Exercice 1139 (Martin)

(a) $\displaystyle\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$

$$(1+x)^{2n} = \underbrace{(1+x)^n}_{\sum_{k=0}^{n}\binom{n}{k}x^k} \underbrace{(1+x)^n}_{\sum_{p=0}^{n}\binom{n}{p}x^p}$$

$$\left[\text{coef devant } x^n \binom{2n}{n}\right] = \sum_{p+k=n}\binom{n}{k}\binom{n}{p} = \sum \binom{n}{k}^2$$

$$C_n = \frac{1}{n+1}\sum\binom{n}{k}^2 = \frac{1}{n+1}\binom{2n}{n}$$

(b) $\displaystyle\frac{C_{n+1}}{C_n} = \frac{n}{n+1}\,\frac{(2n+2)^4}{2n+1} \times \frac{(n!)^2}{(n-1)!^2} = \frac{2(2n+1)n}{(n+1)^2} \longrightarrow 4$

donc $RCV = \frac{1}{4}$

(c) $\displaystyle f(x) = \sum_{n=0}^{\infty} \frac{1}{n+1}\binom{2n}{n}x^n =$

$$= \int \frac{1}{x} \times \sum \binom{2n}{n} x^n \, dx$$

$$= \frac{1}{x}\int \frac{dx}{\sqrt{1-4x}}$$

$$= \frac{1}{2x}\left(1-\sqrt{1-4x}\right)$$

$$C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}$$

$$f(x) = \sum_{n=0}^{+\infty} C_n x^n = x\sum_{n\geq 1} C_{n+1}x^n + 1$$

$$f(x) = x\left(\sum c_n x^n\right)^2 + 1$$

$$\left(f(0) = c_0 = 1\right)$$

$f(x) = x f^2(x) + 1$ donc $x f(x)^2 - f(x) + 1 = 0 \Rightarrow \Delta = 1 - 4x$
> 0

$$f(x) = \frac{1 \pm \sqrt{1-4x}}{2x}$$

donc $f(x) = \dfrac{1 - \sqrt{1-4x}}{2x}$

$$f(x) \underset{x>0}{\approx} \frac{1 \pm (1-2x)}{2x} = \dots \xrightarrow[0]{} +\infty$$