ex 1365 Theodore

$$f(x,y) = \begin{cases} \dfrac{x^2y}{x^2+y^2} & \text{si } (x,y) \neq (0,0) \\ 0 & \text{sinon} \end{cases}$$

Sur $\mathbb{R}^2 \setminus (0,0)$

$u : (x,y) \longmapsto x^2y$
$v : (x,y) \longmapsto x^2+y^2$ | $\mathcal{C}^0$ car polynomiales

et $v$ ne s'annule pas sur $\mathbb{R}^2 \setminus (0,0)$

Donc $f$ est $\mathcal{C}^0$ sur $\mathbb{R} \setminus (0,0)$

Continuité en $(0,0)$ ?

$$\left| f(x,y) - f(0,0) \right| = \left| \frac{x^2y}{x^2+y^2} \right| \leq |y|$$

Donc $(x,y) \to (0,0)$

$|f(x,y) - f(0,0)| \to 0$

Donc $f$ est continue en $(0,0)$

$\forall (x,y) \neq (0,0)$

$$\frac{\partial f}{\partial x} = \frac{2xy(\cancel{x^2}+y^2) - \cancel{x^2y \times 2x}}{(x^2+y^2)^2}$$

$$= \frac{2xy^3}{(x^2+y^2)^2}$$

$$\frac{\partial f}{\partial y} = \frac{x^2(x^2+y) - x^2y \times 2y}{(x^2+y^2)^2}$$

$$= \frac{x^4 + x^2y - 2x^2y^2}{(x^2+y^2)^2}$$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{t \to 0} \frac{f(t,0) - f(0,0)}{t} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{t \to 0} \frac{f(0,t) - f(0,0)}{t} = 0$$

$$\frac{\partial f}{\partial x}(x,x) = \frac{2x^4}{4x^2} = \frac{1}{2} \underset{x \to 0}{\not\longrightarrow} 0$$

$\frac{\partial f}{\partial x}$ n'est pas $\mathcal{C}^0$ en $(0,0)$ Donc $f$ n'est pas $\mathcal{C}^1$

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) = \lim_{t \to 0} \frac{\frac{\partial f}{\partial y}(t,0) - \frac{\partial f}{\partial y}(0,0)}{t} = 0$$

$$\frac{\partial^2 f}{\partial y \partial x}(0,0) = \lim_{t \to 0} \frac{\frac{\partial f}{\partial x}(0,t) - \frac{\partial f}{\partial x}(0,0)}{t} = 0$$

Donc $\boxed{\dfrac{\partial^2 f}{\partial x \partial y}(0,0) = \dfrac{\partial^2 f}{\partial y \partial x}(0,0) = 0}$