<u>Exo 1120 : (Diane)</u>

$$f(x) = \sum_{n \geq 0} \frac{x^n}{\binom{2n}{n}}$$

1) $\dfrac{x^n}{\binom{2n}{n}} \underset{n \to \infty}{\sim} \dfrac{x^n (n!)^2}{(2n!)} \sim \dfrac{x^n\, 2\pi n \left(\frac{n}{e}\right)^{2n}}{\sqrt{4\pi n}\left(\frac{2n}{e}\right)^{2n}}$

$\underset{\infty}{\sim} \dfrac{\sqrt{\pi n}}{2^{2n}}$) RCV $= 4$. donc RCV$(f(x)) = 4$.

(ou alors avec le critère de d'Alembert).

→ <u>recherche EDL</u>.

$$\binom{2n}{n} = \frac{(2n-2)!}{((n-1)!)^2} \frac{(2n-1)2n}{n^2} = \binom{2n-2}{n-1} \frac{4n-2}{n}$$

On pose $a_n = \dfrac{1}{\binom{2n}{n}}$, alors $a_n = \dfrac{n}{4n-2} \dfrac{1}{\binom{2n-2}{n-1}}$

$= \dfrac{n}{4n-2} a_{n-1}$ d'où $(4n-2) a_n = n a_{n-1}$

↳ $4(n-1) a_n x^n = n a_{n-1} x^n \Rightarrow \sum_{n \geq 1} 4(n-1) a_n x^n = \sum_{n \geq 1} n a_{n-1} x^n$.

Or $f'(x) = \sum_{n \geq 1} n x^{n-1} a_n$.

d'où $\sum (4n-2) a_n x^n = x^2 \sum_{n \geq 1} (n-1) x^{n-2} a_{n-1} + x^2 \sum_{n \geq 1} x^{n-2} a_{n-1}$

$x \sum n a_n x^{n-1} - 2f(x) + 2a_0 = x^2 \sum_{\cancel{n \geq 0}\; n \geq 1} n a_n x^{n-1} + x \sum_{\cancel{n \geq 1}\; n \geq 0} x^{\cancel{n-1}\, n} a_{\cancel{n-1}\, n}$

ie $x4f'(x) - 2f(x) + 2 = x^2 f'(x) + x f(x)$.

$\forall x \in]-R, R[, \ (x^2 - 4x)f'(x) + (x+2)f(x) - 2 = 0$.

f est soluO de : $y' + \frac{x+2}{x(x-4)} y = \frac{2}{x(x-4)}$.

$y(x) = C(x) \exp\left(-\int_{x_0}^{x} \frac{t+2}{t(t-4)} dt\right)$.

- $\int \frac{x+2}{x(x-4)} = \int \frac{1}{x-4} + \frac{2}{x(x-4)} dx$

$= \int \left(\frac{1}{x-4} + \left(\frac{1/2}{x} + \frac{1/2}{x-4}\right)\right) dx = \int \left(\frac{3/2}{x-4} - \frac{1/2}{x}\right) dx$

$= \ln\left(\frac{(x-4)^{3/2}}{\sqrt{x}}\right)$

- $C(x) = \int \frac{2}{x(x-4)} e^{\int \frac{x+2}{x^2-4x}} dx + C$.

$= \int \frac{2}{x(x-4)} \times \frac{|x-4|^{3/2}}{\sqrt{x}} dx + C = \overbrace{\int \frac{2\sqrt{|x-4|}}{x^{3/2}} dx}^{= I} + C$

(chgmt : $x = 4\sin^2(t)$ $dx = 8\cos(t)\sin(t) dt$.

$I = \int \frac{4\cos t}{8\sin^3(t)} \times 8 \cos t \sin t \, dt = \cancel{\frac{1}{2}}\,4 \int \frac{\cos^2(t)}{\sin^2(t)} dt$.

$= 4\cancel{\frac{1}{2}} \int \cot^2(t) dt = 4\cancel{\frac{1}{2}} \int (1 + \cot^2(t)) dt - \int 4\cancel{\frac{1}{2}} dt$

$= -4\cancel{\frac{1}{2}} \cot(t) - 4\cancel{\frac{1}{2}} t$.

$\forall x \in]-1, 1[, \ \cot(\arcsin(x)) = \frac{\sqrt{1-x^2}}{x}$

$-\cancel{\tfrac{1}{2}}^{\,4}\arcsin\left(\frac{\sqrt{x}}{2}\right)$

donc $= \pm\cancel{\frac{1}{2}}^{\,4}\,\frac{\sqrt{1-x/4}}{\frac{\sqrt{x}}{2}} = \pm\cancel{\frac{1}{2}}^{\,4}\sqrt{\frac{4-x}{x}} \cdot -\frac{1}{2}\arcsin\left(\frac{\sqrt{x}}{2}\right)$

$-\cancel{\tfrac{1}{2}}^{\,4}\arcsin\left(\frac{\sqrt{x}}{2}\right).$

$$y(x) = \left(\pm\cancel{\frac{1}{2}}^{\,4}\sqrt{\frac{4-x}{x}} + C\right)\frac{\sqrt{x}}{|x-4|^{3/2}}$$

$$y(x) = \pm\cancel{\frac{1}{2}}^{\,4}\,\frac{1}{4-x} + C\,\frac{\sqrt{x}}{(4-x)^{3/2}}$$

~~On doit avoir $f(0) = 1 \Rightarrow$ facteur 2 ?~~

<u>Ex 1171</u> : $\sum_{n\geq 0} \mathrm{tr}(A^n)\, y^n$ (Ulysse)

$|\mathrm{tr}(A^n)| \leq n\,\|A^n\|_\infty \leq \cancel{n \times C\,|||A^n|||}$

~~avec~~ $\cancel{|||A||| = \sup \frac{\|Ax\|}{\|x\|}} = m$

En trigonalisant A (possible dans $\mathbb{C}$).

$$|\mathrm{tr}(A^n)| \leq n\Big(\underbrace{\sup_{\lambda\in Sp(A)}|\lambda|}_{=M}\Big)^n.$$

donc pour $|y| < \frac{1}{M}$, $|yM| < 1$.

donc $R = R_{cv}(\Sigma) \geq \frac{1}{M}$.

$\chi_A(x) = \det(xI_p - A)$.

$$(xI_p - A)^{-1} = \frac{1}{x}\left(I_p - \frac{A}{x}\right)^{-1} = \frac{1}{x}\sum_{n=0}^{+\infty}\left(\frac{A}{x}\right)^n$$

(pour $|x| > M$)