Exercice 1109 : (William)

$a_0 = 0$

$a_1 = 1$

$\forall n \geq 2,\ a_n = \sum_{k=1}^{n-1} a_k a_{n-k}$

$$\rho\Big(\underbrace{\sum_{n=0}^{\infty} a_n x^n}_{S(x)}\Big) = R$$

1- $R > 0$: $S(x) = \sum_{n=0}^{\infty} a_n x^n = 0 + 1 \times x + \sum_{n=2}^{\infty} a_n x^n$

$$S(x) = x + \sum_{n=2}^{\infty} x^k \sum_{k=1}^{n-1} a_k a_{n-k}$$

$$= x + \sum_{n=2}^{\infty} x^n \sum_{k=0}^{n} a_k\, a_{n-k}$$

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{k=0}^{\infty} b_k x^k\right) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_k b_{n-k} x^k$$

$$S(x) = x + \sum_{n=0}^{\infty} x^n \sum_{k=0}^{n} a_k a_{n-k} - x \overbrace{\sum_{k=0}^{n} a_k a_{n-k}}^{=0} - \overbrace{\sum_{k=0}^{\infty} a_k a_{n-k}}^{=0}$$

$$= x + \left(\sum_{k=0}^{\infty} a_k x^k\right)\left(\sum_{k=0}^{\infty} a_k x^k\right)$$

$$= x + S(x)^2$$

Soit $|x| < R$

$S(x)^2 - S(x) + x = 0$

$\Delta = 1 - 4x > 0$ ie $x \leq \frac{1}{4}$

On en déduit $R = \frac{1}{4}$

3. $S(x) = \dfrac{1 \pm \sqrt{1-4x}}{2} \quad \forall x \in \left]-\frac{1}{4}, \frac{1}{4}\right[$

$S(0) = 0$

$$S(x) = \frac{1 - \sqrt{1-4x}}{2}$$

$$S(x) = \frac{1}{2}\left(1 - \sqrt{1-4x}\right)$$

$$= \frac{1}{2}\left[\ \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \times (4x)^n \times \left(+\frac{1}{2}\right) \times \left(-\frac{1}{2}-1\right) \times \text{———} \times \left(+\frac{1}{2} - n + 1\right)\right]$$

$$= \frac{1}{2}\left[\ \sum_{n=1}^{\infty} \frac{4^n x^n}{n!} \times \frac{1}{2^n} \times \underbrace{\prod_{k=0}^{n-2} (2k+1)}_{= \frac{(2n-2)!}{2^{n-1}(n-1)!}}\right]$$

$$= \frac{1}{2}\left[\ \sum_{n=1}^{\infty} \frac{2^n x^n}{n!} \frac{(2n-2)!}{2^n (n-1)!}\right] = \sum_{n=1}^{\infty} \frac{x^n}{n!} \frac{(2n-2)!}{(n-1)!} = \sum_{n=1}^{\infty} \frac{x^n}{n} \times \binom{2n-2}{n-1}$$

4. $S(x) = \sum_{n=0}^{\infty} a_n x^n \Rightarrow a_n = \frac{1}{n} \binom{2n-2}{n-1}$

$P_n = x_1 \times x_2 \times \text{———} \times x_n$

$x_1 x_2 x_3 x_4 = x_1\left((x_2 x_3) x_4\right) = (x_1 \times x_2)(x_3 \times x_4)$

$$C_n = \sum_{k=1}^{n-1} C_k C_{n-k}$$

$C_1 = 1$

$C_2 = 1$

Made with Scanner App