Matho : TD 16/10

Exo 337 : Nicolas

$$M = \begin{pmatrix} P_1(z_1) & \cdots & P_1(z_n) \\ \vdots & & \vdots \\ P_n(z_1) & \cdots & P_n(z_n) \end{pmatrix}$$

$(P_1 - P_n) \in \mathbb{C}_{n-1}[X]^n$

$(z_1 - z_n) \in \mathbb{C}^n$

Q1 $M = f(V(z_1 - z_n))$

avec $V = (z_1 - z_n) = \begin{bmatrix} 1 & \cdots & 1 \\ z_1 & \cdots & z_n \\ & & \\ z_1^{n-1} & \cdots & z_n^{n-1} \end{bmatrix}$

$\forall i \in [\![1, n]\!]$, on pose $P_i(X) = \sum_{j=1}^{n} p_{ij} X^{j-1}$

$$A = \begin{pmatrix} p_{11} & \cdots & p_{1n} \\ \vdots & & \vdots \\ p_{n1} & \cdots & p_{nn} \end{pmatrix} \cdot \underbrace{\begin{pmatrix} 1 & \cdots & 1 \\ z_1 & \cdots & z_n \\ \vdots & & \vdots \\ z_1^{n-1} & \cdots & z_n^{n-1} \end{pmatrix}}_{= V(z_1 - z_n)}$$

Soit $(i,j) \in [\![1, n]\!]^2$

On a $A_{ij} = \sum_{k=0}^{n-1} p_{ik} z_j^k$

$\boxed{A_{ij} = P_i(z_j) = M_{ij}}$

donc $M = A$

$$\det M = \begin{vmatrix} p_{11} & & p_{1n} \\ \vdots & & \vdots \\ p_{n1} & \cdots & p_{nn} \end{vmatrix} \times \det\left[V(z_1 - z_n)\right]$$

$= \prod_{i>j} (z_i - z_j)$

Q2

$$M = \begin{pmatrix} 1 & 2^{n-1} & \cdots & n^{n-1} \\ 2^{n-1} & 3^{n-1} & \cdots & (n+1)^{n-1} \\ \vdots & & & \vdots \\ n^{n-1} & \cdots & & (2n-1)^{n-1} \end{pmatrix}$$

On pose : $(z_1 \cdots z_n) = (1, \cdots, n)$

et $\forall i \in [\![1, n]\!]$, $P_i(X) = (X + i - 1)^{n-1}$

$M = \left(P_i(z_j)\right)_{1 \leq i,j \leq n}$

$$M = \underbrace{\begin{bmatrix} p_{11} & \cdots & p_{1n} \\ \vdots & & \vdots \\ p_{n1} & \cdots & p_{nn} \end{bmatrix}}_{= A} \times V(z_1 \cdots z_n)$$

- det A ?

. Soit $i \in [\![1, n]\!]$. On a : $P_i(X) = (X + i - 1)^{n-1}$

$$= \sum_{k=0}^{n-1} \binom{n-1}{k} X^k (i-1)^{n-1-k}$$

donc $p_{ij} = \binom{n-1}{j-1} (i-1)^{n-1-j+1}$

on a : $\det A = \prod_{j=1}^{n} \binom{n-1}{j-1} \times \det \left((i-1)^{n-j} \right)_{1 \leq i,j \leq n}$

$$B = \left((i-1)^{n-j}\right)_{ij} = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 1 & \cdots & 1 & \vdots \\ 2^{n-1} & \cdots & 2 & \\ \vdots & & & \\ (n-1)^{n-1} & \cdots & (n-1) & 1 \end{pmatrix}$$

Dvlp selon L_1 :

$$\det B = (-1)^{n+1} \begin{vmatrix} 1 & \cdots & 1 \\ 2^{n-1} & \cdots & 2 \\ \vdots & & \vdots \\ (n-1)^{n-1} & \cdots & (n-1) \end{vmatrix}_{(n-1)(n-1)} = (-1)^{n+1} \times (n-1)! \times \begin{vmatrix} 1 & \cdots & 1 \\ 2^{n-1} & \cdots & 2 \\ \vdots & & \\ (n-1)^{n-1} & \cdots & (n-1) \end{vmatrix}$$

Maths : TD 10/10

exo 337 (suite)

$$\det B = (-1)^{m} \times (n-1)! \times \det(V(1, \dots, n-1))$$

$$\det B = (-1)^{m} (n-1)! \times \prod_{1 \leq j < i \leq \underline{n-1}} (i-j)$$

$$\times \prod_{j=1}^{n-1} \binom{n-1}{j-1} \quad \times \prod_{1 \leq j < i \leq \underline{n}} (i-j)$$

$$\text{avec } m = (-1)^{n+1} \cdot \underline{(-1)^{\lfloor \frac{n}{2} \rfloor}}$$

↳ car on inverse les colonnes de la matrice pour se ramener à Vandermonde

Rq: on peut encore simplifier, notamment avec :

$$\prod_{1 \leq j < i \leq n} (i-j) = 1 \times 2 \times \dots \times n-1$$
$$\times 1 \times 2 \times \dots \times n-2$$
$$\times \dots$$
$$= \prod_{j=1}^{n-1} j!$$