373 → 365 → 354 → 384 → 361 → 382

TD du 15/10

Exo 373

Analyse-synthèse :

<u>Analyse</u> : $rg M = r$ $\qquad \exists P, Q \in GL_n(\mathbb{R}),\ M = P J_r Q$

$A = PQ \qquad \det(A - xM) = \underbrace{\det(P)}_{\neq 0} \det(I_n + x J_r) \times \underbrace{\det(Q)}_{\neq 0} = ax+b$

$$\underbrace{\begin{vmatrix} x+1 & & & & \\ & \ddots & & & \\ & & x+1 & & \\ & & & 1 & \\ & & & & \ddots \\ & & & & & 1 \end{vmatrix}}_{r \quad n-r} = \alpha x + \beta \qquad (x+1)^r = \alpha x + \beta$$

$\Rightarrow rg M = 1$

<u>Synthèse</u> : $rg M = 1$ On écrit $M = L \times C$ où $L = (\lambda_1, \dots, \lambda_n)$

$A = (A_1, \dots, A_n)$ (colonnes de A)

$\det(A + xM) = \det(A_1 + \lambda_1 x C ; A_2 + \lambda_2 x C, \dots, A_n + \lambda_n x C)$

$= \det(A_1 ; \dots, A_n + \lambda_n x C) + \lambda_1 x \det(C, \dots, A_n + \lambda_n x C)$

$= \dots$

$= \det(A_1, \dots, A_n) + x \times \sum_{i=1}^{n} \lambda_i \det(A_1, \dots, A_{i-1}, C, A_{i+1}, \dots, A_n)$

$\in \mathbb{K}_1[X]$

d'où CNS : $rg M \leq 1$

Exo 365

Soient $(z_1 \dots z_n) \in \mathbb{C}^n$ et $(P_1 \dots P_n) \in (\mathbb{C}_{n-1}[X])^n$

$$M = \begin{bmatrix} P_1(z_1) & \dots & P_1(z_n) \\ P_2(z_1) & & \vdots \\ \vdots & & \\ P_n(z_1) & \dots & P_n(z_n) \end{bmatrix} \qquad V(z_1 \dots z_n) = \begin{bmatrix} 1 & 1 & \dots & 1 \\ z_1 & z_2 & \dots & z_n \\ \vdots & & & \\ z_1^{n-1} & z_2^{n-1} & \dots & z_n^{n-1} \end{bmatrix}$$

1) Soit $i \in [\![1, n]\!]$, on note $P_i(X) = \sum_{k=1}^{n} \alpha_{i,k} X^{k-1}$

On pose $A = (\alpha_{i,j})_{1 \leq i,j \leq n}$

d'où on montre que $\boxed{M = A \times V(z_1 \dots z_n)}$

Soient $(i,j) \in [\![1, n]\!]^2$ on a :

$$M_{i,j} = P_i(z_j) = \sum_{k=1}^{n} \alpha_{i,k} \underbrace{z_j^{k-1}}_{(V(z_1 \dots z_n))_{k,j}} = (A V_{(z_1 \dots z_n)})_{i,j}$$

donc $M = \begin{bmatrix} \alpha_{1,1} & - & \alpha_{1,n} \\ | & & | \\ \alpha_{n,1} & - & \alpha_{n,n} \end{bmatrix} \times V(z_1 - z_n)$

d'où $\boxed{\det(M) = \begin{vmatrix} \alpha_{1,1} & - & \alpha_{1,n} \\ | & & \\ \alpha_{n,1} & - & \alpha_{n,n} \end{vmatrix} \times \prod_{1 \leq i < j \leq n} (z_j - z_i)}$

2) $M = \begin{bmatrix} 1 & 2^{n-1} & 3^{n-1} & \cdots & n^{n-1} \\ 2^{n-1} & & & & \\ \vdots & & & & \\ n^{n-1} & \cdots & \cdots & & (2n-1)^{n-1} \end{bmatrix} \overset{?}{=} \begin{bmatrix} P_1(z_1) & \cdots & P_1(z_n) \\ \vdots & & \\ P_n(z_1) & \cdots & P_n(z_n) \end{bmatrix}$

On pose $\forall i \in [\![1, n]\!]$

$\begin{cases} z_i = i \\ P_i(X) = (X + i - 1)^{n-1} = \sum_{k=0}^{n-1} \binom{n-1}{k} X^k (i-1)^{n-1-k} \end{cases}$

$= \sum_{k=1}^{n} \binom{n-1}{k-1} (i-1)^{n-k} X^k$

donc on a bien $M = \begin{bmatrix} P_1(z_1) & \cdots & P_1(z_n) \\ \vdots & & \vdots \\ P_n(z_1) & \cdots & P_n(z_n) \end{bmatrix}$

d'où

$\det(M) = \underbrace{\begin{vmatrix} 0 & \text{—} & 0 & 1 \\ \binom{n-1}{0} & \cdots & & \binom{n-1}{n-1} \\ \vdots & & & \vdots \\ \binom{n-1}{0}(n-1)^{n-1} & \cdots & & \binom{n-1}{n-1}(n-1)^0 \end{vmatrix}}_{= \Delta} \times \det(V(1 \cdots n))$

développement par rapport à la 1ère ligne

$\Delta = (-1)^{n+1} \times \begin{vmatrix} \binom{n-1}{0} & \cdots & \binom{n-1}{n-2} \\ \binom{n-1}{0} 2^{n-1} & & \binom{n-1}{n-2} \times 2 \\ \vdots & & \vdots \\ \binom{n-1}{0}(n-1)^{n-1} & \cdots & \binom{n-1}{n-2}(n-1)^1 \end{vmatrix}$

$= (-1)^{n+1} \left(\prod_{k=0}^{n-2} \binom{n-1}{k} \right) \begin{vmatrix} 1 & \text{—} & 1 \\ 2^{n-1} & \text{—} & 2^1 \\ \vdots & & \vdots \\ (n-1)^{n-1} & \cdots & (n-1) \end{vmatrix}_{(n-1)}$

(on factorise par les coeff binomiaux)

on factorise chaque ligne L_i par i

$= (-1)^{n+1} \left(\prod_{k=0}^{n-2} \binom{n-1}{k} \right) \cdot \underbrace{\begin{vmatrix} 1 & \text{—} & 1 \\ 2^{n-2} & \text{—} & 1 \\ \vdots & & | \\ (n-1)^{n-2} & \cdots & 1 \end{vmatrix}} \times (n-1)!$

$= (-1)^{\lfloor \frac{n-1}{2} \rfloor} \begin{vmatrix} 1 & \text{—} & & 1 \\ | & 2 & \cdots & 2^{n-2} \\ | & \vdots & & \vdots \\ 1 & n-1 & & (n-1)^{n-2} \end{vmatrix}$

$$\Delta = (-1)^{n+1} \times (-1)^{\lfloor \frac{n-1}{2} \rfloor} \times (n-1)! \times \left(\prod_{k=0}^{n-2} \binom{n-1}{k}\right) \times \underbrace{\begin{vmatrix} 1 & \cdots & 1 \\ 1 & 2 \cdots & n-1 \\ \vdots & & \vdots \\ 1 & 2^{n-1} & (n-1)^{n-2} \end{vmatrix}}_{\text{Van der Monde}}$$

$$= (-1)^{n+1+\lfloor \frac{n-1}{2} \rfloor} \times (n-1)! \times \left(\prod_{k=0}^{n-2} \binom{n-1}{k}\right) \prod_{1 \leq i < j \leq n-1} (j-i)$$

$$\det M = (-1)^{n+1+\lfloor \frac{n-1}{2} \rfloor} \times (n-1)! \times \left(\prod_{k=0}^{n-2} \binom{n-1}{k}\right)$$

$$\times \left(\prod_{1 \leq i < j \leq n} (j-i)\right) \times \left(\prod_{1 \leq i < j \leq n-1} (j-i)\right)$$

$$= (-1)^{n+1+\lfloor \frac{n-1}{2} \rfloor} (n-1)! \left(\prod_{k=0}^{n-2} \binom{n-1}{k}\right) \left(\prod_{1 \leq i < j \leq n-1} (j-i)^2\right) \underbrace{\prod_{i=1}^{n-1} (n-i)}_{(n-1)!}$$

$$\boxed{\det M = (-1)^{n+1+\lfloor \frac{n-1}{2} \rfloor} (n-1)!^2 \left(\prod_{k=0}^{n-2} \binom{n-1}{k}\right) \prod_{1 \leq i < j \leq n-1} (j-i)^2}$$

Exo 354 :

$A = (\min\{i, j\})_{i,j \in [\![1;n]\!]}$, $n \neq 0$

a) $A = \begin{bmatrix} 1 & \cdots & 1 \\ 1 & 2 \cdots & 2 \\ \vdots & \ddots & \vdots \\ 1 & 2 & n \end{bmatrix}$ $\quad \det A \underset{\substack{C_i \leftarrow C_{i-1} \\ \forall i \in [\![2;n]\!]}}{=} \begin{vmatrix} 1 & & 0 \\ & \ddots & \\ (*) & & 1 \end{vmatrix} = 1$

b) $\det A \neq 0$, $A \in GL_n(\mathbb{K})$.

Inversion du système :
$\begin{cases} f_1 = e_1 + e_2 + \dots + e_n \\ f_2 = e_1 + 2e_2 + \dots + 2e_n \\ \vdots \\ f_n = e_1 + 2e_2 + 3e_3 + \dots + ne_n \end{cases} \iff \begin{cases} f_1 = e_1 + \dots + e_n \\ f_2 - f_1 = e_2 + \dots + e_n \\ \vdots \\ f_n - f_{n-1} = e_n \end{cases}$

Donc $\begin{cases} e_1 = 2f_1 - f_2 \\ e_2 = -f_1 + 2f_2 - f_3 \\ \vdots \\ e_n = f_n - f_{n-1} \end{cases}$

$$A^{-1} = \begin{array}{c} \begin{array}{ccc} e_1 & \cdots & e_n \end{array} \\ \begin{bmatrix} 2 & -1 & & 0 \\ -1 & 2 & \ddots & \\ & \ddots & 2 & -1 \\ 0 & & -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} f_1 \\ \vdots \\ f_n \end{array}$$

AUTRE MÉTHODE : $(i,j) \in [\![1;n]\!]^2$

$X_1 = \sum_{i=1}^{n} E_i = \begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \end{bmatrix}$ $\quad \min(i,j) = X_i^T X_j$

$A = T_- \times T_+$ les lignes de T_+

les lignes de T_+ sont les X_i^T

les colonnes de T_- sont les X_j

$$T_- = \begin{bmatrix} 1 & & 0 \\ | & \ddots & \\ 1 & \cdots & 1 \end{bmatrix} \qquad T_+ = \begin{bmatrix} 1 & \cdots & 1 \\ & \ddots & | \\ 0 & & 1 \end{bmatrix}$$

$$J_+ = \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{bmatrix} \Rightarrow T_+ = \sum_{i=0}^{n-1} J_+^i$$

$$J_- = \begin{bmatrix} 0 & \cdots & 0 \\ 1 & \ddots & \\ 0 & 1 & 0 \end{bmatrix} \Rightarrow T_- = \sum_{i=0}^{n-1} J_-^i$$

$$\begin{aligned} (I_n - J_+)T_+ &= I_n \\ (I_n - J_-)\,T_- &= I_n \end{aligned} \Rightarrow \begin{cases} T_+^{-1} = I_n - J_+ \\ T_-^{-1} = I_n - J_- \end{cases}$$

$$A^{-1} = T_+^{-1} \times T_-^{-1} = (I_n - J_+)(I_n - J_-)$$

$$= I_n - (J_+ + J_-) + J_+ J_- = \begin{pmatrix} 2 & -1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & 2 & -1 \\ 0 & & -1 & 1 \end{pmatrix}$$

Exo 384

Analyse

$(A,B) \in GL_n(\mathbb{C})$, $AB = -BA$

$\det(AB) = \det A \det B = \det(-AB) = (-1)^n \underbrace{\det(A)\det(B)}_{\neq 0} \Rightarrow (-1)^n = 1$

$\Rightarrow n$ doit être pair

Synthèse : $n = 2p$. $A = \begin{bmatrix} I_p & 0 \\ 0 & -I_p \end{bmatrix}$ $\quad \forall i \in [\![1;p]\!]$. $u(e_i) = e_i$

$\forall i \in [\![p+1;n]\!]$, $u(e_i) = -e_i$

$B = (e_{p+1}, \dots, e_n, e_1, \dots, e_p)$

$\mathrm{mat}_B(u) = \begin{bmatrix} -I_p & 0 \\ 0 & I_p \end{bmatrix} = -A \qquad \Rightarrow A \sim -A$

$\Rightarrow \exists B \in GL_n(\mathbb{C})$,

$A = B^{-1}(-A)B \Rightarrow BA = -AB$

Exo 361

$CD = DC$ et $D \in GL_n(\mathbb{K})$. $\det\begin{pmatrix}A&B\\C&D\end{pmatrix}$ et $\det(AD-BC)$?

$\begin{pmatrix}A&B\\C&D\end{pmatrix}\begin{pmatrix}D&0\\0&I_n\end{pmatrix} = \begin{pmatrix}AD&B\\CD&D\end{pmatrix}$; $\det\begin{pmatrix}A&B\\C&D\end{pmatrix}\times\det\begin{pmatrix}D&0\\0&I_n\end{pmatrix} = \det\begin{pmatrix}AD&B\\CD&D\end{pmatrix}$

$C_1 \leftarrow C_1 - C_2 \times C \Rightarrow \begin{vmatrix}A&B\\C&D\end{vmatrix}\times\begin{vmatrix}D&0\\0&I_n\end{vmatrix} = \begin{vmatrix}AD-BC&B\\CD-DC&D\end{vmatrix}$

Or $CD=DC$ donc $\det\begin{pmatrix}A&B\\C&D\end{pmatrix}\times\det(D) = \det(AD-BC)\times\det(D)$

$\Rightarrow$ si D est inversible $\Rightarrow \det\begin{pmatrix}A&B\\C&D\end{pmatrix} = \det(AD-BC)$

~~Si D pas inversible : contre exemple : matrices 2x2 :~~
~~$A = E_{1,1}$, $B = E_{2,2}$, $C = E_{1,2}$, $D = E_{2,1}$~~

~~$\begin{bmatrix}A&B\\C&D\end{bmatrix} = \begin{bmatrix}1&0&0&0\\0&0&0&1\\0&1&0&0\\0&0&1&0\end{bmatrix}$~~ ; $AD-BC=0$
MAIS on n'a pas $CD=DC$

Si D pas inversible, $D-\lambda I_n \in GL_n(\mathbb{K})$ $\forall \lambda \in \underbrace{\mathbb{K}\setminus sp(D)}_{\text{infinité de valeurs}}$

De plus : $(D-\lambda I_n)C = DC - \lambda C = CD - \lambda C = C(D-\lambda I_n)$

Par la question précédente appliquée à $(D-\lambda I_n) \in GL_n(\mathbb{K})$,

on a $\det\begin{pmatrix}A&B\\C&D-\lambda I_n\end{pmatrix} = \det(A(D-\lambda I_n)-BC)$ $\forall \lambda \in \mathbb{K}\setminus sp(D)$

On a deux polynômes en λ qui coincident en une infinité de λ donc ils sont égaux $\forall \lambda \in \mathbb{K}$

Pour $\lambda = 0$, on obtient $\det\begin{pmatrix}A&B\\C&D\end{pmatrix} = \det(AD-BC)$

Rq: c'est faux si on n'a pas $DC=CD$

Exo 382

a) Mq $\forall t \geq 0$, $\det(A^2+tI_n) \geq 0$.

$A^2 + tI_n = \overline{(A+i\sqrt{t}\, I_n)}(A-i\sqrt{t}\, I_n)$

$\det(\overline{C}) = \overline{\det(C)}$; $\det(A^2+tI_n) = \det(A+i\sqrt{t}\,I_n)\det(\overline{A+i\sqrt{t}\,I_n})$
$= |\det(A+i\sqrt{t}\,I_n)|^2 \geq 0$

b) Si $A^2+B^2 = -I_n$ $\Rightarrow$ pour $t \geq 0$, on aurait $A^2+tI_n = -B^2-I_n+tI_n$
$= -(B^2+(1-t)I_n)$

$\underbrace{\det(A^2+tI_n)}_{\geq 0} = \underbrace{(-1)^n}_{-1}\underbrace{\det(B^2+(1-t)I_n)}_{\geq 0}$ pour $0 \leq t \leq 1$

Donc $\det(A^2 + tI_n) = 0 \quad \forall t \in [0;1]$

A^2 aurait une infinité de valeurs propres (les $-t$)

↳ impossible

c) $n = 2m$ $\qquad A^2 + B^2 = -I_{2m}$

$$A = \begin{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \end{bmatrix} , \quad B = 0$$

$A^2 = -I_{2m}$