Ex. 1367 Theodore

$$f \left| \begin{array}{l} [0,\pi]^2 \to \mathbb{R} \\ (x,y) \mapsto \sin x + \sin y + \sin(x+y) \end{array} \right.$$

· Pt critique

$\nabla f(x,y) = (0,0)$ sur $[0,\pi]^2$

$$\frac{\partial f}{\partial x} = \cos x + \cos(x+y)$$

$$\frac{\partial f}{\partial y} = \cos y + \cos(x+y)$$

On a $\begin{cases} \cos x + \cos(x+y) = 0 \\ \cos y + \cos(x+y) = 0 \end{cases}$

$\Rightarrow \cos x = \cos y \Rightarrow x = y$

On a alors $\cos(2x) = -\cos(x)$
$\cos(2y) = -\cos y$

$2x = \pi - x$
ou $2x = x - \pi [2\pi]$ $\Rightarrow$ $x = \frac{\pi}{3}$
$x = \pi$

Donc $(\frac{\pi}{3}, \frac{\pi}{3})$ et (π,π) points critiques

$$\frac{\partial^2 f}{\partial x \partial y} = -\sin(x+y) = \frac{\partial^2 f}{\partial y \partial x} \qquad \frac{\partial^2 f}{\partial x^2} = -\sin x - \sin(x+y)$$

$$\frac{\partial^2 f}{\partial y^2} = -\sin y - \sin(x+y)$$

$$H_f\left(\frac{\pi}{3}, \frac{\pi}{3}\right) = \begin{pmatrix} \sqrt{3} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\sqrt{3} \end{pmatrix}$$

$\det H_f = 3 - \frac{3}{4} > 0$
$\operatorname{tr} H_f = -2\sqrt{3} < 0$
$\operatorname{sp}(H_f(\frac{\pi}{3}, \frac{\pi}{3})) \subset \mathbb{R}^{-*}$

$(\frac{\pi}{3}, \frac{\pi}{3})$ max local strict

Sur la frontière $\forall x$, $f(x,\pi) = \sin(\pi + x) + \sin(x) = 0$
$\forall y$ $f(\pi, y) = 0$

et $f(x,0) = 2\sin(x)$ $f(0,y) = 2\sin y$, $(x,y) \in [0,\pi]^2$

D'où sur la frontière f est maximale en $(0,\frac{\pi}{2})$ et $(\frac{\pi}{2},0)$

et $f(0,\frac{\pi}{2}) = f(\frac{\pi}{2},0) = 2$

Or $f(\frac{\pi}{3},\frac{\pi}{3}) = \frac{3}{2}\sqrt{3} > 2$ donc $(\frac{\pi}{3},\frac{\pi}{3})$ est un max global