Et : $F(1) = -\int_0^{+\infty} \frac{dt}{1(1+t^2)} \arctan t + \int_0^{+\infty} \frac{\arctan(t)^2}{t^2} dt$

Et :

$$\boxed{I = 2F(1) = \pi \ln 2}$$

Ex 990 : Simon

$$J = \int_0^{+\infty} \frac{x}{\operatorname{sh} x} dx \quad , \quad I = \int_0^{+\infty} \frac{x}{\operatorname{ch} x} dx$$

$\rightarrow$ existence et calcul

On a :

$$\forall x \in \mathbb{R}, \quad \operatorname{ch} x = \frac{e^x + e^{-x}}{2}$$

$$I = 2\int_0^{+\infty} \frac{x}{e^x(1+e^{-2x})} = 2\int_0^{+\infty} \left(\sum_{k=0}^{+\infty} x(-1)^k e^{-(2k+1)x} \right) dx$$

$$\frac{1}{1-e^{-2x}} = \sum_{k=0}^{+\infty} (-e^{-2x})^k$$

$$S_n(x) = \sum_{k=0}^{n} (-1)^k 2x e^{-(2k+1)x}$$

$$S_n \xrightarrow{\text{CVS}} \frac{x}{\operatorname{ch} x}$$

$$S_n = \sum_{k=0}^{+\infty} (-1)^k 2x e^{-(2k+1)x} - R_n(x)$$

$$|S_n| \leq \frac{x}{\operatorname{ch} x} + x e^{-(2(n+1)+1)x} \leq x$$

$$I = 2\sum_{k=0}^{+\infty} \int_0^{+\infty} x(-1)^k e^{-(2k+1)x} dx$$

$$= 2\sum_{k=0}^{+\infty} (-1)^k \int_0^{+\infty} x e^{-(2k+1)x} dx$$

$$\int_0^{+\infty} x\, e^{-(2k+1)x}\, dx = \left[\frac{-1}{2k+1} e^{-(2k+1)x} x \right]_0^{+\infty} + \frac{1}{2k+1} \int_0^{+\infty} e^{-(2k+1)x}\, dx$$

$$= \frac{1}{(2k+1)^2}$$

$$\boxed{I = 2 \sum_{k=0}^{+\infty} \frac{(-1)^k}{(2k+1)^2}}$$

↪ calcul non achevé

$$J = \int_0^{+\infty} \frac{x}{\operatorname{sh} x}\, dx$$

$$= \int_0^{+\infty} \frac{2x}{e^{x} - e^{-x}}\, dx$$

$$\frac{1}{1-e^{-2x}} = \sum_{k=0}^{+\infty} e^{-2kx}$$

$$J = \int_0^{+\infty} \sum_{k=0}^{+\infty} \underbrace{2x\, e^{-2(k+1)x}}_{u_k(x)}\, dx$$

- $\sum_{n \geq 0} u_n(x) \xrightarrow{CVS} \frac{x}{\operatorname{sh} x}$
- $\int_0^{+\infty} 2x\, e^{-(2k+1)x}\, dx = \frac{2}{(2k+1)^2}$ tg d'une série cv

$$J = 2 \sum_{k=0}^{+\infty} \int_0^{+\infty} x\, e^{-(2k+1)x}\, dx = 2 \sum_{k=0}^{+\infty} \frac{1}{(2k+1)^2}$$

$$\sum_{k=1}^{+\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$$

$$\sum_{k=1}^{+\infty} \frac{1}{(2k)^2} = \frac{\pi^2}{24}$$

D'où : $\boxed{J = \frac{\pi^2}{4}}$