<u>Exercice 71 (William)</u>

1) B nilpotente $\Leftrightarrow \forall k \in \mathbb{N}^*, \operatorname{tr}(B^k) = 0$.

$$\begin{aligned}\operatorname{tr}(B^n) &= \operatorname{tr}(B^{n-1}B)\\ &= \operatorname{tr}(B^{n-1}(AB^2 - B^2A))\\ &= \operatorname{tr}(B^{n-1}AB^2) - \operatorname{tr}(B^{n+1}A)\\ &= 0.\end{aligned}$$

Soit $d \left| \begin{array}{l} \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_n(\mathbb{C}) \\ X \longmapsto AX - XA \end{array}\right.$

On a $d(B^2) = B$.

Montrons par récurrence que $\forall n \in \mathbb{N}$,
$d(B^{2n}) = n\, d(B^{2n-1})$

<u>$n=1$</u> : $d(B^2) = B$.

<u>Hérédité</u> : Soit $n \in \mathbb{N}^*$
Supposons $d(B^{2n}) = n\, d(B^{2n-1})$.

$AB^{2n} - B^{2n}A = nB^{2n-1}$ $\times) B^2$

$AB^{2n+2} - B^{2n}AB^2 = nB^{2n+1}$.

$+$ Or $B^{2n}(AB^2 - B^2A) = B^{2n+1}$.

D'où $AB^{2n+2} - B^{2n+2}A = (n+1)B^{2n+1}$.

Supposons p pair.

Soit $p \in \mathbb{N}$ tel que $B^{p-1} \neq 0$ et $B^p = 0$, tel que $p = 2k$.

$$d(B^p) = \quad k \, B^{2k-1} = 0.$$

D'où $B^{2k-1} = 0$. Absurde !

Donc $\boxed{p \text{ impair}}$