTRUM

The state information of physical systems is adequately described by the collection of all observed data determined by the functioning of measurement devices in suitably specified experimental environments. Observables are precisely associated with physical quantities that, in principle, can be measured. The mathematical formalization of this procedure relies on the idea of expressing the observables, at least locally, by functions corresponding to measuring devices. Moreover, the usual underlying assumption on the basis of physical theories postulates that our form of observation is represented by coefficients in a number field, which is usually taken to be the field of real numbers. Thus, observables are typically modeled, at least locally, by continuous real-valued functions corresponding to measuring devices.

In this setting, the consideration of the structure of all observables necessitates the imposition of a further requirement pertaining to their algebraic nature. According to this requirement, the set of all observables bears the structure of a commutative, linear associative algebra with the unit over the real numbers, at least locally. The basic fact underlying this requirement is that with any commutative algebra of observables there is a naturally obtained topological space via measurement, which is thought of as the geometric spectrum of this algebra—namely, the space of points that is accessible by means of evaluating this algebra in the real number field. Most important, the observables of the algebra are represented as continuous functions on this geometric spectrum. The crux of this requirement is that any observed geometric spectrum should not be considered ad hoc, but should be associated with evaluating a corresponding algebra of observables via measurement. From a mathematical perspective, this principle has been well demonstrated in a variety of different contexts, and is known as Stone-Gelfand duality in a functional analytic setting or Grothendieck duality in an algebraic geometric setting.¹ In a nutshell, to any commutative algebra

of observables with a unit, there is a naturally associated topological space, namely its geometric spectrum, such that each observable of the algebra becomes a continuous function on the spectrum (at least locally). The basic didactic of this principle when applied, for instance, to the case of smooth manifolds of states utilized in physics, is the following:² smooth algebras of observables allow us to observe smooth geometric spaces, namely smooth manifolds, which are identified with the real-valued spectra of the geometric realization of these smooth algebras' observables. Inversely, the observables are identified with real-valued differentiable functions of these smooth manifolds. We note that these identifications hold up only to isomorphism or equivalence of kind or form. In this manner, a measurement process of the observables of a physical theory can specify the geometric domain of its applicability up to an isomorphic mapping. Thus, the notion of isomorphism demarcates the geometric boundary of observability.

II GROUP ACTIONS AND THE ERLANGEN PROGRAM

The central concept of Klein's Erlangen program is expressed by the thesis that the objective content of a geometric theory is captured by the group of transformations of a space.³ Again, it is instructive here to think of the notion of a space, from a physical perspective and at least locally, as the geometric spectrum of a commutative algebra of observables.

The crucial point of the Erlangen program is that transformation groups constitute an algebraic encoding of a criterion of equivalence for geometric objects. Moreover, a transformation group determines the notion of what is to be a meaningful property of a concrete geometric figure. Therefore, from the Erlangen perspective, a geometric figure may be conceived from an abstract algebraic viewpoint as a manifold acted upon transitively by a group of transformations. The decisive aspect of the criterion of equivalence that a transformation group furnishes is its use in characterizing kinds of geometric figures and not particular instances of these figures. This leads to the idea that geometry, in an abstract sense, refers to kinds of figures that are specified by the transformation group of the space. Each kind can have infinite instantiations; thus, the same geometric form may be manifested in many different ways or else assume multiple concrete realizations. This reveals an important ontological dimension of Klein's program, since a transformation group of a space provides an efficient criterion to abstract a geometric kind from particular geometric instantiations, whereas the specific details of these instantiations, irrespective of their features as instances of a geometric kind, is irrelevant. In light

¹ David Eisenbud, *Commutative Algebra: With a View Toward Algebraic Geometry*, Graduate Texts in Mathematics, vol. 150 (New York: Springer, 1995); Sergei Gelfand and Yuri Manin, *Methods of Homological Algebra*, 2nd ed. (New York: Springer, 2003); Peter T. Johnstone, *Stone Spaces* (Cambridge: Cambridge University Press, 1986).

² Elias Zafiris, "A Sheaf-Theoretic Topos Model of the Physical Continuum and Its Cohomological Observable Dynamics," *International Journal of General Systems* 38, no. 1 (2009).

³ Richard W. Sharpe, *Differential Geometry: Cartan's Generalization of Klein's Erlangen Program* (New York: Springer, 1997).

of this, a geometry is specified by a group and its transitive action on a space, which remarkably can be presented in a purely algebraic way as a group homomorphism from the transformation group to the group of automorphisms of the underlying space. Conceptually speaking, the form of a geometric theory is encoded in the transitive action of a respective transformation group. Different particular geometric configurations are the same in form if, and only if, they share the same transformation group. In other words, the transitive group action provides a precise characterization regarding matters of geometric equivalence.

Mathematically, the above thesis is expressed as the principle of transference, or principle of isomorphism induced by a transitive group action on a space. A transfer of structure takes place by means of an isomorphism, providing different equivalent models of the same geometric theory. Philosophically, underneath lies an Aristotelian conception of space, according to which space is conceived as being matter without form. The form is being enacted by the action of a concrete transformation group. Still, more important, the space itself may be considered as the quotient of the transformation group over a closed subgroup of the former. A change in algebraic form, or else a change of the transformation group, signifies a change in geometry, in the sense that the equivalence criterion encoded in the group action is altered. Thus, moving from a group to a larger one amounts to a change in the resolution unit of figures, expressed as a relaxation of the geometric equivalence criterion involved in the procedure. In effect, the criterion of equivalence serves as a powerful classification principle for geometries in relation to group hierarchies. A crucial aspect of the Erlangen program is that it does not specify which underlying manifolds exist as spectra of corresponding observable algebras, it rather deals with the possible existence of geometric structures on these manifolds in relation to form-inducing transformation groups' action upon them. This naturally leads to a bidirectional relation of dependent-variation between transformation groups and geometric structures on manifolds. This bidirectional relation conveys the information that two spaces cannot have different transformation groups without differing as geometric structures, whereas the converse is clearly false.

III LOCAL GROUP ACTIONS AND GAUGE THEORY

From a physical point of view, geometry is synonymous with measurement—hence, closely related to observation, being in fact the result of it. Group actions can be actually thought of as particular acts of measurement. The general stance toward physical geometry implicated by the Erlangen program assumes that the geometric configuration of states of a physical system and the symmetry group of transformations of those states are regarded as being semantically equivalent via the transitive group action on the space of states.

Modern gauge-invariant physical theories are being mathematically viewed as fiber bundle or Cartan geometries.⁴ Cartan managed to combine Klein's group theoretical conception of geometry with Riemann's infinitesimal metrical viewpoint.⁵ This has been achieved by the introduction of the fundamental notion of a variable connection, which can be made metric-compatible. A connection serves as a covariant derivative of the states establishing the concept of parallelism under transportation from the local to the global level, which is considered to be induced by a physical field. Locally, a connection plays the role of the field's potential, whose observable effects are expressed by a homological tensorial magnitude called the curvature of the connection. The curvature is an observable playing the role of the field's strength. In the setting of gauge theories the transformation group is being modeled locally on the fibers of a principal fiber bundle. Klein geometries are precisely the Cartan geometries whose connections have zero curvature.

In physics, a geometric kind is represented by an equivalence class of state spaces, where a state space includes all possible potential states of a system. The state space is semantically equivalent with the group action space of a symmetry group, at least locally, in the sense that the symmetry group circumscribes the range of possible potential properties that a geometric kind can assume, like in the case of gauge theories. Being a member of a geometric kind, a physical entity can be potentially in any of the possible states locally, although after measurement it is actually in a particular one. This is called "local gauge freedom" and constitutes a concrete physical manifestation of the criterion of equivalence that a local gauge group furnishes in the case of gauge theories.

The conceptual paradigm of gauge theories is very instructive because it convincingly demonstrates that the concept of geometric kinds incorporates the distinction between the potentially possible and the actual. The idea of a geometric kind is precisely articulated by the action of a local symmetry group, which pertains not to spatiotemporal but to qualitative features. The local symmetry group of a gauge theory circumscribes a set of potentially possible states locally and depicts a natural geometric kind via its action. The fiber-bundle formulation of gauge theories captures precisely the formation of geometric kinds under equivalence criteria constituted by actions of local symmetry groups. The base manifold of a fiber bundle equipped with a connectivity structure representing a gauge field plays the role of space-time. Note that space-time is not given a priori, but is an integral part of the existence of matter. It is the carrier of the geometry by which matter is transformed, thus perceived as a structural quality of the dynamic

4 Anastasios Mallios, *Modern Differential Geometry in Gauge Theories*, vol. 1, *Maxwell Fields* (Boston: Birkhäuser, 2006); Anastasios Mallios, *Modern Differential Geometry in Gauge Theories*, vol. 2, *Yang-Mills Fields* (Boston: Birkhäuser, 2009).

5 Sharpe, *Differential Geometry*.

field or dynamic connection between the fibers of the bundle, modeling in turn the local group action. Finally the laws of physics are revealed by the variation of matter, expressed by the dynamic field connection on the fiber space and observed through the curvature of the connection.

IV THE ADVENT OF QUANTUM THEORY

The crucial distinguishing feature of quantum mechanics in relation to all classical theories is that the totality of all physical observables constitutes a global noncommutative algebra, and thus quantum observables are not theoretically compatible.⁶ This simply means that not all observables are simultaneously measurable with respect to a single universal global logical Boolean frame as is the case in all classical theories of physics.⁷ Thus a multiplicity of potential local Boolean frames exists, each one standing for a context of comeasurable observables. Technically speaking, each Boolean frame is a Boolean algebra of projection operators obtained by the simultaneous spectral resolution of a family of compatible observables, represented as self-adjoint operators. Such a family of compatible observables forms a commutative observable algebra whose idempotent elements (projections) constitute a logical Boolean frame. In this way, each local or partial Boolean frame signifies the local logical precondition predication space for the probabilistic evaluation of all the observables belonging to the corresponding commutative observable algebra. Thus, the manifestation of every single observed event in the quantum regime requires taking explicitly into account the specific local Boolean frame with respect to which it is contextualized. Since a single, unique, global Boolean frame does not exist, due to the noncommutativity of the totality of quantum observables, the necessity arises to consider all possible local Boolean frames and their interrelations. It is important to stress that the local/global distinction in quantum mechanics is of a topological nature and does not involve any preexisting set-theoretic space-time background of embedding events.⁸ Actually, by utilizing the Stone-Gelfand representation theorems for Boolean algebras and commutative observable algebras correspondingly, in the setting described above, only

the notion of a local geometric spectrum becomes applicable as co-emergent with the specification and functional role of a local Boolean or commutative observable frame, respectively. The explicit consideration of all potentially possible local Boolean frames entirely covering the factual layer of quantum observable behavior provides a local logical/topological relativization or contextualization of global quantum event structures in local Boolean or commutative algebraic terms. This type of relativization should be best thought of as a process of sieving, or filtering, the factual content of the global quantum geometric spectrum with respect to covering families of partially compatible nested Boolean frames at various logical resolution scales. In category-theoretic terminology, these covering families of local Boolean frames define covering sieves, which are used for the enunciation of an appropriate notion of topology (Grothendieck topology) with respect to which the local/global distinction is formally depicted.⁹

The independence of the local/global distinction implicated in quantum mechanics from any metrical spatiotemporal connotation, as has been the case in classical theories, cannot be overestimated. In the preceding section, where we discussed the conceptualization of physical geometry by means of gauge theory, the geometric fiber spaces are thought of as being soldered over the points of a metrical space-time manifold. If quantum phenomena are actually compatible with the principles of gauge geometry, requiring invariance under the local action of a gauge symmetry group, then the local/global distinction should be disassociated from its restricted metrical spatiotemporal semantic identification and reclaim its original logical/topological semantic role. For this purpose, what is actually required is an efficient method of localization of the observed geometric spectrum that does not depend on the existence of points. In this respect, the notion of a sheaf proves to be indispensable for point-free localization processes, and paves the way for a deeper understanding of quantum theory under a substantially broader gauge-theoretic perspective.¹⁰ A topological approach to quantum mechanics based on a conceptual and technical sheaf-theoretic framework has been presented recently in book form and should be consulted as a standard reference in what follows in this text.¹¹

- 6 Garrett Birkhoff and John von Neumann, "The Logic of Quantum Mechanics," *Annals of Mathematics* 37 (1936): 823–43; Gudrun Kalmbach, *Orthomodular Lattices* (London: Academic Press, 1983); Veeravalli S. Varadarajan, *Geometry of Quantum Theory* (New York: Springer, 1985); John von Neumann, *Mathematical Foundations of Quantum Mechanics* (Princeton, NJ: Princeton University Press, 1955).
- 7 Elias Zafiris, "Boolean Coverings of Quantum Observable Structure: A Setting for an Abstract Differential Geometric Mechanism," *Journal of Geometry and Physics* 50 (2004): 99–114; Elias Zafiris and Vassilios Karakostas, "A Categorical Semantic Representation of Quantum Event Structures," *Foundations of Physics* 43 (2013).
- 8 Elias Zafiris, "Generalized Topological Covering Systems on Quantum Events Structures," *Journal of Physics A: Mathematical and General* 39 (2006): 1485–505; Elias Zafiris, "Sheaf-Theoretic Representation of Quantum Measure Algebras," *Journal of Mathematical Physics* 47 (2006): 92103.

- 9 Elias Zafiris, "Quantum Observables Algebras and Abstract Differential Geometry: The Topos-Theoretic Dynamics of Diagrams of Commutative Algebraic Localizations," *International Journal of Theoretical Physics* 46, no. 2 (2007); Elias Zafiris, "Boolean Information Sieves: A Local-to-Global Approach to Quantum Information," *International Journal of General Systems* 39, no. 8 (2010).
- 10 Saunders MacLane and Ieke Moerdijk, *Sheaves in Geometry and Logic: A First Introduction to Topos Theory* (New York: Springer, 1992); Anastasios Mallios, *Geometry of Vector Sheaves: An Axiomatic Approach to Differential Geometry* (Dordrecht: Kluwer Academic Publishers, 1998); John L. Bell, "From Absolute to Local Mathematics," *Synthese* 69 (1986); Anastasios Mallios, "On Localizing Topological Algebras," *Mathematics* 341, no. 79 (2004).
- 11 Michael Epperson and Elias Zafiris, *Foundations of Relational Realism: A Topological Approach to Quantum Mechanics and the Philosophy of Nature* (Lanham, MD: Lexington Books, 2013).

From a physical viewpoint, a sheaf of observables or states constitutes the natural outcome of a complete bidirectional localization/globalization process. The notion of a sheaf is based solely on the topological form of local/global distinctions and is independent of any smooth, metrical space-time point-manifold substratum.

The sheaf concept essentially expresses gluing conditions—namely, the way by which local structural algebraic information referring to observables or states can be amalgamated compatibly into global ones over a multiplicity of local covering domains of a global space. These local domains may be simply thought of in terms of open loci completely covering a topological space at different levels of spectral resolution. In the case of quantum theory, the local covering domains of a global quantum event space bear a logical semantics, since they can be actually identified as the spectra of local Boolean measurement frames corresponding to complete Boolean algebras of co-measurable quantum observables. In this manner, a sheaf may be thought of as a continuously variable algebraic information structure of observables or states, whose continuous variation is carried over all these local covering domains, such that certain pieces of local information may be glued together compatibly under extension from the local to the global. It is instructive to notice that in comparison to the notion of an algebraic structure of observables or states defined set-theoretically, a sheaf bears an additional intrinsic granulation or crystallization structure of the elements, called sections of the sheaf, specified by the grain of resolution of the covers. For example, in the case of open covers of a topological space the granulation structure is a partial order. In the general case, the granulation structure is thought of as a sieve whose nested holes or variable extent spectral horizons are comprised by the local covering domains. Heuristically, we think of the covers as measures of definability, and the elements of a sheaf restricted to a cover are exactly the members defined to the extent provided by this cover.

One of the most interesting aspects of the general sheaf concept, which is certainly crucial for the understanding of quantum theory, is that it naturally leads to spectral models of space, which do not have a local structure defined by points. In contradistinction, the local structure is generated by the families of covers and reference to points is allowed only contextually—that is, only in relation to all potential covers containing a point. Therefore, the sheaf information conveyed at a point-event of the spectrum is not of a punctual character as in classical physics, but of a germinal character—that is, it bears the semantics of an information seed. This is the case because it requires reference to equivalence classes or germs of all compatible local observable or state information with respect to all covers containing potentially this

point-event if a measurement takes place. Technically, the information germ at a point-event of the spectrum is expressed by an inductive limit synthetic logical procedure carried over these covers.¹²

It needs to be emphasized that the sheaf concept leads to a physically different understanding of the notion of a spectrum space in comparison to the classical set-theoretic one. More precisely, the actualization of each point-event by a measurement is internally related to an observable or state information germ, which contextualizes it with respect to all compatible local covers. At the same time, it constitutes an objective refinement of the spectrum, and therefore alters it globally. In this way, the global spectrum is not fixed and predetermined as in the set-theoretic conceptualization, but it is continuously unfolding and refined by the actualization of new events, correlated historically with their logical antecedents via compatible information germs.

The technical characterization of a sheaf is defined in two steps. The first step involves the functorial organization of the local covering domains' infiltrated observable or state information, meaning that the requirement of compatibility under restriction or reduction from the global to the local level should be satisfied. This process produces a variable algebraic information structure with the prescribed global-to-local compatibility, called a presheaf. The second step involves two processes—namely, the functional localization of the organized information presented in terms of a presheaf, and then its eventual completion by means of gluing locally compatible information from the local to the global. In this manner, the notion of a sheaf incorporates all the necessary and sufficient conditions for the bidirectional compatibility of observable or state information under restriction or reduction from the global to the local, and inversely under extension or induction from the local to the global.

VI THE PROGRAM OF “RELATIONAL REALISM”

Based on the fundamental concepts of sheaf theory, and in confluence with basic philosophical notions of Whitehead's “process theory,”¹³ which proved to be of crucial significance in relation to quantum mechanics,¹⁴ the active research program of “relational realism” has emerged.¹⁵ This program has been developed systematically for a novel interpretation of quantum theory, and has been currently extended toward an approach to quantum gravity and the deep understanding of

¹² Zafiris, “Generalized Topological Covering Systems on Quantum Events Structures”; Zafiris, “Boolean Information Sieves”; Zafiris and Karakostas, “A Categorical Semantic Representation of Quantum Event Structures.”

¹³ Alfred North Whitehead, *Process and Reality: An Essay in Cosmology*, ed. D. Griffin and D. Sherburne (New York: Free Press, 1978).

¹⁴ Michael Epperson, *Quantum Mechanics and the Philosophy of Alfred North Whitehead* (New York: Fordham University Press, 2004).

¹⁵ Epperson and Zafiris, *Foundations of Relational Realism*.

topological order and topological states of matter from first principles.¹⁶ The major novelty of relational realism is the utilization of concepts, methods, and formal techniques from the mathematical fields of category theory, topos theory, and homological algebra for the consistent and non-paradoxical explanation of all peculiar characteristics of the quantum theoretic universe of discourse, together with the logical, physical, and philosophical grounding of all related notions.¹⁷

In particular, the objectives of the research program of relational realism concern the investigation of applicability of categorical and sheaf-theoretic ideas, together with their philosophical implications, in relation to the following perennial issues in quantum theory:

- I. The problem of a revised viable relational realist interpretation of quantum theory superseding the norms of classical realism;
- II. The reevaluation of the globally non-Boolean logical structures of events associated with quantum systems from a categorical and sheaf-theoretic standpoint together with their corresponding truth-value assignments;
- III. The explication of the structure of the part-whole relation in quantum systems by sheaf-theoretic means and its functional role in explaining entanglement correlations being in focus in the domain of quantum information science;
- IV. The understanding of the emergence of classicality from the fundamental quantum description of systems via processes of decoherence modeled categorically;
- V. The elucidation of the notions of global relative topological and geometric phases and their conceptual significance for the explanation of topological order and topological states of matter with applications in solid state, condensed matter physics, and quantum computation;
- VI. The application of the algebraic-topological scheme of “sheaf-theoretic localization” and the gauge field-theoretic method of “extensive connection” for the formulation of background-independent quantum dynamic processes and the study of the issue of singularities from this conceptual standpoint.

In a nutshell, the adjective “relational” in the characterization of the program of relational realism is to be thought of technically not in its

- 16 Anastasios Mallios and Elias Zafiris, “The Homological Kahler-De Rham Differential Mechanism: I. Application in General Theory of Relativity,” *Advances in Mathematical Physics* (2011), doi:10.1155/2011/191083; A. Mallios and E. Zafiris, “The Homological Kahler-De Rham Differential Mechanism: II. Sheaf-Theoretic Localization of Quantum Dynamics,” *Advances in Mathematical Physics* (2011), doi:10.1155/2011/18980.
- 17 Steve Awodey, *Category Theory*, 2nd ed. (Oxford: Oxford University Press, 2010); Bell, “From Absolute to Local Mathematics”; Robert Goldblatt, *Topoi: The Categorical Analysis of Logic*, rev. 2nd ed. (1984; repr., New York: Dover, 2006); MacLane and Moerdijk, *Sheaves in Geometry and Logic*; Eisenbud, *Commutative Algebra*; Gelfand and Manin, *Methods of Homological Algebra*.

usual set-theoretic connotation of a scheme of relations, but in terms of a theoretical and algebraic topological prism of analysis. According to this, the emphasis is on the formation of bidirectional functorial bridges, called adjunctions,¹⁸ between different categories as well as on the establishment of partial or local structural congruence relations between different levels of categorical structure in a natural manner without the intervention of ad hoc choices and artificial conventions. Thus, the target of this analysis is the formation of natural bridges between structural relations and the efficient transfer of difficult problems pertaining to some categorical level into another level where they can be resolved.

VII QUANTUM MECHANICS AS A NON-SPATIOTEMPORAL GAUGE THEORY

A natural question emerging in the sheaf-theoretic setting of understanding quantum event spectra is if quantum phenomena are compatible with the principles of gauge geometry, where the base space of the involved fiber-bundles should not be required to be a space-time point manifold anymore. We remind here that the basic idea of gauge geometry, represented by a fiber bundle geometric structure, requires invariance under the local action of a gauge symmetry group.

It is an astonishing realization that the sheaf and fiber-bundle perspectives can actually be made functorially equivalent under the satisfaction of the mild conditions of the Serre-Swan theorem.¹⁹ According to this theorem, finitely generated projective modules, and thus locally free sheaves of modules called vector sheaves of states, defined over commutative observable algebra sheaves, are equivalent to vector bundles over a paracompact and Hausdorff topological base space.

Notwithstanding this functorial equivalence, sheaves in general can be displayed as fibrations more rich and flexible than fiber bundles, since instead of the local product structure involved in the definition of a fiber bundle, the much weaker condition of a local homeomorphism is required (for the case of topological spaces). From the other side, the set of sections of any vector bundle encoding the physical information of states always forms a vector sheaf of germs.

If we disassociate the semantics of gauge geometry from the usual metrical space-time point manifold base, then general paracompact and Hausdorff topological spaces may be utilized as base spaces of the associated bundles’ geometry. It is instructive to think of these base spaces as topological spaces of control variables. We emphasize that a base topological space of

18 Elias Zafiris, “Rosen’s Modeling Relations via Categorical Adjunctions,” *International Journal of General Systems* 41, no. 5 (2012).

19 Mallios, *Geometry of Vector Sheaves*; Mallios, *Modern Differential Geometry in Gauge Theories*, vol. 1.

control variables serves only as the carrier of a bundle geometric spectrum, and in particular it incorporates the local/global distinction required for the sheaf-theoretic interpretation of this spectrum.

In view of a gauge-theoretic conception of quantum geometric spectra, the following aspects acquire particular significance. First, in the case of fiber-bundle gauge geometry, the fiber over each point of a base space represents the local gauge freedom in the local definition of a physical information attribute. Thus, the vector space over each point of a base topological space of a vector bundle represents the local gauge freedom in the local definition of a state. Second, due to equivalence of vector bundles with vector sheaves of germs a sheaf-theoretic interpretation of the bundle geometric spectrum should be adopted, and in particular state or observable information at a point should be evaluated in terms of germs and not in a punctual way. Third, the sheaf-theoretic interpretation of gauge geometry introduces a subtle distinction between local and global physical information carriers. How are these aspects reflected in the case of quantum mechanics?

In the foundations of quantum mechanics, the widespread opinion is that phases are not important because a state is not actually described by a vector but by a ray or a projection operator so that it can always be removed by a suitable transformation. Moreover, due to the standard probability interpretation of a state vector at a single moment in time, physical significance has been assigned only to the modulus or magnitude of a state vector, whereas its phase has been ignored. Although it is true that the notion of phase can always be gauged away locally, this is not the case globally. Actually all typical global quantum information carriers are relative phases obtained by interference phenomena. These phenomena involve various splitting and recombination processes of beams whose global coherence is measured precisely by some relative phase difference. Generally, a relative phase can be thought of as a global physical attribute measuring the coherence between two distinct histories of events sharing a common initial and final point in the base space of control variables parameterizing the dynamic evolution of a quantum system. We note that due to the functional dependence of the dynamic evolution on the control variables, the state of a quantum system is parameterized implicitly by a temporal parameter through the control variables.

The short discussion above points to the essential non-spatiotemporal gauge nature of quantum geometry. In the simplest case, we may consider the sheaf-theoretic localization of a complex Hilbert space of states over its complex projective Hilbert space in order to obtain a line bundle of states. The line-bundle structure expresses the fact that a quantum state is defined locally up to an arbitrary complex phase, and thus the unitary group of complex phases plays the role of a local symmetry group of a gauge geometric spectrum constituted sheaf-theoretically.

The sheaf-theoretic interpretation of the spectrum implicates the existence of both local and global information carriers. A general

mechanism of a generation of global information carriers in the form of global phase factors of a geometric or topological origin has been initially formulated by Berry and modeled in line-bundle theoretic terms by Simon.²⁰ It has been demonstrated that a quantum system undergoing a slowly evolving (adiabatic) cyclic evolution retains a “global memory” of its motion after coming back to its original physical state. This “memory” is expressed by means of a complex phase factor in the state of the system, called Berry’s phase or the geometric phase. The cyclic evolution, which can be thought of as a periodicity property of the state of a quantum system, is driven by a Hamiltonian bearing an implicit time dependence through a base topological space of control variables. Due to the implicit temporal dependence imposed by the time parameterization of a closed path in the environmental parameters of the control space, this global geometric phase factor is thought of as “memory” of the motion since it encodes the global geometric or topological features of the control space. It should be stressed that a “global memory” is topological or geometric because it depends solely on the topology or geometry of the control space pathway along which the state is transported. It does not depend on the temporal metric duration of the historical evolution nor on the particular form of the dynamics that is applied to the system.

VIII QUANTUM GEOMETRIC SPECTRA

The fixed space-time point manifold independent approach to physical geometry proposed in this paper implies that quantum geometric spectra can be adequately understood only from the prism of a sheaf-theoretic interpretation, which fully utilizes the non-spatiotemporal gauge structure of quantum mechanics. According to this interpretation, global information carriers of quantum systems are encoded either as “global memories” of states during processes of cyclic evolution, or as relative phase differences between distinct histories of events parameterized by the same initial and final points with respect to a base topological space of control variables, through which the temporal evolution is implicitly defined. The particular significance of the concept of global topological and geometric phases, from the viewpoint of the sheaf-theoretic interpretation scheme, is that they mark a distinctive point in the history of quantum theory, where for the first time the significance of global information carriers as distinct entities from local ones is realized and made explicit through precise physical models, which have found

²⁰ M. V. Berry, “Quantal Phase Factors Accompanying Adiabatic Changes,” *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences* 392, no. 1802 (March 8, 1984): 45; Barry Simon, “Holonomy, the Quantum Adiabatic Theorem, and Berry’s Phase,” *Physical Review Letters* 51, no. 24 (December 12, 1983): 2167.

concrete experimental applications, like the quantum Hall effect and topological states of matter—for example, topological insulators.²¹

In particular, global relative phase factors in the gauge-theoretic setting of sheaves are obtained via an integration procedure of local gauge potentials over a contour, represented by a closed path or loop on a base space of control variables, which is implicitly parameterized continuously by an external temporal parameter. This nontrivial geometric or topological information of global significance is measured in terms of global holonomy phase factors via the procedure of lifting closed paths from the base space to the states or observables defined over it according to some parallel transport constraint (like the adiabatic one), technically called a connection. Due to the implicit time dependence parameterizing this procedure, if we continuously trace a loop on the base space, then this loop can be lifted to the implicitly evolving states or observables, which are represented as sections of a vector sheaf or an observable algebra sheaf respectively, over the base space. The particular global transformation undergone, for instance, by a state when it is parallel-transported along a closed curve on the base space is called the holonomy of the connection and is represented by a unitary group element. Thus, in this case the holonomy describes the global state transformation induced by cyclic changes in the controlling variables.

We conclude that although quantum geometric spectra may be locally probed in terms of observables, represented as self-adjoint operators, and their corresponding probabilities of events with respect to an orthonormal basis of eigenstates comprising a Boolean logical frame, so that local phases do not have any measurable significance, globally it is precisely the measurable relative phase differences that maintain the quantum coherence information. A global phase factor is not represented by any self-adjoint operator, but it is represented by means of a holonomy unitary group element, to be thought of as the accumulated “memory” due to periodicity with respect to an environment of control variables. The explicitly different nature of physical information carriers as we make the transition from the local to the global level of describing quantum geometric spectra, and its inverse, requires an adequate interpretation scheme where this distinction is appropriately modeled. The main thesis of this work is that a natural approach to the physical geometry of the quantum regime should be carried out by utilizing the theory of vector sheaves of states equipped with a connection. Thus in this manner quantum geometric spectra admit a sheaf-theoretic interpretation in combination with the non-spatiotemporal gauge structure of quantum mechanics.

21 M. Z. Hasan and C. L. Kane, “Colloquium: Topological Insulators,” *Reviews of Modern Physics* 82 (2010): 3045; M. Z. Hasan and J. E. Moore, “Three-Dimensional Topological Insulators,” *Annual Review of Condensed Matter Physics* 2 (2011); Y. Zhang, Y.-W. Tan, H. L. Stormer, and P. Kim, “Experimental Observation of the Quantum Hall Effect and Berry’s Phase in Grapheme,” *Nature* 438 (2005): 201.

