

Ex 307:

$$A \in M_{3,2}(\mathbb{R})$$

$$B \in M_{2,3}(\mathbb{R})$$

$$AB = \text{Diag}(0, 1, 1) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow BA = I$$

$$ABAB = AB$$

$$\text{rg } AB = 2$$

$$BA \in M_2(\mathbb{R}) \quad AB = A(BA)B$$

$$\Rightarrow \text{rg } AB \leq \text{rg } BA \leq 2$$

$$\Rightarrow \text{rg } BA = 2 \Rightarrow BA \in GL_2(K)$$

$$\underbrace{BABABA}_{AB} = BABA$$

$$(BA)^3 = (BA)^2 \text{ et } BA \in GL_2(\mathbb{R})$$

$$\Rightarrow BA = (BA)^2 \times (BA)^{-2}$$

$$BA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$$

Ex 345:

$$A \in M_n \{0, 1\}$$

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

$C_1 \quad C_n$

$$\text{ROLNT: } \sum_{i=1}^n \alpha_i c_i = 0$$

$$\forall j \in [1, n-1], \sum_{\substack{i=1 \\ i \neq j}}^n \alpha_i = 0 \quad (J), \quad \sum_{i=1}^n \alpha_i = 0 \quad (M)$$

$$\forall j \in [1, n-1], (M) - (J) \Rightarrow \alpha_j = 0$$

$$\Rightarrow A \text{ inversible.}$$

$$\#(1) \geq n^2 - (n-1)$$

Soit B qui possède $n^2 - (n-1) + 1$ "1" dans ses coefficients

$$\Rightarrow \begin{cases} n-2 & \text{"0"} \\ n-1 & \text{colonnes} \end{cases} : \exists c_i, c_j \text{ identiques, } i \neq j.$$

Dans B non inversible

$$\text{Donc } \boxed{\#(1) = n^2 - (n-1)}$$

Ex 312:

$N \in M_n(\mathbb{R})$ nilp.

$$\begin{aligned} a) \quad (I_n - N) \sum_{k=0}^{n-1} N^k &= \sum_{k=0}^{n-1} (N^k - N^{k+1}) \\ &= I_n - N^n \\ &= I_n = [O] \end{aligned}$$

$\Rightarrow I_n - N$ est ℓ -inverse

$$\begin{aligned} b) \quad \sum_{k=0}^{n-1} (k+1) x^k &= \frac{d}{dx} \left(\sum_{k=0}^{n-1} x^{k+1} \right) \\ &= \frac{d}{dx} \left(x \frac{1-x^n}{1-x} \right) \\ &= \frac{1-x^n}{1-x} + x \left(\frac{-nx^{n-1} - (1-x)}{(1-x)^2} \right) \\ &= \frac{(1-x^n)(1-x) + nx^n - x^{n+1} - x}{(1-x)^2} \end{aligned}$$

$$\sum_{k=0}^{n-1} (k+1) x^k = \frac{1-x^n + nx^n}{(1-x)^2}$$

$$\begin{aligned} (N - I_n)^2 \sum_{k=0}^{n-1} (k+1) N^k &= \sum_{k=0}^{n-1} (k+1) N^{k+2} - \sum_{k=0}^{n-1} 2(k+1) N^{k+1} + \sum_{k=0}^{n-1} (k+1) N^k \\ &= \sum_{k=1}^n k N^{k+1} - 2 \sum_{k=0}^{n-1} (k+1) N^{k+1} + \sum_{k=1}^{n-1} (k+2) N^{k+1} \\ &= \cancel{n N^{n+1}} + \cancel{(n-1) N^n} - 2 \cancel{n N^n} - 2 N + \cancel{1 N^0} + 2 N \\ &= I_n \end{aligned}$$

$\Rightarrow (N - I_n)^2$ est ℓ -inverse

Ex 301:

$A \in M_n(\mathbb{C})$

$$\begin{aligned} \varphi_A : \mathbb{R}_{2n^2+4n^2}[X] &\longrightarrow M_n(\mathbb{C}) \\ P &\longmapsto P(A) \end{aligned}$$

φ_A est linéaire

Par th. du rang, $2n^2 + 4n^2 = \dim \text{Ker } \varphi_A + \text{rg } \varphi_A$

Or, $\text{rg } \varphi_A \leq 2n^2 \Rightarrow \dim \text{Ker } \varphi_A \geq 4n^2$

Donc $\exists P \in \mathbb{R}_{2n^2+4n^2}[X]$ tq $P(A) = [O]$

Pour s'en convaincre :

$$M_n(\mathbb{C}) = \mathbb{R}\text{-ev de dim } 2n^2$$

Dont une base est $\{E_{i,j}\}_{1 \leq i,j \leq n} \cup \{iE_{i,i}\}_{1 \leq i \leq n}$

Ex 350:

$$\mathcal{E}_n = \{A \in M_n(\mathbb{R}), \exists \lambda \in \mathbb{R}, A^T A = \lambda I_n\}$$

$$\mathcal{D}_n = \{\dim V, V \text{ sous-esp de } M_n(\mathbb{R}), V \subset \mathcal{E}_n\}$$

$$d_n = \max \mathcal{D}_n$$

Mq d_n est bien défini.

$$\mathcal{I} = \text{Vect}(I_n). \quad (\text{sous-esp de } M_n(\mathbb{R}))$$

$$\text{Soit } \lambda \in \mathbb{R}. \quad (\lambda I_n)^T (\lambda I_n) = \lambda^2 I_n$$

$$\text{Donc } \mathcal{I} \subset \mathcal{E}_n, \dim(\mathcal{I}) = 1 \quad \forall \lambda \in \mathbb{R}$$

$$\Rightarrow \mathcal{D}_n \text{ non vide, } 1 \in \mathcal{D}_n$$

$$\text{De plus } \forall V \text{ sous-esp de } M_n(\mathbb{R}), \dim(V) \leq n^2$$

$$\text{Donc } \mathcal{D}_n \subset [0, n^2] \Rightarrow d_n \text{ bien défini et } d_n \leq n^2$$

• $n=1$. $\dim M_1(\mathbb{R}) = 1$

$$1 \in \mathcal{D}_1 \text{ donc } \boxed{d_1 = 1}$$

• $n=2$. $\dim M_2(\mathbb{R}) = 4$

On sait que $d_2 \leq 4$.

$$\text{Sinon, } \exists V \text{ sous-esp de } M_2(\mathbb{R}), \dim V = 4$$

$$\Rightarrow V = M_2(\mathbb{R})$$

$$\text{Or } M_2(\mathbb{R}) \not\subset \mathcal{E}_2 : E_{11}^T E_{11} = E_{11} \neq \lambda I_2 \quad \forall \lambda \in \mathbb{R}$$

$$\text{Donc } 1 \leq d_2 \leq 3$$

$$I_2 \in \mathcal{E}_2 \quad A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$A^T A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$$

$$\Rightarrow A \in \mathcal{E}_2$$

$$\text{Soit } (\lambda, \mu) \in \mathbb{R}^2. \quad \lambda I_2 + \mu A = \begin{pmatrix} \lambda & \mu \\ -\mu & \lambda \end{pmatrix} = B$$

$$B^T B = \begin{pmatrix} \lambda^2 + \mu^2 & 0 \\ 0 & \lambda^2 + \mu^2 \end{pmatrix} = (\lambda^2 + \mu^2) I_2 \Rightarrow B \in \mathcal{E}_2$$

Donc $\text{Vect}(I_2, A) \subset \mathcal{B}_2$, et (I_2, A) libre
 $\Rightarrow 2 \leq d_2 \leq 3$

Soit $V \subset \mathcal{B}_2$ la sous-espèce par somme:

$$V(A, B) \in V^2, A+B \in V$$

$$\exists (r_1, r_2) \in \mathbb{R}, \begin{cases} A^T A = r_1 I_2 \\ B^T B = r_2 I_2 \end{cases}$$

$$\exists r_3 \in \mathbb{R}, (A+B)^T (A+B) = r_3 I_2$$

$$\underbrace{A^T A}_{r_1 I_2} + \underbrace{A^T B + B^T A}_{r_2 I_2} + \underbrace{B^T B}_{r_2 I_2} = r_3 I_2$$

$$A^T B + B^T A = (r_3 - r_1 - r_2) I_2$$

On pose $D = A^T B = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}$

$$D + D^T = r_2 I_2$$

$$\Rightarrow \begin{cases} d_{11} = d_{22} = \frac{r_2}{2} \\ d_{12} = -d_{21} \end{cases}$$

$$\Rightarrow \exists (\alpha, \beta) \in \mathbb{R}^2, D = \alpha I_2 + \beta \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_M$$

$$A^T B = \alpha I_2 + \beta M$$

$$\rightarrow \text{Si } r_1 = 0, A^T A = 0 \Rightarrow A = 0$$

$$\rightarrow \text{Si } r_1 \neq 0:$$

$$A^T \in GL_2(\mathbb{R})$$

$$\text{Donc } B = A \left(\frac{\alpha}{r_1} I_2 + \frac{\beta}{r_1} M \right)$$

$$\text{Donc } \forall A, B \in V^2, \exists \lambda, \mu \in \mathbb{R}, A = B(\lambda I_2 + \mu M)$$

Par dim $V = 2$ par l'absurde: supposons $\exists (A, B, C) \in V^3, (A, B, C)$ libre

V non, donc $\exists (\lambda_1, \lambda_2, \mu_1, \mu_2) \in \mathbb{R}^4,$

$$A = B(\lambda_1 I_2 + \mu_1 M)$$

$$C = B(\lambda_2 I_2 + \mu_2 M)$$

$\mu_1, \mu_2 \neq 0$ sinon (A, B, C) liée.

$$-\mu_2 A + (\lambda_1 \mu_2 - \lambda_2 \mu_1) B + \mu_1 C = -\lambda_1 \mu_2 B - \mu_1 \mu_2 M + \lambda_1 \mu_1 B - \lambda_2 \mu_1 B + \mu_1 \lambda_2 B + \mu_1 \mu_1 M = 0$$

RLNT car $\mu_1 \neq 0$.

$\hookrightarrow$ absurde car (A, B, C) libre. D'où $d_2 = 2$

$$\bullet \quad n=3 \quad \dim M_3(\mathbb{R}) = 9$$

$$1 \leq d_3 \leq 8$$

$$M_q \quad d_3 = 1$$

On a toujours $\forall V \subset \mathcal{E}_3$ stable par somme

$$\forall (A, B) \in V^2, \quad \underbrace{A^T B + B^T A}_{G_1} = \lambda I_3 \quad \Rightarrow G_1 + G_1^T = \lambda I_3$$

$$\Rightarrow \begin{cases} g_{11} = g_{22} = g_{33} \\ g_{12} = -g_{21} \\ g_{13} = -g_{31} \\ g_{23} = -g_{32} \end{cases}$$

$$\Rightarrow \exists (a, b, c, d) \in \mathbb{R}^4, \quad G = aI_3 + \underbrace{b \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{M_1} + \underbrace{c \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}}_{M_2} + \underbrace{d \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}}_{M_3}$$

$$\text{Si } \lambda_1 = 0, \quad A = 0$$

$$\text{Si } \lambda_1 \neq 0, \quad A \text{ inversible}$$

$$B = A \left(\frac{a}{\lambda_1} I_3 + \frac{b}{\lambda_1} M_1 + \frac{c}{\lambda_1} M_2 + \frac{d}{\lambda_1} M_3 \right)$$

$$\forall (A, B) \in V^2$$

Par l'absurde supposons $d_3 \neq 1$.

$$\exists V \text{ sev de } M_3(\mathbb{R}), \quad V \subset \mathcal{E}_3, \quad \dim V = 2$$

$$\Rightarrow \exists (A, B) \text{ libre dans } V.$$

$$\text{De plus, } \exists (\lambda, \mu_1, \mu_2, \mu_3) \in \mathbb{R}^4,$$

$$A = B (\lambda I_3 + \mu_1 M_1 + \mu_2 M_2 + \mu_3 M_3)$$

$$\Rightarrow \exists i \in \{1, 2, 3\}, \quad \mu_i \neq 0$$

$$\text{Comme } V \text{ sev, } -\lambda B + A \in V$$

$$-\lambda B - A = B \underbrace{(\mu_1 M_1 + \mu_2 M_2 + \mu_3 M_3)}_F$$

$$BF \in V \subset \mathcal{E}_3$$

$$\exists \gamma \in \mathbb{R}, \quad (BF)^T BF = \gamma I_3$$

$$\Leftrightarrow F^T (B^T B) F = \gamma I_3$$

$$B^T B = \lambda_0 I_3$$

$$\Leftrightarrow F^T F = \frac{\gamma}{\lambda_0} I_3 \quad \Rightarrow F \in \mathcal{E}_2$$

$$F = \begin{pmatrix} 0 & \mu_1 \mu_2 \\ -\mu_1 & 0 \mu_3 \\ \mu_1 \mu_3 & 0 \end{pmatrix}$$

$$F^T F = -F^2 = \begin{pmatrix} -\mu_1^2 - \mu_2^2 & -\mu_1 \mu_3 & -\mu_1 \mu_3 \\ -\mu_1^2 - \mu_3^2 & -\mu_2 \mu_1 & -\mu_2 \mu_1 \\ * & -\mu_2^2 - \mu_3^2 & * \end{pmatrix}$$

$$\Rightarrow \mu_1^2 = \mu_2^2 = \mu_3^2 \quad / \text{ Absurde}$$

$$\text{Donc } \boxed{d_3 = 1}$$

Prof:

$$\mathbb{C}_3 \setminus \{0\} \subset GL_3(\mathbb{R})$$

$$\text{car } A^T A = 0 \Rightarrow A = 0$$

si $d_3 \geq 2$, on pourrait trouver

$$A, B \text{ non colinéaires, } (A, B) \in \mathbb{C}_3^2$$

$$\text{et } \det(A, B) \in \mathbb{C}_3 \Rightarrow A, B \neq 0 \Rightarrow A, B \in GL_3(\mathbb{R})$$

$$\Rightarrow A + xB \in \mathbb{C}_3 \quad \forall x \in \mathbb{R}$$

On considère $P(x) = \det(A + xB)$ polynôme

$$= \det(B(B^{-1}A + xI_n))$$

$$= \underbrace{\det B}_{\neq 0} \times \underbrace{\det(B^{-1}A + xI_n)}_{\text{polynôme de deg 3 unitaire.}}$$

$$\Rightarrow \deg P = 3 \Rightarrow \text{P admet une racine } x_0 \text{ réelle}$$

$$\Rightarrow P(x_0) = 0$$

$$\Rightarrow A + x_0 B \notin GL_3(\mathbb{R}), \text{ or } A + x_0 B \in \mathbb{C}_3$$

contradiction

$\neq 0$ car A et B non colinéaires

$$\text{D'où } \boxed{d_3 = 1}$$

Ex 308:

$$a) \text{ Soit } A, B \in M_n(\mathbb{C}), \begin{cases} \text{rang } A = \text{rang } B \\ A^2 B = A \end{cases}$$

$$\text{Mq } B^2 A = B$$

$$\text{Mq } \text{Ker } A = \text{Ker } B:$$

$$\bullet \text{ Ker } B \subset \text{Ker } A$$

$$\bullet \text{ Par th. du rang, } \dim \text{Ker } B = \dim \text{Ker } A$$

$$\Rightarrow \text{Ker } A = \text{Ker } B$$

$$\text{On a } A(AB - I_n) = 0$$

$$\Rightarrow B(AB - I_n) = 0$$

$$\text{D'où } BAB = B$$

• Mg $\text{Ker } B \oplus \text{Im } B = E$

Soit $x \in \text{Ker } B \cap \text{Im } B$

$$\begin{cases} Bx = 0 & \text{et } \exists Y \in M_n(E), \\ x = BY \end{cases}$$

Et $AX = 0$

$$\Rightarrow ABY = 0$$

$$\Rightarrow BABY = 0$$

$$\Rightarrow BY = 0$$

$$\Rightarrow X = 0$$

• Soit $X \in \text{Ker } B$

$$B^2AX = BX$$

• Soit $Y \in \text{Im } B : \exists V \in M_n(E), Y = BV$

$$B^2ABV = B^2V = BY$$

$$B^2AY = BY$$

D'où $B^2A = B$

Ex 334:

E K-es de dim finie, $u \in \mathcal{L}(E)$

Mg $\text{tr}(u) = \text{tr}(u|_{\text{Im } u})$

Soit $u \in \mathcal{L}(E)$. On pose $n = \text{rg } u$

Soit $B = (\underbrace{e_1, \dots, e_n}_S, \underbrace{e_{n+1}, \dots, e_m}_{\text{Ker } u})$ base de E adaptée à $S \oplus \text{Ker } u = E$

Par th. du rang, $F = (u(e_1), \dots, u(e_n))$ base de $\text{Im } u$

$$\begin{aligned} \text{Soit } B' &= \text{mat}(u)_F \\ &= \begin{bmatrix} u(e_1) & \dots & u(e_n) & & \\ & * & & & \\ & & & & \\ & & & & \\ & & & & u(e_n) \end{bmatrix} \end{aligned}$$

TBI: $F'' = (u(e_1), \dots, u(e_n), f_{n+1}, \dots, f_n)$ base de E

$$\text{mat}(u)_{F''} = \begin{bmatrix} B' & * \\ 0 & \bigcirc \end{bmatrix} \begin{matrix} u(e_1) \\ \vdots \\ f_n \end{matrix}$$

D'où $\text{tr } u = \text{tr } B'$
 $= \text{tr}(u|_{\text{Im } u})$

Ex 342:

Soit $M \in M_n(\mathbb{R})$. $S = \{P^{-1}MP, P \in GL_n(\mathbb{R})\}$

$M_0(S = \{\lambda_0 I_n\}, \lambda_0 \neq 0) \Leftrightarrow (\text{tout \u00e9l\u00e9ment de } S \text{ a ses coeff diagonaux nuls.})$

$\Rightarrow$ Direct

$\Leftarrow$ • Si M est de la forme $\lambda_0 I_n, \lambda_0 \neq 0$.

Alors $S = \{\lambda_0 I_n\}$

• Si M n'est pas de cette forme,

$\exists X \in \mathbb{R}^n, (X, MX) \text{ libre}$

$$M \sim \begin{pmatrix} 0 & & \\ & 1 & \\ & & * \end{pmatrix} \begin{pmatrix} X \\ MX \end{pmatrix} \in S$$

$\rightarrow 0$ sur la diag $\Rightarrow$ absurde.

Ex 325:

$A \in M_3(\mathbb{R})$ tq $A^2 = 0$

(si $A=0, A^2=0$)

Analyse: Soit $A \neq 0$ tq $A^2 = 0$

Soit u_A l'ECA \u00e0 A .

Th du rang \u00e0 $u_A|_{\text{Im } u_A}$

$$\text{rg}(u_A|_{\text{Im } u_A}) + \dim \text{Ker}(u_A|_{\text{Im } u_A}) = \text{rg } u_A$$

$$\text{Et } \text{Ker}(u_A|_{\text{Im } u_A}) \subset \text{Ker } u_A$$

$$\dim \text{Ker}(u_A|_{\text{Im } u_A}) \leq \dim \text{Ker } u_A = 3 - \text{rg } u_A$$

$$\text{rg } u_A \leq \text{rg}(u_A^2) + 3 - \text{rg } u_A$$

$$\text{rg } u_A \leq \frac{\text{rg}(u_A^2) + 3}{2}$$

$$\text{rg } u_A \leq 1$$

$$\Rightarrow \text{rg } u_A = 1 \Rightarrow \text{rg } A = 1$$

$$\exists X, Y \in \mathbb{R}^3 \setminus \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\} \text{ tq } A = XY^T, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$A^2 = XY^TXY^T \quad Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$\text{Or } Y^TX = (x_1y_1 + x_2y_2 + x_3y_3)$$

$$A = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \begin{pmatrix} x_1y_1 & * & * \\ * & x_2y_2 & * \\ * & * & x_3y_3 \end{pmatrix}$$

$$\Rightarrow Y^TX = \text{tr } A$$

$$\begin{aligned}
 A^2 &= X \operatorname{tr} A Y^T \\
 &= \operatorname{tr} A X Y^T \\
 &= \operatorname{tr} A \underbrace{X A}_{\neq 0}
 \end{aligned}$$

$$\Rightarrow \operatorname{tr} A = 0$$

D'où $A = 0$ ou $A = (\lambda_1 C \ \lambda_2 C \ \lambda_3 C)$, $C \in \mathbb{R}^3 \setminus \{0\}$
 $\operatorname{tr} A = 0$.

Ex 272:

Soit G sous groupe multiplicatif de $M_n(\mathbb{K})$

Mq toutes les matrices de G ont le même rang.

On note r le nombre.

Soit $A \in G$, $A^{-1} \in G$

$$\left. \begin{aligned}
 \begin{cases} A A^{-1} = I^n \\ A I^n = A \end{cases} &\Rightarrow \begin{cases} \operatorname{rg} A \geq \operatorname{rg} I^n \\ \operatorname{rg} A \leq \operatorname{rg} I^n \end{cases} \end{aligned} \right\} \Rightarrow \operatorname{rg} A = \operatorname{rg} I^n \quad \forall A \in G.$$