Exercice 214

Soit E un $\mathbb{C}$-ev. On note $n = \dim E$. Soit p_1, p_2 2 projecteurs.
Mq $p_1 + p_2$ projecteur ssi $p_1 \circ p_2 = p_2 \circ p_1 = 0$.

$\boxed{\Leftarrow}$ (H) $p_1 \circ p_2 = p_2 \circ p_1 = 0$.

$(p_1 + p_2) \circ (p_1 + p_2) = p_1 \circ p_1 + p_1 \circ p_2 + p_2 \circ p_1 + p_2 \circ p_2$
$= p_1 + p_2$.

$\boxed{\Rightarrow}$ (H) $p_1 + p_2$ projecteur.

$(p_1 + p_2) \circ (p_1 + p_2) = p_1 + p_2 + p_1 \circ p_2 + p_2 \circ p_1$

$\Rightarrow p_1 \circ p_2 + p_2 \circ p_1 = 0$

$\Rightarrow p_1 \circ p_2 + p_1 \circ p_2 \circ p_1 = 0$

$\Rightarrow p_1 \circ p_2 \circ p_1 + p_2 \circ p_1 = 0$

$\Rightarrow p_1 \circ p_2 = p_2 \circ p_1 \Rightarrow p_1 \circ p_2 = p_1 \circ p_2 = 0$.

Exercice 215 :

a) $f^2 = 0 \Rightarrow \operatorname{Im} f \subset \operatorname{Ker} f$. donc $\operatorname{rg} f \leq \dim \operatorname{Ker} f$

Par le th du rang, $\dim E = \operatorname{rg} f + \dim \operatorname{Ker} f \leq 2 \dim \operatorname{Ker} f$.

Donc $\dim \operatorname{Ker} f \geq \frac{\dim E}{2}$.

b) Mq $E = \operatorname{Ker} f \oplus \operatorname{Ker} g$

- Soit $x \in \operatorname{Ker} f \cap \operatorname{Ker} g$. x tq $\begin{cases} g(x) = 0 \\ f(x) = 0 \end{cases}$, $u(x) = 0$ donc $x = \{0_E\}$.

Donc $\operatorname{Ker} f + \operatorname{Ker} g = \operatorname{Ker} f \oplus \operatorname{Ker} g$.

- $\dim \operatorname{Ker} f \oplus \dim \operatorname{Ker} g = \underbrace{\dim \operatorname{Ker} f}_{\geq \frac{\dim E}{2}} + \underbrace{\dim \operatorname{Ker} g}_{\geq \frac{\dim E}{2}}$

$\geq \dim E$.

Donc $\operatorname{Ker} f \oplus \operatorname{Ker} g = E$

- On a $\operatorname{Im} f \subset \operatorname{Ker} f$

Mq $\operatorname{rg} f = \dim \operatorname{Ker} f$.

- $\operatorname{Ker} f = \dim E - \dim \operatorname{Ker} g \geq \frac{\dim E}{2}$.

donc $\dim \operatorname{Ker} f = \dfrac{\dim E}{2}$ $\underset{\text{TDR}}{\Rightarrow}$ $\operatorname{rg} f = \dfrac{\dim E}{2}$.

donc $\operatorname{Im} f = \operatorname{Ker} f$ et $\operatorname{Im} g = \operatorname{Ker} g$.

Exercice 216

$\boxed{\Rightarrow}$ (H) $\operatorname{rg} u = \operatorname{rg} u^2$

Comme $\operatorname{Im} u^2 \subset \operatorname{Im} u \Rightarrow \operatorname{Im} u^2 = \operatorname{Im} u$

$\dim E = \dim \operatorname{Ker} u + \operatorname{rg} u$.

Or, $\operatorname{rg} u = \operatorname{rg} u^2 + \dim \operatorname{Ker} u \cap \operatorname{Im} u. \Rightarrow \dim(\operatorname{Ker} u \cap \operatorname{Im} u) = 0$

$\Rightarrow \operatorname{Ker} u \oplus \operatorname{Im} u$.

On note $u|_{\operatorname{Im} u}$ par formule du rang.

$\dim \operatorname{Im} u = \dim \operatorname{Im} u|_{\operatorname{Im} u} + \dim(\operatorname{Ker} u \cap \operatorname{Im} u)$

$= \dim \operatorname{Im} u^2 = \operatorname{rg} u^2$.

$\left\{\begin{array}{l} \operatorname{Ker} u \oplus \operatorname{Im} u \subset E \\ \text{égalité dimensions} \end{array}\right. \Rightarrow E = \operatorname{Ker} u \oplus \operatorname{Im} u.$

$\boxed{\Leftarrow}$ Supposons $E = \operatorname{Ker} u \oplus \operatorname{Im} u$.

Par formule du rang appliquée à $u|_{\operatorname{Im} u}$.

$\operatorname{rg} u = \operatorname{rg} u^2 + \underset{0}{\underset{\shortparallel}{\dim \operatorname{Ker} u \cap \operatorname{Im} u}}$

$\operatorname{rg} u = \operatorname{rg} u^2$.

Exercice 221 :

$P \in \operatorname{Ker} f$. $f(P) = 0 \Leftrightarrow \dots \Leftrightarrow \underline{P(2) = -P(-2)}$

à copier

Récurrence double

$H(n)$: $P(n) = -P(-n)$

$P(n+1) = 2P(n) - P(n-1)$

$= -2P(-n) + P(-n+1)$

$= -P(-n-1)$

Donc $\forall n \in \mathbb{N}$, $P(n) = -P(-n)$

Donc $P(X) + P(-X) = 0 \Rightarrow P(X) = -P(-X) \Rightarrow P$ est impair

Or mi P' est impair (P impair $\Rightarrow P'$ pair)

Exercice 229.

1)

Supposons f isomorphisme

Soit $g \in H$, $f \circ g \circ f = 0 \Leftrightarrow g \circ f = 0 \Leftrightarrow g = 0$.

Donc $H = \{\tilde{0}\}$.

2)

Soit $g \in H$ $g(\operatorname{Im} f) \subset \operatorname{Ker} f$. Soit S tq $S \oplus \operatorname{Im} f = F$

$$\begin{cases} n = \dim E \\ p = \dim F \\ r = \operatorname{rg} f \end{cases} \quad \text{Soit} \quad \varphi : \begin{cases} H \longrightarrow \mathcal{L}(\operatorname{Im} f, \operatorname{Ker} f) \times \mathcal{L}(S, E) \\ g \longmapsto (g|_{\operatorname{Im} f}, g|_S) \end{cases}$$

isomorphisme

Donc $\dim H = r(n-r) + (p-r)n$.

$$\boxed{\dim H = np - r^2}$$

$g \in H \Leftrightarrow g(\operatorname{Im} f) \subset \operatorname{Ker} f$. $\quad F = S \oplus \overset{r}{\overbrace{\operatorname{Im} f}}$. base $(f_1, \dots, f_r, \dots, f_p)$

$E = \operatorname{Ker} f \oplus G$ base $(\underbrace{e_1, \dots, e_{p-r}}_{\text{base de } \operatorname{Ker} f}, \dots, e_n)$

$$\operatorname{mat}_{(f_1, \dots, f_p)}(g) = \begin{array}{c} \\ n-r \\ \\ \end{array} \begin{array}{c} \begin{array}{ccc} g(f_1) & g(f_r) & g(f_p) \end{array} \\ \left(\begin{array}{c|c} \times & \times \\ \hline 0 & \times \end{array}\right) \\ \underset{p-r}{\longleftrightarrow} \end{array} \begin{array}{l} e_1 \\ \vdots \; p-r \\ e_{p-r} \end{array}$$

$n \times (p-r)$

$\dim H$ = nb de coeff qui varient indépendamment

$= r(n-r) + n(p-r)$

$= np - r^2$.

Exercice 239 :

Soit E $\mathbb{K}$ev. On note $n = \dim E$

$\dim \operatorname{Ker} f \underset{①}{\leq} \dim \operatorname{Ker} f^2 \underset{②}{\leq} 2 \dim \operatorname{Ker} f$.

① $\operatorname{Ker} f \subset \operatorname{Ker} f^2 \Rightarrow \dim \operatorname{Ker} f \leq \dim \operatorname{Ker} f^2$.

② Soit $u = f|_{\operatorname{Im} f}$. $\quad \operatorname{rg} f = \operatorname{rg} u + \dim \operatorname{Ker} u$

$= \operatorname{rg} f^2 + \dim \operatorname{Ker} f \cap \operatorname{Im} u$

~~$\operatorname{rg} u \leq \operatorname{rg} f$~~

$\Rightarrow P' = 0$

Donc $\mathrm{Ker}\, f = \{0\}$. NON

Soit $k \geq 2$.

$f(X^k) = (X+1)^k - 2X^k + (X-1)^k$.

$= k(k-1)X^{k-2} + Q(X)$ $\deg Q < k-2$.

On a $\left|\begin{array}{l} ((X^k)_{k\geq 2}) \text{ base de } E \\ \text{l'image de cette base par } f \text{ est une base de } \mathbb{R}[X] \end{array}\right.$

Donc f isomorphisme de E ds $\mathbb{R}[X]$

Exercice 277 :

$\boxed{\subset}$ $y \in \mathrm{Im} f$: $\exists x \in E$, $y = f(x)$

$f^4(y) = f^5(x) = f^3(f^2(x)) = 0_E$.

$f^4(\underbrace{f(x)}_{= y})$

Donc $y \in \mathrm{Ker}\, f^4 \Rightarrow \mathrm{Im} f \subset \mathrm{Ker} f^4$.

$\boxed{\supset}$ Soit $x \in \mathrm{Ker} f^4$, $f^4(x) = 0_E$.

$f^3(\underbrace{f(x)}_{\in \mathrm{Im} f^2}) = 0_E$.

$\Rightarrow \exists \alpha \in E$, $f(x) = f^2(\alpha)$

$\Rightarrow$ $f(\underbrace{x - f(\alpha)}_{\in \mathrm{Ker} f \Rightarrow \in \mathrm{Ker} f^3 = \mathrm{Im} f^2}) = 0_E$

$\Rightarrow \exists \beta \in E$ $f^2(\beta) = x - f(\alpha)$

$\Rightarrow$ $x = f(\alpha) + f^2(\beta)$

$x = f(\alpha + f(\beta))$

$\Rightarrow x \in \mathrm{Im} f$.

ccl $\mathrm{Im} f = \mathrm{Ker} f^4$.

TDR : à f puis à f^2.

$$\begin{cases} n = \operatorname{rg} f + \dim \operatorname{Ker} f \\ n = \operatorname{rg} f^2 + \dim \operatorname{Ker} f^2 \end{cases}$$

Donc
$$\operatorname{rg} f - \operatorname{rg} f^2 = \dim \operatorname{Ker} f^2 - \dim \operatorname{Ker} f$$
$$= \dim (\operatorname{Ker} f \cap \operatorname{Im} f)$$

$\Rightarrow \dim \operatorname{Ker} f^2 = \dim \operatorname{Ker} f \cap \operatorname{Im} f + \dim \operatorname{Ker} f$.

Or $\dim \operatorname{Ker} f \cap \operatorname{Im} f \leq \dim \operatorname{Ker} f$.

D'où $\boxed{\dim \operatorname{Ker} f^2 \leq 2 \dim \operatorname{Ker} f.}$

Exercice 199 :

Trouver $E = \{ f : \mathbb{N}^* \to \mathbb{N}^* \text{ tq } f\circ f \circ f + f \circ f + f = 3\, Id_E . \}$

__Analyse__ : Soit $f \in E$. Soit $(n, n') \in \mathbb{N}^{*2}$ tq $f(n) = f(n')$.

$\Rightarrow (f \circ f)(n) = (f \circ f)(n')$

$\Rightarrow (f\circ f \circ f)(n) = (f \circ f \circ f)(n')$

$\Rightarrow 3n = 3n'$

$\Rightarrow n = n'$.

Donc f injective.

Mq $f = id_{\mathbb{N}^*}$.

__Init°__ :
$$f\circ f(1) + f \circ f(1) + f(1) = 3.$$

$$\begin{cases} ((f\circ f \circ f)(1) - 1) + ((f\circ f)(1) - 1) + f(1) - 1 = 0. \\ f\circ f : \mathbb{N}^* \to \mathbb{N}^* \quad \Rightarrow (f \circ f \circ f)(1) \geq 1. \\ \quad \Rightarrow (f\circ f)(1) \geq 1. \end{cases}$$

Donc $f(1) = 1$.

Hérédité : Soit $n \in \mathbb{N}^*$ tq $f(k) = k \quad \forall k \in [\![1, n]\!]$

Par l'absurde, on suppose que $f(n+1) = n_0$ avec $n_0 \leq n$

Or, $f(n_0) = n_0 = f(n+1) \underset{\text{injectivité}}{\Rightarrow} n_0 = n+1$ absurde

Idem pour $(f \circ f)(n+1)$
$(f \circ f \circ f)(n+1)$
f

Donc $f(n+1) \geq n+1$
$(f \circ f)(n+1) \geq n+1$.
$(f \circ f \circ f)(n+1) \geq n+1$.

$(f \circ f \circ f)(n+1) + (f \circ f)(n+1) + f(n+1) \geq 3n+3$.

Or, $(f \circ f \circ f)(n+1) - n+1 + f \circ f(n+1) - (n+1) + f(n+1) - f \geq 0$.

Donc $f(n+1) = n+1$.

Exercice 217 : soit $n = \dim E$

* $x \neq 0_E$ $\{x\}$ libre. On prend $(x, e_2, \dots, e_n)$ base de E.

$\begin{cases} f(x) = y. \\ f(e_i) = 0 \quad \forall i \in [\![2, n]\!] \end{cases}$

* $x = 0_E$ → $y = 0_E$

* → $y \neq 0_E$ $\nexists f \in \mathcal{L}(E, F)$ tq $f(x) = y$.

Exercice 220 :

$\boxed{\Leftarrow}$ (H) $f(F) \subset F$

Soit $x \in E$, $\exists! (w, y) \in F \times G$ tq $x = w + y$.

$q \circ p \circ f(x) = q \circ \underbrace{f(w)}_{\in F} = 0 \Rightarrow q \circ f \circ p = 0$.

$\boxed{\Leftarrow}$ (H) $q \circ p \circ f = 0$.

Soit $x \in F$, $f(x) = f \circ p(x)$

$q \circ f(x) = q \circ f \circ p(x) = 0 \Rightarrow f(x) \in F$

Exercice 238 :

V un ev de dim° finie $a, b \in \mathcal{L}(V)$, $u \in \mathcal{L}(E)$ tq $u \neq 0$. $\begin{cases} u \circ a = a \circ u \\ u \circ b = b \circ u \end{cases}$

$(u \circ a)(V) = (a \circ u)(V) \subset u(V)$

$u(V)$ est stable par a idem par b.

$u(V) = V$ ou $u(V) = 0$.

Or u non nul donc $u(V) = V$.

$(u \circ a)(\operatorname{Ker} u) = (a \circ u)(\operatorname{Ker} u) = \{0\}$.

Donc $a(\operatorname{Ker} u) \subset \operatorname{Ker} u$. idem pour b.

$\operatorname{Ker} u = V$ ou $\operatorname{Ker} u = \{0\}$.

Or u non nul donc $\operatorname{Ker} u \neq V$ donc $\operatorname{Ker} u = \{0\}$.

Exercice 227 :

$F \oplus S = G \oplus S = E \Rightarrow \dim F = \dim G$.

Lemme 1 : A, B 2 sev de E : $A \cup B$ sev de E ssi $A \subset B$ ou $B \subset A$.

$\boxed{\Leftarrow}$ OK

$\boxed{\Rightarrow}$ contraposée $x \in A \setminus B$ et $y \in B \setminus A$. $x, y \in A \cup B$, $x + y \notin A \cup B$.

$(a_1, \ldots, a_p)$
$(b_1, \ldots, b_p)$ libre de vecteurs de E.

$p \leq n - 1$.

A, B

$A \cup B \subset E \Rightarrow A \cup B \neq E \Rightarrow \exists x \in E \setminus (A \cup B)$ $x \neq 0_E$.

$(a_1, \ldots, a_p, x)$ $x \notin \operatorname{Vect}(a_i)$

Exercice 214

Soit E un $\mathbb{C}$-ev. On note $n = \dim E$. Soit p_1, p_2 2 projecteurs

Mq $p_1 + p_2$ projecteur $\underline{ssi}$ $p_1 \circ p_2 = p_2 \circ p_1 = 0$.

$\boxed{\Leftarrow}$ (H) $p_1 \circ p_2 = p_2 \circ p_1 = 0$.

$(p_1 + p_2) \circ (p_1 + p_2) = p_1 \circ p_1 + p_1 \circ p_2 + p_2 \circ p_1 + p_2 \circ p_2$

$= p_1 + p_2$.

$\boxed{\Rightarrow}$ (H) $p_1 + p_2$ projecteur.

$(p_1 + p_2) \circ (p_1 + p_2) = p_1 + p_2 + p_1 \circ p_2 + p_2 \circ p_1$

$\Rightarrow p_1 \circ p_2 + p_2 \circ p_1 = 0$

$\Rightarrow p_1 \circ p_2 + p_1 \circ p_2 \circ p_1 = 0$

$\Rightarrow p_1 \circ p_2 \circ p_1 + p_2 \circ p_1 = 0$

$\Rightarrow p_1 \circ p_2 = p_2 \circ p_1 \Rightarrow p_1 \circ p_2 = p_1 \circ p_2 = 0$.

Exercice 215 :

a) $f^2 = 0 \Rightarrow \mathrm{Im} f \subset \mathrm{Ker} f$ donc $\mathrm{rg}\, f \leq \dim \mathrm{Ker} f$

Par le th du rang, $\dim E = \mathrm{rg} f + \dim \mathrm{Ker} f \leq 2 \dim \mathrm{Ker} f$.

Donc $\dim \mathrm{Ker} f \geq \frac{\dim E}{2}$.

b) Mq $E = \mathrm{Ker} f \oplus \mathrm{Ker} g$

- Soit $x \in \mathrm{Ker} f \cap \mathrm{Ker} g$. $\underline{u}$ tq $\begin{cases} g(x) = 0 \\ f(x) = 0 \end{cases}$, $u(x) = 0$ donc $x = \{0_E\}$.

Donc $\mathrm{Ker} f + \mathrm{Ker} g = \mathrm{Ker} f \oplus \mathrm{Ker} g$.

- $\dim \mathrm{Ker} f \oplus \dim \mathrm{Ker} g = \underbrace{\dim \mathrm{Ker} f}_{\geq \frac{\dim E}{2}} + \underbrace{\dim \mathrm{Ker} g}_{\geq \frac{\dim E}{2}}$

$\geq \dim E$.

Donc $\mathrm{Ker} f \oplus \mathrm{Ker} g = E$

- On a $\mathrm{Im} f \subset \mathrm{Ker} f$

Mq $\mathrm{rg} f = \dim \mathrm{Ker} f$.

- $\mathrm{Ker} f = \dim E - \dim \mathrm{Ker} g \geq \frac{\dim E}{2}$.

donc $\dim \operatorname{Ker} f = \frac{\dim E}{2}$. $\underset{\text{TDR}}{\Rightarrow}$ $\operatorname{rg} f = \frac{\dim E}{2}$.

donc $\operatorname{Im} f = \operatorname{Ker} f$ et $\operatorname{Im} g = \operatorname{Ker} g$.

Exercice 216

$\boxed{\Rightarrow}$ (H) $\operatorname{rg} u = \operatorname{rg} u^2$

Comme $\operatorname{Im} u^2 \subset \operatorname{Im} u \Rightarrow \operatorname{Im} u^2 = \operatorname{Im} u$

$\dim E = \dim \operatorname{Ker} u + \operatorname{rg} u$.

Or, $\operatorname{rg} u = \operatorname{rg} u^2 + \dim \operatorname{Ker} u \cap \operatorname{Im} u$. $\Rightarrow \dim(\operatorname{Ker} u \cap \operatorname{Im} u) = 0$

$\Rightarrow \operatorname{Ker} u \oplus \operatorname{Im} u$.

On note $u|_{\operatorname{Im} u}$ par formule du rang.

$\dim \operatorname{Im} u = \dim \operatorname{Im} u|_{\operatorname{Im} u} + \dim(\operatorname{Ker} u \cap \operatorname{Im} u)$

$= \dim \operatorname{Im} u^2 = \operatorname{rg} u^2$.

$\begin{cases} \operatorname{Ker} u \oplus \operatorname{Im} u \subset E \\ \text{égalité dimensions} \end{cases} \Rightarrow E = \operatorname{Ker} u \oplus \operatorname{Im} u$.

$\boxed{\Leftarrow}$ Supposons $E = \operatorname{Ker} u \oplus \operatorname{Im} u$.

Par formule du rang appliquée à $u|_{\operatorname{Im} u}$.

$\operatorname{rg} u = \operatorname{rg} u^2 + \underbrace{\dim \operatorname{Ker} u \cap \operatorname{Im} u}_{\overset{\shortparallel}{0}}$

$\operatorname{rg} u = \operatorname{rg} u^2$.

Exercice 221 :

$P \in \operatorname{Ker} f$. $f(P) = 0 \Leftrightarrow \dots \Leftrightarrow \underline{P(2) = -P(-2)}$

à copier

Récurrence double

$H(n)$: $P(n) = -P(-n)$

$P(n+1) = 2P(n) - P(n-1)$

$= -2P(-n) + P(-n+1)$

$= -P(-n-1)$

Donc $\forall n \in \mathbb{N}$, $P(n) = -P(-n)$

Donc $P(X) + P(-X) = 0 \Rightarrow P(X) = -P(-X) \Rightarrow P$ est impair

De m̂ P' est impair (P impair $\Rightarrow P'$ pair)

Exercice 229.

1) Supposons f isomorphisme

Soit $g \in H$, $f \circ g \circ f = 0 \iff g \circ f = 0 \iff g = 0$.

Donc $H = \{\tilde{0}\}$.

2) Soit $g \in H$ $\quad g(\mathrm{Im} f) \subset \mathrm{Ker} f$. Soit S tq $S \oplus \mathrm{Im} f = F$

$\begin{cases} n = \dim E \\ p = \dim F \\ r = \mathrm{rg} f \end{cases}$ Soit $\varphi : \begin{cases} H \longrightarrow \mathcal{L}(\mathrm{Im} f, \mathrm{Ker} f) \times \mathcal{L}(S, E) \\ g \longmapsto (g|_{\mathrm{Im} f}, g|_S) \end{cases}$ isomorphisme

Donc $\dim H = r(n-r) + (p-r)n$.

$\boxed{\dim H = np - r^2}$

$g \in H \iff g(\mathrm{Im} f) \subset \mathrm{Ker} f$. $\qquad F = S \oplus \overbrace{\mathrm{Im} f}^{r}$. base $(f_1, \dots, f_r, \dots, f_p)$

$E = \underbrace{\mathrm{Ker} f}_{p-r} \oplus G$ $\quad$ base $(\underbrace{e_1, \dots, e_{p-r}}_{\text{base de } \mathrm{Ker} f}, \dots, e_n)$

$$\mathrm{mat}_{(f_1, \dots, f_p)}(g) = \begin{array}{c} \begin{array}{ccc} g(f_1) & g(f_r) & g(f_p) \end{array} \\ \left(\begin{array}{c|c} \times & \times \\ \hline 0 & \times \end{array}\right) \end{array} \begin{array}{l} e_1 \\ e_{n-r} \end{array}$$

$n \times (p-r)$

$\dim H =$ nb de coeff qui varient indépendamment

$= r(n-r) + n(p-r)$

$= np - r^2$.

Exercice 239 :

Soit E $\mathbb{K}$ev. On note $n = \dim E$

$\dim \mathrm{Ker} f \underset{(1)}{\leq} \dim \mathrm{Ker} f^2 \underset{(2)}{\leq} 2 \dim \mathrm{Ker} f$.

(1) $\mathrm{Ker} f \subset \mathrm{Ker} f^2 \implies \dim \mathrm{Ker} f \leq \dim \mathrm{Ker} f^2$.

(2) Soit $u = f|_{\mathrm{Im} f}$. $\qquad \mathrm{rg} f = \mathrm{rg}\, u + \dim \mathrm{Ker}\, u$

$= \mathrm{rg} f^2 + \dim \mathrm{Ker} f \cap \mathrm{Im}\, u$

~~$\mathrm{rg}\, u \leq \mathrm{rg} f$~~

$\Rightarrow P' = 0$

Donc $\text{Ker } f = \{0\}$. NON

Soit $k \geq 2$.

$f(X^k) = (X+1)^k - 2X^k + (X-1)^k$.

$= k(k-1)X^{k-2} + Q(X)$ $\deg Q < k-2$.

On a $\left| \begin{array}{l} ((X^k)_{k\geq 2}) \text{ base de } E \\ \text{l'image de cette base par } f \text{ est une base de } \mathbb{R}[X] \end{array} \right.$

Donc f isomorphisme de E dans $\mathbb{R}[X]$

Exercice 277 :

$\boxed{\subset}$ $y \in \text{Im} f$: $\exists x \in E$, $y = f(x)$

$f^4(y) = f^5(x) = f^3(f^2(x)) = 0_E$.

$f^4(\underbrace{f(x)}_{y})$

Donc $y \in \text{Ker } f^4 \Rightarrow \text{Im} f \subset \text{Ker} f^4$.

$\boxed{\supset}$ Soit $x \in \text{Ker} f^4$, $f^4(x) = 0_E$.

$f^3(\underbrace{f(x)}_{\in \text{Im} f^2}) = 0_E$.

$\Rightarrow \exists \alpha \in E$, $f(x) = f^2(\alpha)$

$\Rightarrow$ $f(\underbrace{x - f(\alpha)}) = 0_E$

$\in \text{Ker} f \Rightarrow \in \text{Ker} f^3 = \text{Im} f^2$.

$\Rightarrow \exists \beta \in E$ $f^2(\beta) = x - f(\alpha)$

$\Rightarrow$ $x = f(\alpha) + f^2(\beta)$

$x = f(\alpha + f(\beta))$

$\Rightarrow x \in \text{Im} f$.

Ccl $\text{Im} f = \text{Ker} f^4$.

TDR : à f puis à f^2.

$$\begin{cases} n = \text{rg} f + \dim \text{Ker} f. \\ n = \text{rg} f^2 + \dim \text{Ker} f^2. \end{cases}$$

Donc

$$\text{rg} f - \text{rg} f^2 = \dim \text{Ker} f^2 - \dim \text{Ker} f.$$
$$= \dim (\text{Ker} f \cap \text{Im} f)$$

$\Rightarrow \dim \text{Ker} f^2 = \dim \text{Ker} f \cap \text{Im} f + \dim \text{Ker} f.$

Or $\dim \text{Ker} f \cap \text{Im} f \leq \dim \text{Ker} f.$

D'où $\boxed{\dim \text{Ker} f^2 \leq 2 \dim \text{Ker} f.}$

Exercice 199 :

Trouver $E = \{ f : \mathbb{N}^* \to \mathbb{N}^*$ tq $f \circ f \circ f + f \circ f + f = 3 \, Id_E. \}$

<u>Analyse</u> : Soit $f \in E$. Soit $(n, n') \in \mathbb{N}^{*2}$ tq $f(n) = f(n')$.

$\Rightarrow (f \circ f)(n) = (f \circ f)(n')$

$\Rightarrow (f \circ f \circ f)(n) = (f \circ f \circ f)(n')$

$\Rightarrow 3n = 3n'$

$\Rightarrow n = n'.$

Donc f injective.

Mq $f = id_{\mathbb{N}^*}$.

<u>Init°</u> : $f \circ f(1) + f \circ f(1) + f(1) = 3.$

$$\begin{cases} ((f \circ f \circ f)(1) - 1) + ((f \circ f)(1) - 1) + f(1) - 1 = 0. \\ f \circ f : \mathbb{N}^* \to \mathbb{N}^* \quad \Rightarrow (f \circ f \circ f)(1) \geq 1. \\ \qquad \Rightarrow (f \circ f)(1) \geq 1. \end{cases}$$

Donc $f(1) = 1.$

<u>Hérédité</u> : Soit $n \in \mathbb{N}^*$ tq $f(k) = k$ $\forall k \in [\![1, n]\!]$

Par l'absurde, on suppose que $f(n+1) = n_0$ avec $n_0 \leq n$

Or, $f(n_0) = n_0 = f(n+1) \underset{\text{injectivité}}{\Rightarrow} n_0 = n+1$ <u>absurde</u>

Idem pour $(f \circ f)(n+1)$
$(f \circ f \circ f)(n+1)$
f

Donc
$f(n+1) \geq n+1$
$(f \circ f)(n+1) \geq n+1$.
$(f \circ f \circ f)(n+1) \geq n+1$.

$(f \circ f \circ f)(n+1) + (f \circ f)(n+1) + f(n+1) \geq 3n+3$.

Or, $(f \circ f \circ f)(n+1) - n+1 + f \circ f(n+1) \cancel{(n+1)} + f(n+1) - f \geq 0$.

Donc $f(n+1) = n+1$.

Exercice 217 : soit $n = \dim E$

* $x \neq 0_E$ $\{x\}$ libre. On prend $(x, e_2, \ldots, e_n)$ base de E.

$\begin{cases} f(x) = y. \\ f(e_i) = 0 \ \forall i \in [\![2, n]\!] \end{cases}$

* $x = 0_E$ $\rightarrow y = 0_E$

* $\rightarrow y \neq 0_E$ $\nexists f \in \mathcal{L}(E, F)$ tq $f(x) = y$.

Exercice 220 :

$\boxed{\Leftarrow}$ (H) $f(F) \subset F$

Soit $x \in E$, $\exists! (w, y) \in F \times G$ tq $x = w + y$.

$q \circ p \circ f(x) = q \circ \underbrace{f(w)}_{\in F} = 0 \Rightarrow q \circ f \circ p = 0$.

$\boxed{\Rightarrow}$ (H) $q \circ p \circ f = 0$.

Soit $x \in F$, $f(x) = f \circ p(x)$

$q \circ \underbrace{f(x)} = q \circ f \circ p(x) = 0. \Rightarrow f(x) \in F$

Exercice 238 :

V un ev de dim° finie $a, b \in \mathcal{L}(V)$, $u \in \mathcal{L}(E)$ tq $u \neq 0$. $\begin{cases} u \circ a = a \circ u \\ u \circ b = b \circ u \end{cases}$

$(u \circ a)(V) = (a \circ u)(V) \subset u(V)$

$u(V)$ est stable par a idem par b.

$u(V) = V$ ou $u(V) = 0$.

Or u non nul donc $u(V) = V$.

$(u \circ a)(\mathrm{Ker}\, u) = (a \circ u)(\mathrm{Ker}\, u) = \{0\}$.

Donc $a(\mathrm{Ker}\, u) \subset \mathrm{Ker}\, u$. idem pour b.

$\mathrm{Ker}\, u = V$ ou $\mathrm{Ker}\, u = \{0\}$.

Or u non nul donc $\mathrm{Ker}\, u \neq V$ donc $\mathrm{Ker}\, u = \{0\}$.

Exercice 227 :

$F \oplus S = G \oplus S = E \Rightarrow \dim F = \dim G$.

Lemme 1 : A, B 2 sev de E : $A \cup B$ sev de E ssi $A \subset B$ ou $B \subset A$.

$[\Leftarrow]$ OK

$[\Rightarrow]$ contraposée $x \in A \setminus B$ et $y \in B \setminus A$. $x, y \in A \cup B$, $x + y \notin A \cup B$.

$(a_1, \ldots, a_p)$
$(b_1, \ldots, b_p)$ libre de vecteurs de E.

$p \leq n-1$.

A, B

$A \cup B \subset E \Rightarrow A \cup B \neq E \Rightarrow \exists x \in E \setminus (A \cup B)$ $x \neq 0_E$.

$(a_1, \ldots, a_p, x)$ $x \notin \mathrm{Vect}(a_i)$