$\boxed{1227}$ Ulysse

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s}$$

a) $\mathbb{P}(X=n) = \frac{\lambda}{n^s}$

$$\begin{aligned}\mathbb{P}(X\in \mathbb{N}^*) &= 1\\ &= \mathbb{P}\Big(\bigcup_{n\in\mathbb{N}^*}(X=n)\Big)\\ &= \sum_{n=1}^{+\infty} \mathbb{P}(X=n)\\ &= \lambda\,\zeta(s) \qquad \Rightarrow \boxed{\lambda = \frac{1}{\zeta(s)}}\end{aligned}$$

b) $A_p = (p \mid X)$ ↪ divise

soit $J \subset \mathbb{N}^*$ fini

$$\begin{aligned}\mathbb{P}\Big(\bigcap_{i\in J} A_i\Big) &= \mathbb{P}\Big(\bigcap_{i\in J}(i\mid X)\Big)\\ &= \mathbb{P}\Big(\prod_{i\in J} i \mid X\Big)\\ &= \mathbb{P}\Big(\bigcup_{n\in\mathbb{N}^*} X = P_J\cdot n\Big)\\ &= \sum_{n=1}^{+\infty} \frac{\lambda}{(P_J\cdot n)^s}\\ &= \frac{1}{(P_J)^s}\cdot\lambda\cdot\sum_{n\in\mathbb{N}^*}\frac{1}{n^s}\\ &= \frac{1}{(P_J)^s}\\ &= \prod_{i\in J}\frac{1}{i^s} = \prod_{i\in J}\mathbb{P}(A_i)\end{aligned}$$

car $\mathbb{P}(A_i) = \sum_{n\in\mathbb{N}^*}\mathbb{P}(X=ni) = \sum_{n\in\mathbb{N}^*}\frac{\lambda}{(ni)^s} = \frac{1}{i^s}$

c) $\mathbb{P}\Big(X\in\Big(\bigcup_{p\in\mathcal{P}} A_p\Big)\Big) = 1$

$\mathbb{P}\Big(X\in\Big(\bigcap_{p\in\mathcal{P}}\overline{A_p}\Big)\Big) = \mathbb{P}(X=1) = \lambda$

$$\prod_{p\in\mathcal{P}}\mathbb{P}(\overline{A_p}) = \prod_{p\in\mathcal{P}}(1-\mathbb{P}(A_p)) = \prod_{p\in\mathcal{P}}\Big(1-\frac{1}{p^s}\Big) = \frac{1}{\zeta(s)}$$

$\ln(\zeta(s)) = \sum_{k=1}^{+\infty} -\ln\Big(1-\frac{1}{p_k^s}\Big)$ de même nature que $\sum_{k=1}^{+\infty}\frac{1}{p_k^s}$

$\xrightarrow[s\to 1]{} +\infty$

$\zeta(s) \sim \frac{1}{s-1}$ d'où $\sum_{k=1}^{+\infty}\frac{1}{p_k}$ diverge