Exercice 1366 Theodore

$$f \left| \begin{array}{l} \mathbb{R} \times \mathbb{R}_+^* \longrightarrow \mathbb{R}_+ \\ (x,y) \longmapsto y(x^2 + \ln(y)^2) \end{array} \right.$$

On cherche les points critiques

$$\nabla f(x,y) = (0,0) \quad \text{sur } \mathbb{R} \times \mathbb{R}_+^*$$

$$\frac{\partial f}{\partial x} = 2xy \qquad \frac{\partial f}{\partial y} = x^2 + \ln(y)^2 + y \times \frac{1}{y} \times 2\ln(y)$$

$$= x^2 + (\ln y)^2 + 2\ln(y)$$

$$\begin{cases} 2xy = 0 \\ x^2 + (\ln y)^2 + 2\ln y = 0 \end{cases} \Leftrightarrow \begin{cases} x = 0 \quad \text{car } y \in \mathbb{R}_+^* \\ \ln y(\ln y + 2) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x = 0 \\ y = 1 \text{ ou } \ln y = -2 \\ \qquad\quad y = e^{-2} \end{cases}$$

$$(x,y) = (0,1) \text{ ou } (x,y) = (0, e^{-2})$$

Calcul de $H_f(0,1)$

$$\frac{\partial^2 f}{\partial x \partial y} = 2x \qquad \frac{\partial^2 f}{\partial y \partial x} = 2x$$

$$\frac{\partial^2 f}{\partial x^2} = 2y \qquad \frac{\partial^2 f}{\partial y^2} = \frac{2(\ln(y) + 1)}{y}$$

$$H_f(0,1) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \in \mathcal{S}^{++}$$

$(0,1)$ est un min local strict

$$H_f(0, e^{-2}) = \begin{pmatrix} \frac{2}{e^2} & 0 \\ 0 & -2e^2 \end{pmatrix} \quad \Bigg| \quad sp(H_f) \not\subset \mathbb{R}_+^* \quad \not\subset \mathbb{R}_-^*$$

Donc $(0, e^{-2})$ est un point selle

On a $f(\langle \mathbb{R} \times \mathbb{R}_+^* \rangle) \subset \mathbb{R}_+$

et $f(0,1) = 0$

Donc $(0,1)$ est un minimum global strict