Category Theoretic Generalization of the Complementarity Principle in Quantum Mechanics

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Abstract

This study aims to provide a generalization of the complementarity principle in quantum theory through the establishment of partial structural congruence relations between the quantum and Boolean kinds of event structure. Specifically, on the basis of the existence of a categorical adjunction between the category of quantum event algebras and the category of presheaves of variable Boolean event algebras, we establish a twofold complementarity scheme consisting of a generalized/global and a restricted/local conceptual dimension, where the latter conception is subordinate to and constrained by the former. In this respect, complementarity is no longer understood just as a relation between mutually exclusive experimental arrangements or contexts of co-measurable observables, as envisaged by the original conception, but it is primarily comprehended as a relation between two hierarchically different structural kinds of event structure that can be brought into partial congruence by means of the established adjunction. It is further argued that the proposed category-theoretic scheme of complementarity naturally advances a contextual realist philosophical stance towards our deeper understanding of the microphysical nature of reality.

1 On the Structure and Function of Complementarity

The concept of complementarity was introduced in quantum mechanics by Niels Bohr in his lecture at the International Congress of Physics, held in the Italian city of Como, in 1927, although the essential ideas reach back to 1925. He proposed the complementarity doctrine as the new framework for the ordering of physical experience in the microscopic domain. Apart from being conceived as an inseparable aspect of quantum theory, Bohr himself envisaged his views on complementarity as forming also part of a wider regulative epistemological principle concerning an unavoidable analysis of the limitations imposed on the applicability of the concepts by which our experiences are described. As he puts it, "...the lessons taught us by recent developments in physics regarding the necessity of a constant extension of the frame of concepts appropriate for the classification of new experiences leads us to a general epistemological attitude which might help us to avoid apparent conceptual difficulties in other fields of knowledge as well" (Bohr 1937, p. 289). In this respect, complementarity according to Bohr arises, in its most general form, whenever a sharp distinction between the (observing) subject and the (observed) object cannot be made. The closing sentence of his Como lecture is telling: "I hope, however, that the idea of complementarity is suited to characterize the situation, which bears a deep-going analogy to the general difficulty in the formation of human ideas, inherent in the distinction between subject and object" (Bohr 1928, p. 590).

In spite of the numerous articles Bohr wrote on the topic, striving to develop the concept of complementarity into a definite interpretative viewpoint, he never gave an exact definition or a sharp formulation. If restricted to the domain of applicability of quantum theory, nonetheless, a definition may be surmised from several of his statements. In this respect, Bohr's varying phraseology and his lack of explicit reflection on the continuous refinements of the complementarity concept is recognized as another hindrance (e.g., Jammer 1974; Beller 1999). Louis de Broglie, for instance, described Bohr as "the Rembrandt of contemporary physics", with a "predilection for 'obscure clarity'" (quoted in Honner 1987, p. 24). Einstein too, after decades of debating the conceptual foundations of quantum theory with Bohr, insisted that the sharper meaning of complementarity eluded him (Einstein 1949, p. 674). Recent years, however, have seen a re-appreciation of Bohr's views of quantum theory (e.g., Dieks 2017 and references therein), and the present study is an attempt of clarifying, generalizing and applying the complementarity concept to the contemporary investigations of the interpretation of quantum mechanics, aiming at its further fruitful development.

Bohr expressed the concept of complementarity in various terms on different occasions, according to the physical situation under consideration. In the early phase of his interpretation of quantum physics, he used complementarity to characterize a binary relation between, for example, *physical pictures* such as the particle picture and the wave picture, or *physical variables* such as position and momentum, or *physical descriptions* such as the space-time description and the dynamical description based on the consideration of conservation laws. As a view about the nature of physical pictures, complementarity declares that each atomic system has a wave aspect and a particle aspect both of which being equally fundamental from a classical viewpoint but yet *mutually incompatible* within a single quantum phenomenon. As a view about the assignment of physical variables, it designates that conjugate quantities, like position and momentum, cannot be ascribed *exact simultaneous values*; the corresponding quantitative limits are to be specified by Heisenberg's uncertainty relations. As a view about physical descriptions, it asserts that there are *mutually exclusive conditions* for the unambiguous use of the kinematical concepts involving detailed space-time coordination and dynamical concepts involving conservation laws of momentum and energy (the latter sometimes being referred to as a "requirement of causality"). The incompatibility between spatio-temporal $(t, \mathbf{r})$ and dynamical concepts $(E, \mathbf{p})$ is traced back to the unavoidable presence of Planck's quantum of action in the atomic domain, imposing an in principle uncontrollable transfer of momentum and energy to fixed scales and synchronized clocks needed to define the reference frame for specifying exact trajectories.

In his later writings, demarcated essentially by the time of his reply to the EPR argument, Bohr's viewpoint of complementarity gradually converges towards a preferred expression of complementarity as a relationship between phenomena demanding mutually exclusive experimental arrangements. No doubt, Bohr's concept of a quantum phenomenon forms the logical basis for his expounded notion of complementarity. According to Bohr, the fundamental difference with respect to the analysis of phenomena in classical and in quantum physics is that in the latter the 'measured system', the 'measuring apparatus' and their 'mutual interaction' constitute an indivisible whole in such a manner that analysis into disjoint elements is not admissible without destroying or altering the original phenomenon. "The essential wholeness of a proper quantum phenomenon', writes Bohr, "finds indeed logical expression in the circumstance that any attempt at its well-defined subdivision would require a change in the experimental arrangement incompatible with the appearance of the phenomenon itself" (Bohr 1937, p. 291). In this respect, he strongly advocated that the use of the word "phenomenon" must exclusively refer to "observations obtained under specified circumstances, including an account of the whole experiment" (Bohr 1948, p. 317). Thus, for Bohr, a quantum phenomenon is not fully specified until the whole experimental procedure is determined:

> It is certainly far more in accordance with the structure and interpretation of the quantum mechanical symbolism, as well as with elementary epistemological principles, to reserve the word "phenomenon" for the comprehension of the effects observed under given experimental conditions.
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> These conditions, which include the account of the properties and manipulation of all measuring instruments essentially concerned, constitute in fact the only basis for the definition of the concepts by which the phenomenon is described (Bohr 1939, p. 316).

The general epistemological thesis advanced by Bohr towards this direction, especially as regards the ordering and subsequent synthesis of phenomena, consists in *conditionalizing* the meaning of theoretical concepts to the context of a possible experiment. It is only by knowing the experimental conditions that the concepts used for the description of experimental results can become relevant in the context of a physical inquiry.

In this connection, it is after his study of the EPR incompleteness argument that Bohr clearly realized that an unambiguous interpretation of the quantum mechanical formalism can be achieved only by an equally unambiguous description of the phenomena, with the experimental arrangement specified in classical terms, at least to the degree that it functions as a measuring instrument:

> In the system to which the quantum mechanical formalism is applied, it is of course possible to include any intermediate auxiliary agency employed in the measuring processes. ... The only significant point is that in each case some ultimate measuring instruments, like the scales and clocks which determine the frame of space-time coordination on which, in the last resort, even the definitions of momentum and energy quantities rest must always be described entirely on classical lines, and consequently kept outside the system subject to quantum mechanical treatment.
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> ... It is just in this sense that phenomena defined by different concepts, corresponding to mutually exclusive experimental arrangements, can unambiguously be regarded as complementary aspects of the whole obtainable evidence concerning the objects under investigation (Bohr 1939, pp. 316-317).

With this terminology, the complementary aspects between conjugate variables, wave-particle pictures, or physical descriptions, as previously specified, can be derived from a primary notion of complementarity, by referring to phenomena appearing under mutually exclusive experimental arrangements, constituting also Bohr's ultimate formulation on the issue. In Bohr's own words:

> The impossibility of combining phenomena observed under different experimental arrangements into a single classical picture implies that such apparently contradictory phenomena must be regarded as complementary in the sense that, taken together, they exhaust all well-defined knowledge about the atomic objects (Bohr 1962, p. 25).

Bohr's preceding expression of "apparently contradictory phenomena", elsewhere being characterized as "apparently incompatible" (e.g., Bohr 1937, p. 291), when referred to complementary aspects of the same object, can only mean incompatible on classical considerations alone. This is rendered particularly clear in the following passage:

> Information regarding the behavior of an atomic object obtained under definite experimental conditions may, however, according to a terminology often used in atomic physics, be adequately characterized as complementary to any information about the same object obtained by some other experimental arrangement excluding the fulfillment of the first conditions. Although such kinds of information cannot be combined into a single picture by means of ordinary concepts, they represent indeed equally essential aspects of any knowledge of the object in question which can be obtained in this domain (Bohr 1938, p. 26).

Any two phenomena, then, can be called complementary if, by referring to the same object, they are *mutually exclusive* in the sense that the two corresponding experimental arrangements employed for the manifestation of the phenomena and the information which they yield are incompatible, while at the same time being *jointly necessary* for an exhaustive as possible description of the object concerned as allowed by quantum theory.

As an illustration, let us consider concisely Bohr's characteristic example of complementary phenomena involving the measurement of the position and momentum observables. The important point in Bohr's thinking is that position and momentum cannot both be sharply defined in one and the same measurement context. For, the context needed for a measurement of the position observable is not compatible with that needed for a measurement of the momentum observable. This suggests that they should be measured independently, using mutually exclusive experimental arrangements, thus rendering the notions of position and momentum complementary in the sense that they are not simultaneously applicable. Yet, this holds true for any conjugate pair of observables A and B satisfying the canonical commutation relation $[A, B]_- = i\hbar I$. The existence of complementary physical quantities and relations is ubiquitous, reflecting an essential feature of the quantum mechanical formalism. The historiographical record of the conceptual development of quantum physics, however, reveals that there is no clear consensus in the literature as to the exact meaning of Bohr's complementarity principle (e.g., Faye and Folse 1994; Howard 2004). Nonetheless, if projected into the formalism of conventional Hilbert space quantum mechanics, we arrive at a characterization of the functioning of complementarity in quantum theory that may attract general acceptance. On this basis, the structure of the hard core of complementarity may be decomposed as follows:

(i) A quantum mechanical observable is well-defined within the experimental arrangement or measurement context serving to measure that observable.

(ii) Upon specifying an experimental arrangement and, therefore, selecting an observable to be measured, atomic propositions referring to the properties of a quantum mechanical system acquire, within the associated measurement context, *determinate* truth values, thus, forming a Boolean subalgebra in the non-Boolean logical structure of the system.

(iii) Complementary or incompatible quantum mechanical observables correspond to different *mutually exclusive* experimental arrangements, defining different aspects of reality, which cannot be unified in a single classical Boolean picture.

(iv) Each complementary aspect of reality, thus defined, is *jointly necessary* for a comprehensive account of the quantum phenomenon.

It is worth noting that in recent years the complementarity concept has been shown to underlie many aspects of quantum information technology, ranging from quantum superdense coding (Coles 2013) to information complementarity (Zhu 2015) and from the security of quantum cryptography (Gisin et al. 2002) to quantitative complementarity relations in interferometry (Bramon et al. 2004) and delayed-choice experiments (Ma et al. 2016). In this respect, our attempt to generalize and accommodate the idea of complementarity within a formal description of quantum mechanics may have implications in all preceding areas.

In this work we aim to establish that the concept of complementarity, pertaining to any non-Boolean quantal physical description, is mathematically formalized, conceptually further clarified and physically generalized, through the categorical notion of *adjunction* that has been recently proved to exist between the *category of quantum event algebras* and the *category of presheaves on Boolean event algebras* (XX 2001; see also XX & YY (2013) for a more detailed treatment including in addition the involved logical aspects). We will demonstrate, in particular, that the notion of categorical adjunction embodies the semantics of complementarity and, furthermore, explicates its functioning through a twofold scheme consisting of a *generalized/global* and a *restricted/local* conceptual dimension.

To this task, we proceed, in view of the proposed category theoretic framework, to a generalization of quantum complementarity in the following sense. At the first level of generalization, instead of considering a quantum mechanical observable as an impenetrable unit, we focus, via the spectral representation theorem, on the *complete Boolean subalgebra* generated by the projection operators in the spectral measure of the observable, for instance, the complete Boolean subalgebra associated with the position observable, or the analogous subalgebra associated with momentum, and so on. Instead of referring to conjugate pairs of observables, the complementarity concept now concerns the relationship between the *mutually incompatible* Boolean subalgebras in the non-Boolean logical structure of a quantum system. Instead of adhering to 'mutual exclusiveness' in an absolute sense by restricting complementary phenomena only to those corresponding to 'maximally incompatible' observables, so that measuring one yields no information whatsoever about a possible measurement result of the other, we adopt a weaker version of complementarity allowing *partial compatibility of overlapping* Boolean subalgebras associated to spectral measures of quantum mechanical observables.

It is, however, the second level of generalization that is unique to the proposed approach. Our category theoretic analysis of quantum complementarity is based on the existence of *partial structural congruences* between the *quantum* and *Boolean kinds* of event structure. Through the existence of a pair of *adjoint functors* between the category of presheaves of the Boolean kind of structure and the category of the quantum kind of structure, we establish a twofold complementarity scheme which constitutes an instance of the concept of adjunction. It should be remarked that adjointness, and the related pair of adjoint functors, is a concept of fundamental logical and mathematical importance contributed to mathematics by category theory (e.g., Awodey 2010, p. 207; Goldblatt 2006, p. 438).

The typical mathematical structure associated with the conception of events is a structure endowed with an ordering relation. In the Hilbert space formalism of quantum theory, events are considered as closed subspaces of a separable, complex Hilbert space, H, corresponding to a quantum system. Then, the quantum event structure is identified with the lattice of closed linear subspaces of H, ordered by inclusion, and carrying an orthocomplementation operation which is given by the orthogonal complement of the closed subspaces, thus forming a complete, atomic, orthomodular lattice. In this framework, due to the one-to-one correspondence between the set of all closed subspaces and the set of all projection operators of H, the equivalence of events and atomic propositions is made literal (e.g., Rédei 1998; Dalla Chiara et al. 2004). In effect, a non-Boolean logical structure is induced which has its origin in quantum theory. On the contrary, the logic underlying the event or propositional structure of classical physics is Boolean, in the sense that the algebra of propositions of a classical system is isomorphic to the lattice of subsets of phase space, a Boolean lattice that can be interpreted semantically by a two-valued truth-function. This means that to every classical mechanical proposition one of the two possible truth values 1 (true) and 0 (false) can be assigned. From a physical standpoint, this is essentially due to the fact that classical physics views objects-systems as bearers of determinate properties. Accordingly, in a Boolean propositional structure, all propositions, being mutually compatible, are simultaneously decidable.

The complementarity scheme developed in the sequel is based on the physically significant observation that whereas the totality of all propositions referring to properties of a quantum mechanical system and, henceforth, the totality of all associated experimental/empirical facts can only be represented in a globally non-Boolean logical structure, the acquisition of every single fact depends on a locally Boolean context. An intuitive understanding of this insight may be advanced by pointing out that although Kochen-Specker's (1967) theorem in quantum logic excludes the possibility of a *global* two-valued truth-functional assignment on the non-Boolean logical structure of a quantum system, nonetheless, there always exists a *local* two-valued truth-functional assignment with respect to a complete Boolean algebra of projection operators on the Hilbert space of the system. More precisely, each self-adjoint operator representing an observable has associated with it a Boolean subalgebra which is identified with the Boolean algebra of projection operators belonging to its spectral decomposition. Hence, given a set of observables of a quantum system, there always exists a complete Boolean algebra of projection operators, namely, a local Boolean subalgebra of the global non-Boolean event algebra of a quantum system with respect to which a local two-valued truth-functional assignment is meaningful, if and only if the given observables are commutative, or, physically speaking, simultaneously measurable. Consequently, the possibility of local two-valued truth-functional assignments of the global non-Boolean event algebra of a quantum system points to the assumption that *complete Boolean algebras furnish the role of local Boolean logical frames for contextual true/false value assignments* (YY & XX 2017). In this respect, a Boolean algebra in the lattice of quantum events picked by an observable to be measured instantiates locally a physical context, which may serve as a *reference frame relative to which results of measurement are being coordinatized*, setting, henceforth, the conceptual ground for the development of the ideas of complementarity. The modeling scheme we propose in order to implement this idea in a universal way, so that the *global structure of a quantum system* to be modeled categorically as a *reflection of the category of presheaves of variable local Boolean frames*, uses the technical apparatus of categorical sheaf theory (Mac Lane and Moerdijk 1992, Borceux 1994, Awodey 2010).

2 Universal Solution to the Functorial Representation of a Quantum Event Algebra via Boolean Frames

2.1 Generalized conceptual framework

The preceding analysis reveals the significant fact that the acquisition of every single experimental event in the quantum domain requires taking explicitly into consideration the complete Boolean algebra of projection operators, which spectrally resolves the observable that this event refers to. It should be underlined that such a complete Boolean algebra bears the status of a logical structural invariant characterizing a whole algebra of observables commuting with the considered one, and thus potentially may give rise to the same event. In other words, a complete Boolean algebra of projections is the pertinent *logical structural invariant* of a whole commutative algebra of observables, in the sense that it instantiates the simultaneous spectral resolution of all these commuting observables. Since there exist complementary or incompatible quantum mechanical observables non-commuting with any considered commutative algebra of observables, there exists a *multiplicity* of possible Boolean algebras of orthogonal projections furnishing an invariant only in the context of all those commuting observables that can be simultaneously resolved spectrally.

In this manner, the conceptual framework surrounding the notion of complementarity should pertain to the fact that in the quantum domain there does not exist a unique and universal logical structural invariant with respect to which all possible observables can be spectrally resolved simultaneously, but, in contradistinction, there is a multiplicity of such invariants attached only to commutative subalgebras of observables. Therefore, although a quantum event structure is globally non-Boolean, it can be qualified spectrally only in terms of Boolean event structures functioning as logical structural invariants of co-measurable families of observables. Since these invariants are not global, it seems reasonable to consider them as local ones, where the locality requirement refers precisely to the physical context of all commuting observables that can be simultaneously resolved spectrally by an invariant of this form. We conclude that a complete Boolean algebra of projection operators in its function as a spectral invariant of a commutative subalgebra of quantum observables furnishes the role of a Boolean logical frame with respect to which a quantum event can be qualified and lifted to the empirical level. Thus, the consideration of each local Boolean frame serves as a natural pre-condition for establishing a local invariant logical structure for the event evaluation of all co-measurable observables in this frame. Due to the absence of a global, uniquely defined Boolean frame, it is necessary to consider all possible local ones together with their interrelations. From this perspective, the crucial problem is if there exists a universal way to specify a quantum event structure via the literal adjunction of local spectral invariants to it, objectified by variable local Boolean frames, under their intended physical interpretation as logical probing frames for the manifestation and concomitant classification (or contextualization) of quantum events.

We advance the view that the conceptual net of ideas enveloping the notion of complementarity pertains exactly to the possibility of existence of a universal solution to the preceding problem. The existence of a universal solution essentially renders the global orthomodular lattice structure of quantum events empirically inert without the adjunction of local spectral invariants to it. The role of these invariants is to induce partial or local structural congruences with Boolean event structures pertaining to all typical contexts of measurement. Intuitively, the multiplicity of applicable local Boolean frames allows the filtration and separation of several resolution sizes and types of quantum observable grain depending on the qualification of the corresponding spectral orthogonal projections. The objective of a universal solution is to derive the non-directly accessible quantum kind of event structure by means of all possible partial structural congruences with the directly accessible Boolean kind of event structure, forced by means of adjoining local spectral invariants as probing frames to the former. In this setting, the major role is subsumed by all possible structural relations allowed among the probing Boolean frames, the spectra of which may be disjoint or nested or overlapping and interlocking together non-trivially. It is the realization that distinct Boolean frames may have non-trivial intersections, or, more generally, non-trivial pullback compatibility relations with respect to a global quantum event structure that extends the semantics of complementarity beyond the standard standpoint concerning disjoint Boolean frames. In turn, the universality of such a viable solution poses the necessity to formulate the problem in functorial terms, i.e., non-dependent on the artificial choice of particular Boolean frames. It has been first proposed in (XX 2001) that the subtlety of a complete Boolean algebra of orthogonal projections, in its function to act as a local logical frame in a global quantum event algebra, as well as the many-to-one correspondence with the notion of a quantum event, requires a category theoretic framework of interpretation, based on the aforementioned notion of partial structural congruence.

The basic conceptual steps involved in the realization of the suggested approach may be summarized as follows. First, we introduce the notion of a *covering scheme* of a quantum event algebra consisting of epimorphic families of local Boolean probing frames. From the physical point of view, this attempt amounts to analyze or 'coordinatize' the information contained in a quantum event algebra L by means of structure preserving morphisms $B \to L$, having as their domains locally defined Boolean event algebras B. Of course, any single map from a Boolean domain to a quantum event algebra does not suffice for a complete determination of the latter's information content. In fact, it contains only the amount of information related to a particular Boolean frame, prepared for a specific kind of measurement, and inevitably is constrained to represent exclusively the abstractions associated with it. This problem is confronted by employing a sufficient amount of maps, organized in terms of *covering sieves*, from the coordinatizing Boolean domains to a quantum event algebra so as to cover it completely. These maps furnish the role of local *Boolean covers* for the filtration of the information associated with a quantum structure of events, in that, their domains B provide Boolean coefficients associated with typical measurement situations of quantum observables. The local Boolean covers capture, in essence, separate, complementary features of the quantum system under investigation, thus providing a structural decomposition of a quantum event algebra in the descriptive terms of Boolean probing frames. Technically, this is described by an action of a category of variable local Boolean frames on a global quantum event algebra, forming a *presheaf*. Second, the coherent realization of the multitude of various possible different coordinatizations of a quantum event algebra demands the satisfaction of *partial compatibility* between *overlapping* local Boolean covers. This guarantees an efficient pasting code between different localizations of a quantum algebra of events. Technically, this is described by the notion of a *Boolean localization functor*, or, equivalently, by a *structure sheaf* of Boolean coefficients of a quantum event algebra. Category-theoretically, one may think of a Boolean localization functor as consisting of epimorphic families or diagrams of Boolean covers interconnected together targeting a quantum event algebra. Third, we establish the necessary and sufficient conditions for the isomorphic representation of quantum event algebras in terms of Boolean localization functors, giving rise to sheaves of variable local Boolean frames.

As pointed out in Sect. 1, the key distinguished method used in order to establish these conditions is based on the existence of a *categorical adjunction*, consisting of a pair of adjoint functors, between the category of (pre)sheaves of variable local Boolean frames and the category of quantum event algebras. In essence, this

Boolean frames-quantum adjunction provides a bi-directional and functorial translational mechanism of encoding and decoding information between the Boolean and quantum kind of structures respecting their distinctive form. The crucial notion of adjunction, which does not have a set-theoretic counterpart, provides the mathematical means of *relating relations* and, thus, establishing *local* or *partial structural congruences* between categories of quantum information structures and categories of sufficiently understood Boolean information structures. It is precisely this category-theoretic fact which allows a proper generalization of the quantum complementarity principle by *relating internally* and *bi-directionally* the variable and local Boolean with the quantum global level of structure.

2.2 Functorial representation of quantum event algebras via Boolean frames

Category theory provides a general theoretical framework for the study of structured systems in terms of their *mutual relations* and *admissible transformations*. Contrary to the atomistic approach of set theory which crucially depends on the concept of elements-points and the membership relationship of a variable x in a set X, $x \in X$, in category theory the notion of morphism or arrow undertakes primary role. A morphism, for instance, $f : A \to B$ in a category $\mathcal{C}$ expresses one of the many possible ways in which the object A relates to the object B within the context of category $\mathcal{C}$. However counterintuitive it may initially appear, in category theory the nature of the objects is a *derivative* aspect of the patterns described by the morphisms or mappings that connect the objects. In fact, an object can be completely classified and uniquely derived up to an isomorphism by the network of all morphisms – the structure-preserving relations – targeting this object within the same category. Most significantly, as previously underlined, this does not exclude the inter-level relational determination of objects belonging to other categorical species of structure, under the condition that there exists a bi-directional functorial correlation between them, formulated in the language of adjunctions. It is precisely the latter development that gradually introduced into category theory a paradigm change in understanding structures of general types and paved the way for forming bridges between seemingly unrelated mathematical disciplines.

Evidently, besides the consideration of morphisms between objects in a category, there also exist, at the next higher level, structure-preserving mappings between categories, namely functors, and, further on, mappings between functors called natural transformations. It is interesting to note that Mac Lane, one of the two inventors of category theory, once observed that the notion of "category has been defined in order to be able to define "functor" and "functor" has been defined in order to be able to define "natural transformation"" (Mac Lane 1998, p. 18). Mac Lane's remark apart from signifying the importance of the latter notion, which, in fact, historically preceded the formulation of the notions of category and functor, it also points to the fact that category theory is inherently hierarchical. It is able to formally define and operate with both structures of systems and structures of structures encompassing layers of increasing abstraction and complexity. Thus, essential categorical notions and constructions present themselves in ascending levels of generality and depth: category, functor, natural transformation, adjointness, higher-order categories, etc.

The relationships between the morphisms in a category or between the functors in a functor category provide the patterns that allow to capture the *shared structure* or *generic common properties* existing between different kinds of systems. The innovative conceptual spirit of category theory provides the framework from within we can analyze the shared structure of abstract kinds of complex systems in terms of the structure-preserving transformations existing between them. The basic categorical principles that we adopt in the subsequent analysis are summarized as follows:

(i) To each kind of mathematical structure used to represent a system, there corresponds a *category* whose objects have that structure and whose arrows or morphisms preserve it.

(ii) To any natural construction on structures of one kind, yielding structures of another kind, there corresponds a *functor* from the category of the first specified kind to the category of the second. The implementation of this principle is associated with the fact that a construction is not merely a function from objects of one kind to objects of another kind, but must preserve the essential structural relationships among objects, that is, identity morphisms and compositions.

(iii) To each natural translation between two functors having identical domains and codomains, there corresponds a *natural transformation*.

In view of the aforementioned principles, the functorial representation of a quantum event algebra in terms of suitably qualified functors of Boolean probing frames requires distinct notions of Boolean and quantum categorical event structures, respectively.

A *Boolean categorical event structure* is a small category, denoted by $\mathcal{B}$, which is called the category of Boolean event algebras. The objects of $\mathcal{B}$ are complete Boolean algebras of events and the arrows are the corresponding Boolean algebraic homomorphisms.

A *quantum categorical event structure* is a locally small co-complete category, denoted by $\mathcal{L}$, which is called the category of quantum event algebras. The objects of $\mathcal{L}$ are quantum event algebras, i.e., complete orthomodular lattices of events, and the arrows are quantum algebraic homomorphisms.

According to the preliminary conceptual analysis in Sect. 2.1, we consider a *Boolean modeling* or *shaping functor* of $\mathcal{L}$, $\mathbf{M} : \mathcal{B} \to \mathcal{L}$, which assigns to each Boolean event algebra in $\mathcal{B}$ (constitutive of a model category) the underlying quantum event algebra from $\mathcal{L}$, and to each Boolean homomorphism the underlying quantum algebraic homomorphism. Hence, $\mathbf{M}$ acts as a forgetful functor, not taking into account the extra Boolean structure of $\mathcal{B}$. Because of the fact that an opposite-directing functor from $\mathcal{L}$ to $\mathcal{B}$ is not feasible, since a quantum event algebra cannot be realized within any Boolean event algebra, we seek for an extension of $\mathcal{B}$ into a larger categorical environment where such a realization becomes possible. This extension should conform with the intended physical semantics of adjoining a multiplicity of Boolean spectral invariants to a quantum event algebra, objectified as probing Boolean frames of the latter. In this perspective, the global information encoded in a quantum event algebra is expected to be recovered in a structure-preserving way by an appropriate sheaf-theoretic construction gluing together categorical diagrams of locally variable Boolean frames. As shown in the sequel, a topological gluing construction of this form can only take place by extending the categorical level of $\mathcal{B}$ to the categorical level of diagrams in $\mathcal{B}$, forming, in turn, the functor category $\mathbf{Sets}^{\mathcal{B}^{op}}$, which actually constitutes the free completion of $\mathcal{B}$ under colimits. The existence of colimits expresses in category theoretical language the basic intuition that a complex object may be conceived as arising from the structured interconnection of partially or locally defined information carriers in a specified covering system.

It is apparent that the realization of this extension process requires initially the construction of the *functor category of presheaves of sets on Boolean event algebras*, denoted by $\mathbf{Sets}^{\mathcal{B}^{op}}$, where we remind that $\mathcal{B}^{op}$ is the opposite category of $\mathcal{B}$. The objects of $\mathbf{Sets}^{\mathcal{B}^{op}}$ are all functors, $\mathbf{P} : \mathcal{B}^{op} \to \mathbf{Sets}$, with morphisms all natural transformations between such functors. Each object $\mathbf{P}$ in the category of presheaves $\mathbf{Sets}^{\mathcal{B}^{op}}$ is a contravariant set-valued functor on $\mathcal{B}$, called a *presheaf of sets* on $\mathcal{B}$ (e.g., Borceux 1994, p. 195). The functor category of presheaves on Boolean event algebras, $\mathbf{Sets}^{\mathcal{B}^{op}}$, provides the instantiation of a structure known as *topos*. A topos exemplifies a well-defined notion of a universe of variable sets. It can be conceived as a local mathematical framework corresponding to a generalized model of set theory or as a generalized topological space (e.g., Bell 1988/2008).

In order to obtain a clear understanding of the structure of the functor category $\mathbf{Sets}^{\mathcal{B}^{op}}$, we note that a functor $\mathbf{P}$ in $\mathbf{Sets}^{\mathcal{B}^{op}}$ can be thought of as constructing an image of $\mathcal{B}$ in $\mathbf{Sets}$ contravariantly or as a contravariant translation of the "language" of $\mathcal{B}$ into that of $\mathbf{Sets}$. Thus, a functor $\mathbf{P}$ is a structure-preserving morphism of these categories, that is, it preserves compositions and identities. Given another such contravariant functor $\mathbf{Q}$ of $\mathcal{B}$ into $\mathbf{Sets}$ we need to compare them. This can be done by assigning, for each object B in $\mathcal{B}$, a transformation, $\tau_B : \mathbf{P}(B) \to \mathbf{Q}(B)$, which compares the two images of the object B. Not any morphism will do, however, as we intend the construction to be parametric in B rather than ad hoc. Since B is an object in $\mathcal{B}$, whereas, $\mathbf{P}(B)$ is in $\mathbf{Sets}$, we cannot link them by a morphism. Rather, the goal is that the transformation should respect the morphisms of $\mathcal{B}$ or, in other words, the interpretations of $f : B \to C$ by $\mathbf{P}$ and $\mathbf{Q}$ should be compatible with the transformation under τ. Then τ is a *natural transformation* in the functor category of presheaves, $\mathbf{Sets}^{\mathcal{B}^{op}}$, on Boolean event algebras.

Alternatively, an object $\mathbf{P}$ of $\mathbf{Sets}^{\mathcal{B}^{op}}$ may be understood as a right action of the category $\mathcal{B}$ on a set of observables, which is partitioned into a variety of Boolean spectral kinds parameterized by the Boolean event algebras B in $\mathcal{B}$. Such an action $\mathbf{P}$ is equivalent to the specification of a *diagram* in $\mathcal{B}$, to be thought of as a $\mathcal{B}$-variable set forming a presheaf $\mathbf{P}(\mathrm{B})$ on $\mathcal{B}$. For each Boolean algebra B of $\mathcal{B}$, $\mathbf{P}(\mathrm{B})$ is a set, and for each Boolean homomorphism $f : C \to B$, $\mathbf{P}(f) : \mathbf{P}(B) \to \mathbf{P}(\mathrm{C})$ is a set-theoretic function, such that, if $p \in \mathbf{P}(\mathrm{B})$, the value $\mathbf{P}(f)(p)$ for an arrow $f : C \to B$ in $\mathcal{B}$ is called the *restriction* of p along f and is denoted by $\mathbf{P}(f)(p) = p \cdot f$. From a physical viewpoint, the purpose of introducing the notion of a presheaf $\mathbf{P}$ on $\mathcal{B}$, in the environment of the functor category $\mathbf{Sets}^{\mathcal{B}^{op}}$, is to identify an element of $\mathbf{P}(\mathrm{B})$, that is, $p \in \mathbf{P}(\mathrm{B})$, with an event observed by means of a physical procedure over a Boolean domain cover for a quantum event algebra. As further analyzed in Sect. 3.2, this identification forces the interrelation of observed events, over all Boolean probing frames of the base category $\mathcal{B}$, to fulfil the requirements of a uniform and homologous fibred structure.

In the categorical setting of the functor category, $\mathbf{Sets}^{\mathcal{B}^{op}}$, it becomes possible to realize a quantum event algebra L in $\mathcal{L}$ in terms of a distinguished presheaf of Boolean event algebras, which emerges due to the invariant spectral capacity of the latter to act locally as logical probing frames of a quantum event algebra. These Boolean frames of an L in $\mathcal{L}$ are objectified as L-targeting morphisms in $\mathcal{L}$,

$$\psi_B : \mathbf{M}(B) \to L, \tag{2.1}$$

being interrelated by the operation of presheaf restriction. This means explicitly that for each Boolean homomorphism $f : C \to B$, if $\psi_B : \mathbf{M}(B) \to L$ is a Boolean frame of L, the corresponding Boolean frame $\psi_C : \mathbf{M}(C) \to L$ is given by the restriction or pullback of ψ_B along f, denoted by $\psi_B \cdot f = \psi_C$. Thus, we obtain a contravariant presheaf functor,

$$\mathbf{R}(L)(-) = Hom_{\mathcal{L}}(\mathbf{M}(-), L) \ , \tag{2.2}$$

called the *functor of Boolean frames* of L. Since the physical interpretation of the functor $\mathbf{R}(L)(-)$ refers to the functorial realization of a quantum event algebra L in $\mathcal{L}$ in terms of Boolean algebras B in $\mathcal{B}$, we think of $\mathbf{R}(L)(-)$ as the *variable Boolean spectral functor* of L through the structured multitudes of local Boolean frames adjoined to it. In this manner, a Boolean event algebra B, induced locally by the measurement of a quantum observable via the spectral decomposition of the corresponding self-adjoint operator, acts as a partial coordinatizing logical probing frame of a quantum event algebra L in terms of Boolean coefficients.

It is now important to stress that each Boolean algebra B in $\mathcal{B}$ gives naturally rise to a contravariant representable Hom-functor $\mathbf{y}^B := \mathbf{y}[B] := Hom_{\mathcal{B}}(-, B)$. This functor defines a $\mathcal{B}$-variable set, or, equivalently, a presheaf of sets on $\mathcal{B}$ for each B in $\mathcal{B}$. Concomitantly, the functor $\mathbf{y}$ is a full and faithful functor from $\mathcal{B}$ to the contravariant functors on $\mathcal{B}$, i.e.,

$$\mathbf{y} : \mathcal{B} \longrightarrow \mathbf{Sets}^{\mathcal{B}^{op}}, \tag{2.3}$$

defining the Yoneda embedding $\mathcal{B} \hookrightarrow \mathbf{Sets}^{\mathcal{B}^{op}}$ (Mac Lane and Moerdijk 1992, p. 26; Awodey 2010, pp. 187-189). According to the Yoneda Lemma, there exists an injective (one-to-one) correspondence between elements of the set $\mathbf{P}(B)$ and natural transformations in $\mathbf{Sets}^{\mathcal{B}^{op}}$ from $\mathbf{y}[B]$ to $\mathbf{P}$ and this correspondence is natural in both $\mathbf{P}$ and B, for every presheaf of sets $\mathbf{P}$ in $\mathbf{Sets}^{\mathcal{B}^{op}}$ and Boolean algebra B in $\mathcal{B}$. The functor category of presheaves of sets on Boolean event algebras $\mathbf{Sets}^{\mathcal{B}^{op}}$ is a complete and co-complete category. Thus, the Yoneda embedding $\mathbf{y} : \mathcal{B} \longrightarrow \mathbf{Sets}^{\mathcal{B}^{op}}$

constitutes the intended free completion of $\mathcal{B}$ under colimits of diagrams of Boolean event algebras.

The deep meaning of this fact is that if we consider a Boolean modeling functor $\mathbf{M} : \mathcal{B} \to \mathcal{L}$ there exists precisely one corresponding uniquely defined, up to isomorphism, colimit-preserving functor $\widehat{\mathbf{M}} : \mathbf{Sets}^{\mathcal{B}^{op}} \to \mathcal{L}$, such that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{B} \arrow[d, tail, "\mathbf{y}"'] \arrow[dr, "\mathbf{M}"] & \\
\mathbf{S}ets^{\mathcal{B}^{op}} \arrow[r, dotted, shift left, "\mathbf{L}"] & \mathcal{L} \arrow[l, dotted, shift left, "\mathbf{R}"]
\end{tikzcd}$$

Consequently, every morphism from a Boolean algebra B in $\mathcal{B}$ to a quantum event algebra L in $\mathcal{L}$ factors uniquely through the functor category $\mathbf{Sets}^{\mathcal{B}^{op}}$, and the consideration of the colimit-preserving functor $\widehat{\mathbf{M}} : \mathbf{Sets}^{\mathcal{B}^{op}} \to \mathcal{L}$ is crucial for understanding the underlying structure of L in $\mathcal{L}$ in terms of diagrams of Boolean probing frames.

The physical significance of the functor $\widehat{\mathbf{M}}$ is unraveled by the fact that it plays the role of a left adjoint $\mathbf{L}$ and, thus, colimit-preserving functor from $\mathbf{Sets}^{\mathcal{B}^{op}}$ to $\mathcal{L}$. More precisely, the functor $\widehat{\mathbf{M}} := \mathbf{L}$ is the left adjoint of the *categorical adjunction* between the categories $\mathbf{Sets}^{\mathcal{B}^{op}}$ and $\mathcal{L}$,

$$\mathbf{L} : \mathbf{Sets}^{\mathcal{B}^{op}} \to \mathcal{L}, \tag{2.4}$$

whilst the right adjoint functor,

$$\mathbf{R} : \mathcal{L} \to \mathbf{Sets}^{\mathcal{B}^{op}}, \tag{2.5}$$

is physically interpreted as the *Boolean realization functor* of $\mathcal{L}$ in terms of variable local probing frames, functioning as natural contexts for measurement of observables. The existence of the functorial relations (2.4) and (2.5) designates the fact that a quantum event algebra L in $\mathcal{L}$ can be expressed in terms of structured multitudes of interlocking local Boolean frames capable of carrying all the information encoded in the former.

Henceforth, the problem of establishing a functorial representation of a quantum event algebra via Boolean frames has a universal solution, which is provided by the left adjoint functor $\mathbf{L} : \mathbf{Sets}^{\mathcal{B}^{op}} \to \mathcal{L}$ to the Boolean realization functor $\mathbf{R} : \mathcal{L} \to \mathbf{Sets}^{\mathcal{B}^{op}}$. In other words, the existence of the left adjoint functor $\mathbf{L}$ paves the way for an explicit reconstruction of quantum event algebras by means of appropriate diagrams of Boolean probing frames in a structure-preserving manner, based on partial congruences between the Boolean and quantum kinds of event structure. Since our interpretative scheme of complementarity is based on this pair of adjoint functors, it is useful to express their bi-directional correspondence in the form of the following theorem, formulated and proved in (XX 2001), see also (XX & YY 2013).

Theorem. *There exists a categorical adjunction between the categories* $\mathbf{Sets}^{\mathcal{B}^{op}}$ *and* $\mathcal{L}$*, called the Boolean frames–quantum adjunction. Specifically, there exists a pair of adjoint functors* $\mathbf{L} \dashv \mathbf{R}$ *as follows:*

$$\mathbf{L} : \mathbf{Sets}^{\mathcal{B}^{op}} \overset{\longleftarrow}{\longrightarrow} \mathcal{L} : \mathbf{R}. \tag{2.6}$$

Equivalently, there exists a bijection, which is natural in both $\mathbf{P}$ *in* $\mathbf{Sets}^{\mathcal{B}^{op}}$ *and* L *in* $\mathcal{L}$*,*

$$Hom_{\mathbf{Sets}^{\mathcal{B}^{op}}}(\mathbf{P}, \mathbf{R}(L)) \cong Hom_{\mathcal{L}}(\mathbf{LP}, L), \tag{2.7}$$

abbreviated as

$$Nat(\mathbf{P}, \mathbf{R}(L)) \cong Hom_{\mathcal{L}}(\mathbf{LP}, L). \tag{2.8}$$

Corollary. *The left adjoint functor of the Boolean frames-quantum adjunction,* $\mathbf{L} : \mathbf{Sets}^{\mathcal{B}^{op}} \to \mathcal{L}$*, is defined for each presheaf* $\mathbf{P}$ *in* $\mathbf{Sets}^{\mathcal{B}^{op}}$ *as the colimit* $\mathbf{L}(\mathbf{P})$*:*

$$\mathbf{L}(\mathbf{P}) = Colim\{\int(\mathbf{P}, \mathcal{B}) \xrightarrow{\int_{\mathbf{P}}} \mathcal{B} \xrightarrow{\mathbf{M}} \mathcal{L}\} . \tag{2.9}$$

The colimit $\mathbf{L}(\mathbf{P})$ is explicitly constructed in Sect. 3.2 in relation to the physically important case of our interest, $\mathbf{P} = \mathbf{R}(L)$, namely, when the presheaf functor $\mathbf{P}$ of Boolean event algebras represents the functor of Boolean frames $\mathbf{R}(L)$ of a quantum event algebra L.

The Boolean frames - quantum adjunction, specified by the pair of adjoint functors $\mathbf{L} \dashv \mathbf{R}$, formalizes categorically the process of encoding and decoding information between diagrams of Boolean event algebras B and quantum event algebras L via the action of Boolean probing frames $\psi_B : \mathbf{M}(B) \to L$. In general, the existence of an adjunction between two categories always gives rise to a family of *universal morphisms*, called *unit* and *counit* of the adjunction, one for each object in the first category and one for each object in the second. In this way, each object in the first category induces a certain property in the second category and the universal morphism carries the object to the universal for that property. Most significantly, every adjunction extends to an *adjoint equivalence* of certain subcategories of the initial functorially correlated categories. It is precisely this category-theoretic fact which determines the *necessary* and *sufficient conditions* for the isomorphic representation of a quantum event algebra L in $\mathcal{L}$ by means of suitably qualified functors of Boolean probing frames.

We note for this purpose that every categorical adjunction is completely characterized by the unit and counit natural transformations, acquiring the status of universal mapping properties (Awodey 2010, pp. 208-215). In relation to the Boolean frames-quantum adjunction, for any presheaf $\mathbf{P}$ of Boolean event algebras in the functor category $\mathbf{Sets}^{\mathcal{B}^{op}}$, the unit of the adjunction is defined by

$$\delta_{\mathbf{P}} : \mathbf{P} \longrightarrow \mathbf{RLP}. \tag{2.10}$$

On the other side, for each quantum event algebra L in $\mathcal{L}$, the counit is defined by

$$\epsilon_L : \mathbf{LR}(L) \longrightarrow L. \tag{2.11}$$

As further analyzed in Sect. 3.3, the representation of a quantum event algebra L in $\mathcal{L}$, in terms of the functor of Boolean frames $\mathbf{R}(L)$ of L, is full and faithful if and only if the counit of the Boolean frames-quantum adjunction is a quantum algebraic isomorphism, that is structure-preserving, injective and surjective. In turn, the counit of the Boolean frames-quantum adjunction is a quantum algebraic isomorphism if and only if the right adjoint functor $\mathbf{R}$ is full and faithful.

We argue in the sequel that the prescribed universal solution incorporates the semantics of a twofold complementarity scheme generalizing the original notion of complementarity as it has been exemplified in the introductory part of this work.

3 Twofold Complementarity Scheme via the Boolean Frames – Quantum Adjunction

3.1 Standard vs generalized conception of complementarity

The standard formulation of the notion of complementarity pertains to conjugate quantum observables, like position and momentum, which cannot be sharply specified simultaneously in the context of the same experimental arrangement. At this point, it is instructive to recall from Sect. 1 Bohr's definition of the word phenomenon to refer exclusively to observations obtained under specific circumstances, including the account of the whole experiment. Thus, for methodological purposes, we may adapt Bohr's concept of phenomenon as a referent of the assignment of an observable to a quantum system in the context of a Boolean frame of measurement. Since the concept of a Boolean frame is qualified, by definition, as a probing frame of a quantum event algebra, it is natural to associate the semantics of complementarity with their function. In this way, every Boolean probing frame is constitutive of a phenomenon and every event is conditioned by at least one Boolean frame.

The original conception of complementarity from this viewpoint pertains to any two mutually incompatible Boolean frames, which correspond physically to mutually exclusive experimental arrangements. These Boolean frames neither share any common projections nor have any common refinement and, thus, may be considered as totally disjoint. We claim that the requirement of total disjointness represents an extreme case, which precludes the physically important realistic possibility of partial compatibility between various probing frames. For this reason, we consider a weaker version of complementarity that refers to partially compatible Boolean frames, which may have a non-trivial overlap, or a common refinement, or, more generally, they are compatible under a pullback operation. In order to substantiate the proposed generalized notion of complementarity, including the former one as a special case, we need to examine in detail the semantics of the derived universal solution to the problem of functorial representation of a quantum event algebra in terms of Boolean logical probes.

As a prelude to our considerations, it is significant to stress that the proposed generalized notion of complementarity does not only pertain to the mutual relations between any two Boolean event algebras considered merely as Boolean subalgebras of a given quantum event algebra, but it pertains also to their mutual relations in their capacity to jointly act in a compatible way as Boolean probing frames of an otherwise empirically inaccessible quantum event algebra. It is the latter interpretation that is implemented by the functorial representation of a quantum event algebra in contradistinction to the former one which requires only the subalgebra relation. This differentiation is crucial because it implicates that complementarity is not only a relation between Boolean probes, or contexts of co-measurable observables, but it is primarily a relation between two different structural kinds of event structure that can be brought into partial congruence by means of the established adjunction. This essentially means that the quantum structural kind can be inductively constituted by the adjunction of coherently inter-connected families of partially congruent Boolean probing frames covering it entirely. In this manner, the mutual relations between any two Boolean event algebras in their capacity to function jointly as probing frames of a quantum event algebra are constrained globally by the pair of adjoint functors acting as inverse conceptual bridges between two different categorical event structures. Therefore, these mutual relations are globally conditioned by their universal factorization through the action of the left adjoint $\mathbf{L}$ on the functor of Boolean frames $\mathbf{R}(L)$ of L. In a nutshell, one should refrain from identifying, at the risk of conflating, the notion of contextuality pertaining to the structural kind of Boolean event algebras with the notion of complementarity pertaining to the partial functorial congruence between the Boolean and quantum kinds of event structure.

It will become clear in the sequel that the notion of contextuality, if transplanted functorially from the Boolean to the quantum kind via the modeling functor $\mathbf{M}$, gives rise to a *restricted/local* conceptual dimension of complementarity, which, nevertheless, is subordinate to the *generalized/global* dimension. As explicitly shown in Sect. 3.3, this subordination is expressed by the Boolean frames – quantum adjunction via the intervention of a covering functor of a quantum event algebra L, objectified as an appropriate subfunctor $\mathbf{T}$ of the functor of Boolean frames $\mathbf{R}(L)$ of L, i.e., $\mathbf{T} \hookrightarrow \mathbf{R}(L)$. Consequently, the proposed weakening of the former rigid notion of complementarity, implicated by the pertinent pair of adjoint functors, is qualified by means of a twofold scheme consisting of a local and a global conceptual dimension, where, importantly, the former conception is not ad-hoc but is functorially constrained by the latter.

3.2 Global conceptual dimension of complementarity

The global dimension of complementarity is formally characterized by the counit natural transformations of the identity functor in the category of quantum event algebras $\mathcal{L}$. As pointed out in the preceding section, for each quantum event algebra L in $\mathcal{L}$, the counit is defined as follows:

$$\epsilon_L : \mathbf{LR}(L) \longrightarrow L. \tag{3.1}$$

Then, the counit natural transformation ϵ_L defines the *spectral constitution* of a quantum event algebra L in $\mathcal{L}$ via the *colimiting–interconnection of Boolean probing frames* of L, whose domains are partially congruent Boolean event algebras to L.

It is clear from the counit expression (3.1) that the functorial representation of a quantum event algebra L in $\mathcal{L}$ through the category $\mathbf{Sets}^{\mathcal{B}^{op}}$ requires an explicit calculation of the colimit $\mathbf{LR}(L)$, when the functor on which it acts is the presheaf functor of Boolean frames $\mathbf{R}(L)$ of L. Let us initially note that in order to calculate, in general, the colimit $\mathbf{L}(\mathbf{P})$ for any presheaf functor $\mathbf{P}$ in $\mathbf{Sets}^{\mathcal{B}^{op}}$, it is necessary to specify the *index* or *parameterizing* category corresponding to the functor $\mathbf{P}$, which is defined over the base category of Boolean event algebras $\mathcal{B}$. This index category is called the *category of elements of the presheaf* $\mathbf{P}$, denoted by $\int(\mathbf{P}, \mathcal{B})$, and defined as follows: it has objects all pairs (B, p) and morphisms $(\acute{B}, \acute{p}) \rightarrow (B, p)$ are those morphisms $u : \acute{B} \rightarrow B$ of $\mathcal{B}$ for which $p \cdot u = \acute{p}$, that is, the restriction or pullback of p along u is $\acute{p}$. Projection on the second coordinate of $\int(\mathbf{P}, \mathcal{B})$ defines a functor $\int_{\mathbf{P}} : \int(\mathbf{P}, \mathcal{B}) \rightarrow \mathcal{B}$ called the *split discrete fibration* induced by $\mathbf{P}$, where $\mathcal{B}$ is the base category of the fibration, as in the diagram below.

$$
\begin{tikzcd}
\int(\mathbf{P}, \mathcal{B}) \arrow[d, "\int_{\mathbf{P}}"'] & \\
\mathcal{B} \arrow[r, "\mathbf{P}"] & \mathbf{Sets}
\end{tikzcd}
$$

Notice that the fibers of this fibration are categories in which the only arrows are identity arrows, i.e., they are actually sets. Then, if B is an object of $\mathcal{B}$, the inverse image under $\int_{\mathbf{P}}$ of B is simply the set $\mathbf{P}(B)$, although its elements are written as pairs so as to form a disjoint union.

From a physical viewpoint, the split discrete fibration induced by $\mathbf{P}$ provides a well-defined notion of a uniform homologous fibred structure in the following sense. Firstly, by the arrows specification defined in the category of elements of $\mathbf{P}$, any element p, determined over the reference locus B, is homologously related with any other element $\acute{p}$ over the reference locus $\acute{B}$, and so on, by variation over all the reference loci of the base category $\mathcal{B}$. Secondly, all the elements p of $\mathbf{P}$ of the same sort B, namely, the elements determined over the same reference locus B, are uniformly equivalent to each other, since all the arrows in $\int(\mathbf{P}, \mathcal{B})$ are induced by lifting arrows from the base $\mathcal{B}$.

The task of calculating the colimit $\mathbf{LR}(L)$, when the presheaf functor $\mathbf{P}$ represents the functor of Boolean frames $\mathbf{R}(L)$ of a quantum event algebra L, is simplified by the observation that there exists an underlying, colimit-preserving, faithful functor from the category $\mathcal{L}$ to the category $\mathbf{Sets}$. Thus, we can calculate the colimit by means of set-valued equivalence classes in $\mathbf{Sets}$, under the constraint that the derived set of equivalence classes carries the structure of a quantum event algebra.

In this connection, it is instructive to recall from Sect. 2.2 that the Boolean realization functor of the category of quantum event algebras $\mathcal{L}$ into the category of presheaves of Boolean event algebras $\mathbf{Sets}^{\mathcal{B}^{op}}$ is defined by

$$
\mathbf{R} : \mathcal{L} \rightarrow \mathbf{Sets}^{\mathcal{B}^{op}} . \tag{3.2}
$$

Then, the functor of Boolean frames $\mathbf{R}(L)$ of a quantum event algebra L in $\mathcal{L}$ is the *image* of the realization functor $\mathbf{R}$, evaluated at L, into the category of presheaves of Boolean event algebras, $\mathbf{Sets}^{\mathcal{B}^{op}}$, i.e.,

$$\mathbf{R}(L) : \mathcal{B}^{op} \to \mathbf{Sets} \, . \tag{3.3}$$

Thus, the functor $\mathbf{R}(L)$ of L is actually an object in the category of presheaves $\mathbf{Sets}^{\mathcal{B}^{op}}$, representing L in the environment of the topos of presheaves over the base category $\mathcal{B}$.

It is important to stress that the Boolean realization functor of a quantum categorical event structure $\mathcal{L}$ is completely determined by the action of the functor of Boolean frames, for each quantum event algebra L in $\mathcal{L}$, on the objects and arrows of the category of Boolean event algebras $\mathcal{B}$. It follows immediately from Eq. (2.2) that its action on a Boolean algebra B in $\mathcal{B}$ is given by

$$\mathbf{R}(L)(B) = Hom_{\mathcal{L}}(\mathbf{M}(B), L) \, , \tag{3.4}$$

while its action on a Boolean homomorphism $x : D \longrightarrow B$ in $\mathcal{B}$, for $v : \mathbf{M}(B) \longrightarrow L$ is defined by

$$\mathbf{R}(L)(x) : Hom_{\mathcal{L}}(\mathbf{M}(B), L) \longrightarrow Hom_{\mathcal{B}}(\mathbf{M}(D), L) \, ,$$

$$\mathbf{R}(L)(x)(v) = v \circ x \, . \tag{3.5}$$

As already pointed out, the functor of Boolean frames $\mathbf{R}(L)$ of a quantum event algebra L in $\mathcal{L}$ forms a presheaf of sets on Boolean event algebras B in $\mathcal{B}$. Thus, in complete analogy with the preceding analysis concerning any presheaf functor $\mathbf{P}$ in $\mathbf{Sets}^{\mathcal{B}^{op}}$, we consider the corresponding category of elements $\int(\mathbf{R}(\mathbf{L}), \mathcal{B})$, specified as follows: it has objects all pairs (B, ψ_B), where B is a Boolean event algebra and $\psi_B : \mathbf{M}(B) \to L$ is a Boolean probing frame of L defined over B. The morphisms of $\int(\mathbf{R}(\mathbf{L}), \mathcal{B})$, denoted by $(\acute{B}, \psi_{\acute{B}}) \to (B, \psi_B)$, are those Boolean event algebra homomorphisms $u : \acute{B} \to B$ of the base category $\mathcal{B}$ for which $\psi_B \cdot u = \psi_{\acute{B}}$, that is, the restriction or pullback of the Boolean frame ψ_B along u is $\psi_{\acute{B}}$. Then, the projection functor $\int_{\mathbf{R}(L)} : \int(\mathbf{R}(\mathbf{L}), \mathcal{B}) \to \mathcal{B}$, namely, the split discrete fibration induced by the functor of Boolean frames of L, where $\mathcal{B}$ is the base category of the fibration, provides, as previously explained, a uniform and homologous fibred representation of quantum events in terms of Boolean probing frames for the measurement of observables. The notion of functional dependence incorporated in this representation forces a global quantum event algebra to be partitioned into sorts parameterized by the Boolean frames of the base category $\mathcal{B}$. In this way, the functioning of a Boolean localization scheme on a global structure of quantum events is represented by means of a fibred construct, understood geometrically as a variable set of possible events over the base category of local Boolean probing frames conditioning the actualization of events.

Consequently, the index category corresponding to the functor $\mathbf{R}(L)$ is the category of its elements $\mathcal{I} \equiv \int(\mathbf{R}(\mathbf{L}), \mathcal{B})$, whence the functor $[\mathbf{M} \circ \int_{\mathbf{R}(L)}]$ defines the diagram $\mathcal{I} \to \mathcal{L}$ over which the colimit should be calculated. Hence, we obtain:

$$\mathbf{LR}(L) = \mathbf{L_M}(\mathbf{R}(L)) = Colim\{ \int (\mathbf{R}(\mathbf{L}), \mathcal{B}) \xrightarrow{\int_{\mathbf{R}(L)}} \mathcal{B} \xrightarrow{\mathbf{M}} \mathcal{L} \dashrightarrow \mathbf{Sets} \} \, . \tag{3.6}$$

It has been shown in (XX & YY 2013) that the sought colimit is equivalent to the definition of the following tensor product structure,

$$\mathbf{LR}(L) \cong \mathbf{R}(L) \otimes_{\mathcal{B}} \mathbf{M}\,, \tag{3.7}$$

formed by the set-valued functors,

$$\mathbf{R}(L) : \mathcal{B}^{op} \to \mathbf{Sets}, \qquad \mathbf{M} : \mathcal{B} \to \mathbf{Sets}\,, \tag{3.8}$$

where the contravariant functor $\mathbf{R}(L)$ is considered as a right $\mathcal{B}$-module and the covariant functor $\mathbf{M}$ as a left $\mathcal{B}$-module, in complete analogy with the algebraic definition of the tensor product of a right $\mathcal{B}$-module with a left $\mathcal{B}$-module over a ring of coefficients $\mathcal{B}$ (e.g., Lam 1999). We call this construction the *functorial tensor product decomposition of the colimit in the category of elements* of $\mathbf{R}(L)$ induced by the Boolean modeling functor $\mathbf{M} : \mathcal{B} \to \mathcal{L}$ of $\mathcal{L}$.

Thus, for a Boolean probing frame $\psi_B \in \mathbf{R}(L)(B)$, $v : \acute{B} \to B$ and $\acute{q} \in \mathbf{M}(\acute{B})$, the elements of the set $\mathbf{R}(L) \otimes_{\mathcal{B}} \mathbf{M}$ are all of the form $\chi(\psi_B, q)$, which can be written in a suggestive notation as,

$$\chi(\psi_B, q) = \psi_B \otimes q, \quad \psi_B \in \mathbf{R}(L)(B),\ q \in \mathbf{M}(B)\,, \tag{3.9}$$

such that,

$$\psi_B \cdot v \otimes \acute{q} = \psi_B \otimes v(\acute{q}), \quad \psi_B \in \mathbf{R}(L)(B),\ \acute{q} \in \mathbf{M}(\acute{B}),\ v : \acute{B} \to B. \tag{3.10}$$

This state of affairs gives rise to the conclusion that the set $\mathbf{R}(L) \otimes_{\mathcal{B}} \mathbf{M}$ is actually the quotient of the set $\sum_B \mathbf{P}(B) \times \mathbf{M}(B)$, for $B \in \mathcal{B}$, by the smallest equivalence relation generated by the above equations, whence the elements $\psi_B \otimes q$ of the quotient set $\mathbf{R}(L) \otimes_{\mathcal{B}} \mathbf{M}$ are the equivalence classes of this relation. Since the morphisms $\psi_B : \mathbf{M}(B) \to L$ denote Boolean probing frames of L, encoded as elements in the associated index category, $\mathcal{I} \equiv \int(\mathbf{R}(\mathbf{L}), \mathcal{B})$, and q denotes a projection operator $q \in \mathbf{M}(B)$, then, the quotient set $\mathbf{R}(L) \otimes_{\mathcal{B}} \mathbf{M}$ is actually a set of equivalence classes of pointed Boolean frames.

Henceforth, the function of the left adjoint of the Boolean frames-quantum adjunction, $\mathbf{L} : \mathbf{Sets}^{\mathcal{B}^{op}} \to \mathcal{L}$, when it acts on the functor of Boolean frames $\mathbf{R}(L)$ of a quantum event algebra L, is defined, for each L in $\mathcal{L}$, as the colimit $\mathbf{LR}(L)$ of Eq. (3.6), which, in turn, is equivalent to the quotient set $\mathbf{R}(L) \otimes_{\mathcal{B}} \mathbf{M}$. Furthermore, the latter set is naturally endowed with the structure of a quantum event algebra, thus, completing the construction of the left adjoint colimit in $\mathcal{L}$ via the category of $\mathbf{Sets}$.

The embedding of a quantum event algebraic structure into the quotient set $\mathbf{R}(L) \otimes_{\mathcal{B}} \mathbf{M}$ is specified as follows. First, the orthocomplementation operator is defined by the assignment:

$$(\psi_B \otimes q)^* := \psi_B \otimes q^*\,. \tag{3.11}$$

Second, the unit element is defined as:

$$\mathbf{1} := \psi_B \otimes 1\,. \tag{3.12}$$

Third, two equivalence classes in the quotient set $\mathbf{R}(L) \otimes_{\mathcal{B}} \mathbf{M}$ can be ordered if and only if they have a common refinement. Consequently, the partial order structure is defined by the assignment,

$$(\psi_B \otimes q) \preceq (\psi_C \otimes r) \,, \tag{3.13}$$

if and only if,

$$d_1 \preceq d_2 \,, \tag{3.14}$$

where we have made the following identifications,

$$\begin{aligned} (\psi_B \otimes q) &= (\psi_D \otimes d_1) \,, \\ (\psi_C \otimes r) &= (\psi_D \otimes d_2) \,, \end{aligned} \tag{3.15}$$

with d_1, $d_2 \in \mathbf{M}(D)$, according to the pullback diagram,

$$\begin{tikzcd} \mathbf{M}(D) \arrow[r, "\beta"] \arrow[d, "\gamma"] & \mathbf{M}(B) \arrow[d, "\alpha"] \\ \mathbf{M}(C) \arrow[r, "\lambda"] & L \end{tikzcd}$$

such that $\beta(d_1) = q$, $\gamma(d_2) = r$, and $\beta : \mathbf{M}(D) \to \mathbf{M}(B)$, $\gamma : \mathbf{M}(D) \to \mathbf{M}(C)$ denote the pullback of $\alpha : \mathbf{M}(B) \to L$ along $\lambda : \mathbf{M}(C) \to L$ in the category of quantum event algebras. Apparently, the ordering relation between any two equivalence classes of pointed Boolean frames in the set $\mathbf{R}(L) \otimes_{\mathcal{B}} \mathbf{M}$ requires the existence of pullback compatibility conditions between the corresponding Boolean frames. This is a crucial requirement that will be also utilized in the next section for establishing the local conceptual dimension of complementarity.

We conclude that the spectral constitution of a quantum event algebra L in $\mathcal{L}$ via the Boolean frames–quantum adjunction is based on the action of the endofunctor $\mathbf{G}$ on $\mathcal{L}$, defined by

$$\begin{aligned} \mathbf{G} := \mathbf{LR} : \mathcal{L} \to \mathbf{Sets}^{\mathcal{B}^{op}} \to \mathcal{L} \,, \\ L \to \mathbf{R}(L) \to \mathbf{LR}(L) \to L \,, \end{aligned} \tag{3.16}$$

which may be called the *global spectral constitution endofunctor* of a quantum categorical event structure $\mathcal{L}$ via Boolean probing frames.

In particular, if the counit universal morphism of the Boolean frames-quantum adjunction, evaluated at L, for each quantum event algebra L in $\mathcal{L}$, is expressed in terms of equivalence classes of pointed Boolean frames, viz. $\mathbf{LR}(L) \cong \mathbf{R}(L) \otimes_{\mathcal{B}} \mathbf{M}$, we obtain:

$$\epsilon_L : \mathbf{R}(L) \otimes_{\mathcal{B}} \mathbf{M} \longrightarrow L \,. \tag{3.17}$$

Thus, the counit ϵ_L fits into the following diagram:

$$
\begin{tikzcd}
\mathbf{R}(L)\otimes_{\mathcal{B}}\mathbf{M} \arrow[dr, "\epsilon_L"] & \\
\mathbf{M}(B) \arrow[u, "\psi_B\otimes(-)"] \arrow[r, "\psi_B"] & L
\end{tikzcd}
$$

Accordingly, for every Boolean frame $\psi_B : \mathbf{M}(B) \to L$ the projection operator $q \in \mathbf{M}(B)$ is mapped to an event in L *only through its factorization via the equivalence class* $\psi_B \otimes q$ of pointed Boolean frames, or, equivalently,

$$\epsilon_L([\psi_B \otimes q]) = \psi_B(q), \qquad q \in \mathbf{M}(B)\,. \tag{3.18}$$

This epitomizes the semantical role of the global conceptual dimension of complementarity as a relation between the Boolean and the quantum structural kinds of event structure that can be brought into partial congruence by means of the established adjunction.

3.3 Local conceptual dimension of complementarity

The local dimension of complementarity is formally characterized by the conditions that make the counit natural transformations of the identity functor in the category of quantum event algebras $\mathcal{L}$ an isomorphism. In turn, the counit isomorphism expresses the property of invariance of a quantum event algebra under encoding in terms of appropriate families of Boolean event algebras via probing frames in $\mathbf{R}(L)$, and then decoding back by means of the action of the left adjoint on the former, denoted by $\mathbf{LR}(L)$.

We recall that, in view of the global dimension of complementarity, the counit ϵ_L defines the spectral constitution of a quantum event algebra L in $\mathcal{L}$ via the colimit of the functor of Boolean probing frames of L, whose domains are partially congruent Boolean event algebras to L. This colimit $\mathbf{LR}(L)$ is actually an object in $\mathcal{L}$, qualified as partially ordered, if additionally there exist pullback compatibility conditions between the Boolean frames that may be ordered.

We notice that if the counit evaluated at L is an isomorphism, then L can be considered as a fixed point of the corresponding global spectral constitution endofunctor of L via Boolean probing frames. In general, the counit natural transformation ϵ_L is a natural isomorphism if and only if the right adjoint functor of the Boolean frames–quantum adjunction is full and faithful, or, equivalently, if and only if the cocone from the functor $\mathbf{M} \circ \int_{\mathbf{R}(L)}$ to L is universal for each L in $\mathcal{L}$. In the latter case, we characterize the functor $\mathbf{M} : \mathcal{B} \to \mathcal{L}$ as a dense Boolean modeling functor. On this account, therefore, it is important to specify the necessary and sufficient conditions which make the counit ϵ_L to be an isomorphism. This requirement leads to the notion of *sheaf-theoretic localization* of L through the probing frames $\psi_B : \mathbf{M}(B) \to L$, giving subsequently rise to the local conceptual dimension of complementarity.

As a first step, we emphasize that if the counit natural transformation ϵ_L at L can be suitably restricted to an isomorphism $\mathbf{R}(L)\otimes_{\mathcal{B}}\mathbf{M} \cong L$, then the quantum structural information of L can be completely encoded and classified logically through equivalence classes of pointed Boolean frames. For this purpose, we need to impose appropriate conditions on families of Boolean frames in $\mathbf{R}(L)$, which are going to function as local Boolean covers of L. The requirements qualifying such restricted families of Boolean frames as local Boolean covers of L are the following: First, they should constitute a *minimal generating class* of Boolean frames instantiating a subfunctor $\mathbf{T}$ of the functor of Boolean frames $\mathbf{R}(L)$ of L. Second, they should jointly form an *epimorphic family* covering L entirely on their overlaps. Third, they should be *compatible* under refinement operations, or, more generally, pullback conditions in L. Fourth, they should be *transitive* such that subcovers of covers of L can be qualified as covers themselves. We formulate these notions as follows.

A *functor of Boolean coverings* for a quantum event algebra L in $\mathcal{L}$ is defined as a subfunctor $\mathbf{T}$ of the functor of Boolean frames $\mathbf{R}(L)$ of L, i.e., $\mathbf{T} \hookrightarrow \mathbf{R}(L)$. For each Boolean algebra B in $\mathcal{B}$, a subfunctor $\mathbf{T} \hookrightarrow \mathbf{R}(L)$ is equivalent to an algebraic right ideal or spectral sieve of quantum homomorphisms $\mathbf{T} \triangleright \mathbf{R}(L)$, defined by the requirement that, for each B in $\mathcal{B}$, the set of elements of $\mathbf{T}(B) \subseteq [\mathbf{R}(L)](B)$ is a set of Boolean frames $\psi_B : \mathbf{M}(B) \to L$ of $\mathbf{R}(L)(B)$, called *Boolean covers of* L, satisfying the following property:

{If $[\psi_B : \mathbf{M}(B) \to L] \in \mathbf{T}(B)$, viz. it is a Boolean cover of L, and $\mathbf{M}(v) : \mathbf{M}(\acute{B}) \to \mathbf{M}(B)$ in $\mathcal{L}$ for $v : \acute{B} \to B$ in $\mathcal{B}$, then $[\psi_B \circ \mathbf{M}(v) : \mathbf{M}(\acute{B}) \to \mathcal{L}] \in \mathbf{T}(B)$, viz. it is also a Boolean cover of L}.

A family of Boolean covers $\psi_B : \mathbf{M}(B) {\longrightarrow} L$, B in $\mathcal{B}$, is the *generator of a spectral sieve of Boolean coverings* $\mathbf{T}$, if and only if, this sieve is the smallest among all containing that family. The spectral sieves of Boolean coverings for an L in $\mathcal{L}$ constitute a partially ordered set under inclusion of subobjects. The minimal sieve is the empty one, namely, $\mathbf{T}(B) = \emptyset$ for all B in $\mathcal{B}$, whereas the maximal sieve is the set of all probing Boolean frames of L for all B in $\mathcal{B}$, considered as Boolean covers.

We recall from Sect. 3.2 that the ordering relation between any two equivalence classes of pointed Boolean frames in the set $\mathbf{R}(L)\otimes_{\mathcal{B}}\mathbf{M}$ requires the existence of pullback compatibility conditions between the corresponding Boolean frames. Henceforth, if we consider a functor of Boolean coverings $\mathbf{T}$ for a quantum event algebra L, we require that the generating family of Boolean covers they belong to is compatible under pullbacks.

Then, the *pullback or categorical overlap* of any pair of Boolean covers $\psi_B : \mathbf{M}(B) {\longrightarrow} L$, B in $\mathcal{B}$, and $\psi_{\acute{B}} : \mathbf{M}(\acute{B}) {\longrightarrow} L$, $\acute{B}$ in $\mathcal{B}$, with common codomain a quantum event algebra L, consists of the Boolean cover $\mathbf{M}(B) \times_L \mathbf{M}(\acute{B})$, together with the two arrows $\psi_{B\acute{B}}$ and $\psi_{\acute{B}B}$, called projections, as shown in the diagram:

$$
\begin{tikzcd}
\mathbf{M}(\acute{\acute{B}}) \arrow[dr, "u"] \arrow[drr, "h"] \arrow[ddr, "g"'] & & \\
& \mathbf{M}(B)\times_L \mathbf{M}(\acute{B}) \arrow[r, "\psi_{B,\acute{B}}"] \arrow[d, "\psi_{\acute{B},B}"] & \mathbf{M}(B) \arrow[d, "\psi_B"] \\
& \mathbf{M}(\acute{B}) \arrow[r, "\psi_{\acute{B}}"] & L
\end{tikzcd}
$$

The square *commutes* and for any Boolean domain object $\mathbf{M}(\acute{\acute{B}})$ or event algebra $\acute{\acute{B}}$ in $\mathcal{B}$ and arrows h and g that make the outer square commute, there is a unique $u : \mathbf{M}(\acute{\acute{B}}) \longrightarrow \mathbf{M}(B)\times_L \mathbf{M}(\acute{B})$ that makes the whole diagram commutative. Hence, we obtain the condition: $\psi_{\acute{B}} \circ g = \psi_B \circ h$.

We emphasize that if ψ_B and $\psi_{\acute{B}}$ are injective morphisms, then their pullback is isomorphic with the intersection $\mathbf{M}(B)\times_L \mathbf{M}(\acute{B})$. Then, we can define the *gluing* or *pasting map* between Boolean probing frames on their overlap, which is an isomorphism, as follows:

$$\Omega_{B,\acute{B}} : \psi_{\acute{B}B}(\mathbf{M}(B)\times_L \mathbf{M}(\acute{B})) \longrightarrow \psi_{B\acute{B}}(\mathbf{M}(B)\times_L \mathbf{M}(\acute{B})) \ ,$$

$$\Omega_{B,\acute{B}} = \psi_{B\acute{B}} \circ {\psi_{\acute{B}B}}^{-1} \ . \tag{3.19}$$

An immediate consequence of the preceding definition is the satisfaction of the following *cocycle conditions*: (i) $\Omega_{B,B} = 1_B$, $1_B := id_B$, (ii) $\Omega_{B,\acute{B}} \circ \Omega_{\acute{B},\acute{\acute{B}}} = \Omega_{B,\acute{\acute{B}}}$ if $\mathbf{M}(B) \cap \mathbf{M}(\acute{B}) \cap \mathbf{M}(\acute{\acute{B}}) \neq 0$, and (iii) $\Omega_{B,\acute{B}} = \Omega^{-1}{}_{\acute{B},B}$ if $\mathbf{M}(B) \cap \mathbf{M}(\acute{B}) \neq 0$.

Thus, the gluing isomorphism $\Omega_{B,\acute{B}}$ between any two Boolean frames of a spectral sieve $\mathbf{T}(L)$ assures that $\psi_{\acute{B}B}(\mathbf{M}(B)\times_L \mathbf{M}(\acute{B}))$ and $\psi_{B\acute{B}}(\mathbf{M}(B)\times_L \mathbf{M}(\acute{B}))$ probe L on their common refinement in a compatible way. This provides the sought criterion of *qualifying the local conceptual dimension of complementarity with respect to a spectral sieve* $\mathbf{T}(L)$, under the proviso that the family of all Boolean covers $\psi_B : \mathbf{M}(B) \longrightarrow L$, for variable B in $\mathcal{B}$, generating this spectral sieve, jointly form an epimorphic family covering L completely,

$$T_L : \sum\nolimits_{(B_j,\psi_j:\mathbf{M}(B_j)\to L)} \mathbf{M}(B_j) \twoheadrightarrow L, \tag{3.20}$$

where T_L is an epimorphism in $\mathcal{L}$ with codomain a quantum event algebra L.

A sieve adjoined to a quantum event algebra L constitutes a *Boolean localizing spectral sieve of* L, or, equivalently, *a functor of Boolean localizations of* L, if and only if it is closed with respect to an epimorphic family of Boolean covers of L and the above cocycle conditions are satisfied. The conceptual significance of a Boolean localizing spectral sieve of L lies on the fact that the functor of Boolean probing frames $\mathbf{R}(L)$ becomes a *structure sheaf of local Boolean coefficients* when restricted to it. Then, we can show that for a dense epimorphic family of Boolean covers in a Boolean localization functor $\mathbf{T}$ of L, the counit of the Boolean frames-quantum adjunction is restricted to a quantum algebraic isomorphism, that is structure-preserving, injective and surjective (XX 2004). In turn, the right adjoint functor of the adjunction restricted to a Boolean localization functor is full and faithful. This argument can be formalized more precisely in topos theoretic terminology using the technical means of the subcanonical Grothendieck topology consisting of epimorphic families of covers on the base category of Boolean event algebras. For details the interested reader should consult (XX 2006). Consequently, $\mathcal{L}$ becomes a reflection of the category of sheaves of variable local Boolean frames $\mathbf{Sets}^{\mathcal{B}^{op}}$, constituting a topos, and the structure of a quantum event algebra L in $\mathcal{L}$ is encoded and preserved by the action of a family of Boolean frames if and only if this family forms a Boolean localization functor of L.

4 Conceptual and Philosophical Aspects of the Twofold Complementarity Scheme

The existence of a categorical adjunction between the category of quantum event algebras and the category of (pre)sheaves of variable local Boolean frames incorporates the semantics of a twofold complementarity scheme, generalizing the original notion of complementarity, by giving rise to two interdependent conceptual routes of global and local dimension. The global dimension of complementarity acquires a meaning by relating two hierarchically different structural kinds, namely, the Boolean and the quantum kinds of event structure that can be brought into partial congruence through the established adjunctive correspondence, consisting of the pair of adjoint functors $\mathbf{L} \dashv \mathbf{R}$,

$$\mathbf{L} : \mathbf{Sets}^{\mathcal{B}^{op}} \underset{\longrightarrow}{\longleftarrow} \mathcal{L} : \mathbf{R}, \tag{4.1}$$

and the natural bijection,

$$Nat(\mathbf{P}, \mathbf{R}(L)) \cong Hom_{\mathcal{L}}(\mathbf{LP}, L). \tag{4.2}$$

The essential functioning of the Boolean frames - quantum adjunction in generalizing the complementarity concept is made transparent if we consider that it provides an amphidromous translation mechanism of encoding and decoding information between the Boolean and quantum kind of structures respecting their distinctive form. Thus, if we think of $\mathbf{Sets}^{\mathcal{B}^{op}}$ as the categorical universe of variable local Boolean frames modeled in $\mathbf{Sets}$, and of $\mathcal{L}$ as the categorical universe of quantum event structures, then the left adjoint functor $\mathbf{L} : \mathbf{Sets}^{\mathcal{B}^{op}} \to \mathcal{L}$ signifies a translational code of information from the level of local Boolean algebras to the level of global quantum event algebras, whereas the Boolean realization functor $\mathbf{R} : \mathcal{L} \to \mathbf{Sets}^{\mathcal{B}^{op}}$ signifies a translational code in the inverse direction. In general, the content of the information cannot remain completely invariant with respect to translating from one categorical universe to another, and conversely. Note, in this respect, that the functors **L** and **R** are not inverses, since neither **LR** nor **RL** need be isomorphic to an identity functor. However, there remain two alternatives for a variable set over local Boolean frames **P**, standing for a presheaf functor **P** in $\mathbf{Sets}^{\mathcal{B}^{op}}$, to exchange information with a quantum algebra L. Either the content of information is transferred in quantum terms with the colimit in the category of elements of **P** translating, represented as the quantum morphism $\mathbf{LP} \to L$, or the content of information is transferred in Boolean terms with the functor of Boolean frames of L translating, represented correspondingly as the natural transformation $\mathbf{P} \to \mathbf{R}(L)$. In the first case, from the perspective of L, information is being received in quantum terms, while in the second, from the perspective of **P**, information is being sent in Boolean terms. Then, the natural bijection of Eq. (4.2) corresponds to the assertion that these two distinct ways of information transfer are equivalent. Thus, the fact that these two functors are adjoint underlines an amphidromous dependent variation, safeguarding that the global information encoded in a quantum kind of event structure is retrievable in a structure-preserving manner by all possible partial structural congruences with the Boolean kind of event structure that, in addition, is empirically directly accessible.

The global dimension of complementarity, therefore, concerns primarily the fact that an object behaving in terms of the quantum kind of structure is possible to be communicated in its entirety through a multilevel structure of coherently interconnected families of Boolean probing frames that, by virtue of the established adjunctive correspondence, have the potential of unfolding its meaning and, simultaneously, preserving it consistently, thus providing a complementarity-based conception of the process of quantum becoming.

The local dimension of complementarity acquires a meaning by relating at the same level multitudes of partially ordered Boolean probing frames, used as localization contexts for the measurement of observables pertaining to a quantum system and, subsequently, integrating them in Boolean localization functors. The latter may be understood as the semantic carriers of networks of internal relations among Boolean frames with respect to which the targeted quantum event structure is consistently and completely covered. The Boolean frames, interconnected in localization functors, act like complementary pattern recognition mechanisms such that complementary aspects of the same quantum system are being formed in relation to the abstractions associated with the preparation of an experimental context for extracting information about the system under investigation. In quantum mechanics the relation between the global theoretical structure and its various empirical substructures is indeed such that, depending on the type of experimental context a quantum system is brought to interact, different manifested aspects of the system are disclosed, impossible to be combined into a single picture as in classical physics, although only one type of system is concerned. Thus, by virtue of the proposed categorical approach to quantum mechanics, a quantum event structure can only be unfolded through structured multitudes of interconnected Boolean probing frames and the twofold complementarity scheme safeguards the correct communicability both at the local dimension, relating different Boolean frames at the same level, and the global dimension, linking the Boolean and the quantum hierarchical levels by means of the established categorical adjunction. Furthermore, as shown in Sect. 3.3, if the counit of the Boolean frames - quantum adjunction is restricted to a Boolean localization functor, then, it is reduced to an isomorphism and, subsequently, acquires the status of a closure condition expressing the preservation of the total qualitative information content, embodied in a quantum event structure, through the bi-directional process of unfolding in terms of coherently interconnected families of Boolean probing frames and then enfolding back. In this way, the global structural information of a quantum event algebra can be captured homomorphically or, by restriction to a Boolean localization functor, can be completely constituted (up to isomorphism) by means of gluing together the information collected in all compatible Boolean frames in the form of appropriate equivalence classes (by the colimit construction).

In view of the preceding considerations, therefore, and in relation to philosophical matters, it is instructive to note primarily that the proposed twofold complementarity scheme elevates to an epistemological dictum the physically significant fact that, in contrast to classical physics, values of quantum mechanical quantities cannot, in general, be attributed to a quantum object as inherently possessed, intrinsic properties. Whereas in classical physics, nothing prevented one from considering *as if* the phenomena reflected intrinsic properties, in quantum physics, even the *as if* is precluded. Indeed, quantum phenomena are not stable enough across series of measurements of non-commuting, complementary observables in order to be treated as direct reflections of invariable properties. In terms of the structural component of quantum theory, this is due to functional relationship constraints that govern the algebra of quantum mechanical observables, as revealed, for instance, by Kochen-Specker's theorem. It is no longer possible, not even *in principle*, to assign to a quantum system non-contextual properties corresponding to *all* possible measurements. This means that it is not possible to assign a definite unique answer to every single yes-no experimental question, represented by a projection operator, independent of which subset of mutually commuting projection operators one may consider it to be a member. Accordingly, well-defined values of quantum observables can, in general, be regarded as pertaining to an object under investigation only within a local Boolean frame involving the experimental conditions. Creating a preparatory Boolean environment for an object to interact with a measuring arrangement does not determine which event will take place. It does determine, however, the *kind* of event that will take place. It forces the outcome, whatever it is, to belong to a certain definite Boolean sublattice of events for which the standard measurement conditions are invariant. Such a set of standard conditions for a definite kind of measurement constitutes a set of necessary and sufficient conditions for the occurrence of an event of the selected kind. Under the scope of the latter conditions we articulate meaningful statements that the properties attributed to quantum objects are part of physical reality. Consequent upon that the exemplification of quantum objects is a *context-dependent* issue with the experimental procedure supplying the physical context for their realization. The introduction of the latter operates as a formative factor on the basis of which a quantum object manifests itself. The classical idealization of sharply individuated objects possessing intrinsic properties and having an independent reality of their own breaks down in the quantum domain. Although possible in classical physics, in quantum mechanics we can no longer display the whole of nature in one view. It would be illusory to search for an overall frame by virtue of which one may utter 'this' or 'that', 'really is' independently of a context of reference. In contrast, therefore, to an immutable and panoptical 'view from nowhere' of the traditional paradigm, quantum mechanics describes physical reality in a substantially context-dependent manner.

The view that the properties possessed by observed objects are context-dependent departs from the naive metaphysical vision of a world of self-autonomous, independently existing objects but is perfectly compatible with the objectivity of scientific knowledge (YY 2012). Empirically confirmed manifestations of quantum systems cannot be meaningfully conceived of as absolute bare particulars of reality, enjoying intrinsic individuality or intertemporal identity. Instead, they represent local or partial carriers of patterns or properties which arise in interaction with their relevant experimental context. Thus, the resulting contextual object is the quantum object exhibiting a particular property with respect to a certain experimental situation. The contextual character of property ascription implies, however, that a state-dependent property of a quantum object is not a well-defined property that has been possessed *prior* to the object's entry into an appropriate context. This also means that not all contextual properties can be ascribed to an object at once. As already remarked, one and the same quantum object does exhibit several possible contextual manifestations in the sense that it can be assigned several definite incommensurable or complementary properties only with respect to distinct experimental arrangements which may mutually exclude each other. Thus, in contradistinction to a mechanistic or naive realistic perception, we arrive at the following general conception of an object in quantum mechanics. According to this, a quantum object – as far as its state-dependent properties are concerned – constitutes a totality defined by all the possible relations in which this object may be involved. Quantum objects, therefore, are viewed as carriers of inherent dispositional properties. This means that ascribing a property to a quantum object means recognizing this object an *ontic potentiality* to produce effects whenever it is involved in various possible relations to other things in nature or whenever it is embedded within an appropriate experimental context. Henceforth, in our categorical approach to quantum mechanics, a global quantum event structure is not conceptualized as an a priori existing set-theoretic structure, already completed in itself, but it is constituted in a continuous process of extension from the local to the global level by actualization of new potential facts with respect to local Boolean frames. For, each quantum event actualized relative to a particular reference frame serves as a datum for subsequent potential actualizations, thus instantiating simultaneously a bundle of potential relations referring to this frame. Importantly, all these potential relations, namely, all relations among observables at the local level are captured by the internal relations among their underlined Boolean frames and are extended to the global quantum level through the sheaf-theoretic gluing conditions of Boolean localization functors.

The key philosophical meaning of this approach implies, therefore, the view that the quantum world can be consistently approached and comprehended through a multilevel structure of overlapping Boolean frames, which interlock, in a category-theoretical environment, to form a coherent picture of the whole in a nontrivial way. By virtue of this scheme of interpretation, the classical realist assumption that knowledge of an object is achieved by forming a representation of that object as an immutable substance possessing intrinsic properties is rejected and, subsequently, replaced by the possibility of formulating local or partial contextual theoretical structures permitting different or overlapping physical descriptions, grounded on the same actually existing object, where the sameness is precisely characterized by the constraint of satisfying collectively and globally the closure condition of the counit isomorphism. Consequently, the proposed category-theoretic analysis of complementarity reveals that the core of the quantum structure of reality is to be sought not in its internal constitution as a set-theoretical entity endowed with inherent qualities, but rather in the form of its relationship with the Boolean kind of structure through the established network of adjoint functors between the category of presheaves of variable overlapping Boolean frames and the category of quantum event algebras. In this respect, it is only natural to assert, additionally, from the viewpoint of theory construction, that our categorical approach to quantum mechanics provides the appropriate mathematical substratum for developing a structured view of scientific theorizing in the sense of comprehending a theory not just as a class of empirical models simpliciter, as a structureless set of "models of the data", but also establishing mappings between these models allowing thereby their coherent embedding in a global theoretical structure.

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