Exercice 1046 (William)

(a) $g : x \longmapsto \displaystyle\int_0^\infty \frac{\sin(tx)}{t} e^{-t}\, dt$

$u = tx$
$du = x\,dt$

$$g(x) = \int_0^\infty \frac{\sin u}{u} e^{-u/x)}\, du = f\left(\frac{1}{x}\right)$$

$\Rightarrow$ $g = f \circ h$ où $h : x \longmapsto \frac{1}{x}$

$\mathcal{D}_g = \mathbb{R}_+^*$ (d'après 1045) $\quad g \in \mathcal{C}^1(\mathbb{R}_+^*, \mathbb{R})$

$$g'(x) = \int_0^\infty \sin(tx)\, e^{-t}\, dt = \mathrm{Re}\left(\int_0^\infty e^{itx} e^{-t}\, dt\right)$$

et $$\int_0^\infty e^{t(ix-1)}\, dt = \left[\frac{e^{t(ix-1)}}{ix-1}\right]_0^\infty$$

$$= \frac{-1}{ix-1}$$

$$= \frac{+(ix+1)}{x^2+1} \quad \Rightarrow \quad g'(x) = \frac{+1}{x^2+1}$$

Donc $g(x) = -\arctan(x) + \overbrace{g(0)}^{=0}$

(b) D'après 1045, f existe, $f(x) = g\left(\frac{1}{x}\right)$

On a $f(0) = \displaystyle\int_0^\infty \frac{\sin t}{t}\, dt = \lim_{x \to 0} g\left(\frac{1}{x}\right) = \lim_{x \to 0} \arctan\left(\frac{1}{x}\right) = \frac{\pi}{2}$

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