Exercice 931

a) Soit $n \in \mathbb{N}$, $x \in [0,1]$, $f_n(x) = \dfrac{2^n x}{1+2^n n x^2}$

$f_n(0) = 0 \quad \forall n \in \mathbb{N}$

$x \in]0,1]$ $\quad f_n(x) \underset{n\infty}{\sim} \dfrac{2^n x}{2^n n x^2} \sim \dfrac{1}{nx} \to 0$

Donc $f_n \xrightarrow{cvs} 0$ sur $[0,1]$.

b) $I_n = \int_0^1 f_n(x)\,dx = \int_0^1 \dfrac{2^n x}{1+n2^n x^2}\,dx = \int_0^1 \dfrac{1}{2n} \times \dfrac{2^{n+1} n x}{1+n2^n x^2}\,dx$

$= \dfrac{1}{2n}\left[\ln(1+n2^n x^2)\right]_0^1$

limite I_n cf d)

$$\boxed{I_n = \dfrac{\ln(1+n2^n)}{2n}}$$

c) Convergence uniforme

f_n est dérivable sur $[0,1]$

$f_n'(x) = \dfrac{2^n(1+n2^n x^2) - 2^n x(2x n 2^n)}{(1+n2^n x^2)^2}$

$= \dfrac{2^n + n2^{2n}x^2 - 2^{2n}x^2 n \times 2}{(1+n2^n x^2)^2}$

$= \dfrac{2^n - n2^{2n}x^2}{(1+n2^n x^2)^2}$

- $f_n'(x_n) = 0 \iff x_n^2 = \dfrac{1}{n2^n}$

$\iff x_n = \dfrac{1}{\sqrt{n}\,2^{n/2}}$

Donc f_n atteint un maximum en $x_n = \dfrac{1}{\sqrt{n}\,2^{n/2}}$

Donc $\|f_n\|_\infty^{[0,1]} = f_n(x_n)$

$$= \frac{2^n \times \frac{1}{\sqrt{n}\,2^{n/2}}}{1 + 2^n n \times \frac{1}{n 2^n}}$$

$$= \frac{2^{n/2}}{2\sqrt{n}} \xrightarrow[n\infty]{} +\infty \quad \text{par croissance comparée.}$$

Donc f_n ne converge pas uniformément.

d) DAS I_n

On a $I_n = \frac{\ln(1+n2^n)}{2n}$

$$= \frac{\ln(2^n n) + \ln\left(1 + \frac{1}{n2^n}\right)}{2n}$$

$$= \frac{n\ln(2)}{2n} + \frac{\ln(n)}{2n} + \underbrace{\frac{\ln\left(1+\frac{1}{n2^n}\right)}{2n}}_{o\left(\frac{\ln(n)}{n}\right)}$$

<u>D'où</u> : $I_n = \frac{\ln(2)}{n} + \frac{\ln(n)}{2n} + o\left(\frac{\ln(n)}{n}\right)$