$\boxed{\text{Ex 241}}$ $(e_1 \dots e_n)$ base de E

Mq [illegible] supp de $\{0\} \times F$ [illegible] $\{(x, f(x))\}$

Soit $x \in E$: $(x, f(x)) \in E \times F$

Soit G [illegible] tq $\{0\} \times F \oplus G = E \times F$

$\exists (a, c) \in G^2$ tq $(a ; b + c) = (x, y)$ $(y \in F)$

$(0, c) \in \{0\} \times F$

Donc $\exists d \in F$, $(x, f) \in G$

$\forall i \in [\![1, n]\!]$, $\exists f_i \in F$, $(e_i ; f_i) \in G$

$\rightarrow$ $\forall i \in [\![1, n]\!]$, $f(e_i) = f_i$

Soit $x \in E$, $(f_1, f_2) \in F^2$ tq $(x, f_1), (x, f_2) \in G$

donc $(0, f_1 - f_2) \in G \cap \{0\} \times F$

Donc $f_1 = f_2$: $\forall x \in E$, $\exists ! f_x \in F$, $(x, f_x) \in G$

or $x = \sum_{k=1}^{m} \lambda_k e_k$ et $f(x) = \sum \lambda_k f_k$

donc $(x, f_x) = \sum_{k=1}^{m} \lambda_k (e_k, f_k)$

$= \left(\sum \lambda_k e_k, \sum \lambda_k f_k \right)$

Donc $f_x = \sum_{k=1}^{m} \lambda_k f_k = f(x)$

et $G = \{(x, f(x)) ; x \in E\}$

$\boxed{\text{Ex 246}}$ $T : P \mapsto P(x+1)$

$H = \{ u \in \mathcal{L}(\mathbb{R}[X]) ; T \circ u = 0 \}$

- $\mathbb{R}[T] \subset H$
- $D_k : P \mapsto P^{(k)}$

} conviennent.

$P = (a_0 \ldots a_n, 0 \ldots)$

$u(P) = (u(a_0) \ldots u(a_n), 0 \ldots)$

<u>Analyse</u> Soit $u \in H$ et $P \in \mathbb{R}_n[X]$

$$P = \sum_{i=0}^{n} a_i X^i$$

$$Q = \sum_{i=0}^{k} q_i X^i$$

$$u(Q) = \sum u(q_i) X^i$$

<u>On a</u> : $(u \circ T)(P) = u(T(P)) = u(P(X+1))$

$$= T(u(P)) = T\left(\sum u(a_i) X^i\right)$$

$$= \sum u(a_i)(X+1)^i$$

<u>Or</u> $(X+1)^i = \sum_{k=0}^{i} \binom{i}{k} X^k$

<u>Donc</u> $P(X+1) = \sum_{i=0}^{n} \sum_{k=0}^{i} \binom{i}{k} a_i X^i$

$$= \sum_{i=0}^{n} \sum_{k=0}^{n} \binom{i}{k} a_i X^k$$

$$= \sum_{k=0}^{n} X^k \sum_{i=0}^{n} \binom{i}{k} a_i$$

donc $u(T(P)) = \sum X^k \, u\left(\sum_{i=0}^{n} \binom{i}{k} a_i\right)$

or $\sum u(a_i)(X+1)^i = \sum_{k=0}^{n} X^k \sum_{i=k}^{n} u(a_i) \binom{i}{k}$

Par identification : $\sum_{i=k}^{n} u(a_i) \binom{i}{k} = u\left(\sum_{i=k}^{n} \binom{i}{k} a_i\right)$

<u>Alternative</u>

$\delta : T \longmapsto T - id$ (dérivée discrète)

$P \longmapsto P(X+1) - P$

δ nilpotente : $\begin{cases} \deg \delta(P) = \deg P - 1 \\ \mathrm{Cd}\, (\delta(P)) = d \times \mathrm{Cd}(P) \text{ si } \deg P = d \end{cases}$

$\begin{cases} \delta^{n+1} = 0 \\ \delta^n \neq 0 \end{cases} \rightarrow \delta$ nilpotente d'indice n

(Ex 208) : $\mathcal{C}(\delta) = (Id \cdots \delta^n)$

Or $\mathcal{C}(t) = \mathcal{C}(T)$

$u \circ T = T \circ u \Rightarrow u \circ \delta = t \circ u$

car u commute avec $\underline{\underline{Id_E}}$

donc $\mathcal{C}(T) = \text{vect}(Id_E, \delta, \ldots, \delta^n)$

$= \boxed{\text{vect}(Id_E, T, \ldots, T^n)}$

$\boxed{\underline{Ex\ 234}}$

a) Soit $(u_n), (v_n) \in E_p$, $\lambda \in \mathbb{C}$

$+ (\lambda u_n + v_n)_{n \in \mathbb{N}} = \lambda (u_n)_{n \in \mathbb{N}} + (v_n)_{n \in \mathbb{N}}$

$+ 0 \in E_p$

Donc E_p est un $\mathbb{C}$eV

$(v_n^{(k)})_{n \in \mathbb{N}} : v_t = \begin{cases} 1 \text{ si } i \equiv k\, [p] \\ 0 \text{ else} \end{cases}$

$(v_n^{(k)}) \in E_p$

Soit $\lambda_1 \ldots \lambda_p \in \mathbb{C}$ tq

$A_n = (\lambda_1 v^{(1)} + \ldots + \lambda_p v^{(p)}) = 0$

$A_0 = \lambda_p = 0$

$A_1 = \lambda_1 = 0 \Rightarrow \lambda_1 = \ldots = \lambda_p = 0$: libre

$\underline{\text{Gén}}$: $\underline{\text{Soit } (w_n) \in E_p}$ $\begin{cases} \lambda_0 = w_0 \\ \vdots \\ \lambda_{p-1} = w_{p-1} \end{cases} \Rightarrow (\lambda_0 v_0 + \ldots + \lambda_p v_p) = w_n$

Donc $(v_0 \ldots v_{p-1})$ base de E_p

b) $(u_n^{(k)})_{n \in \mathbb{N}} = \left(e^{\frac{2ik\pi}{p}}\right)^n$

Soit $k \in \mathbb{N}$, $n \in \mathbb{N}$

$+ u_{n+p}^{(k)} = \left(e^{\frac{2ik\pi}{p}}\right)^{n+p} = \left(e^{\frac{2ik\pi}{p}}\right)^n \times \underset{1}{\underline{e^{2ik\pi}}} = u_n^{(k)}$

$\Rightarrow$ dimension ok