Exercice 905 (Ulysse)

$$\sum_{k=1}^{n} f(k) = \int_1^n f(x)\,dx + \frac{1}{2}(f(1) - f(n)) + \int_1^n (\{x\}-1) f'(x)\,dx$$

f est $C^1(\mathbb{R},\mathbb{R})$. on fait une IPP.

~~Somme~~ Soit $n \in \mathbb{N}$,

$$\int_1^n f(x) \times 1\,dx = [x f(x)]_1^n - \int_1^n x f'(x)\,dx$$

$$= n f(n) - 1 f(1) - \int_1^n x f'(x)\,dx$$

$$= n f(n) - 1 f(1) - \int_1^n (\lfloor x \rfloor + \{x\}) f'(x)\,dx$$

$$= n(f(n)) - 1f(1) - \int_1^n \lfloor x \rfloor f'(x)\,dx + \int_1^n \{x\} f'(x)\,dx$$

$$= n f(n) - 1 f(1) - \sum_{k=1}^{n-1} \int_k^{k+1} k f'(x)\,dx - \int_1^n (\{x\} - \frac{1}{2}) f'(x)\,dx$$

$$- \underbrace{\int_1^n \frac{1}{2} f'(x)\,dx}_{-\frac{1}{2}(f(n)-f(1))}$$

$\cancel{= n f(n) - f(1) - \Sigma}$

$$= n f(n) - f(1) - \underbrace{\sum_{k=1}^{n-1} k(f(k+1) - f(k))}_{\sum_{k=1}^{n} ((k+1) f(k+1) - k f(k) - f(k+1)) = n f(n) - f(1) - \sum_{k=2}^{n} f(k)} + \int_1^n (\{x\} - \frac{1}{2}) f'(x)\,dx$$

$$- \frac{1}{2}(f(n) - f(1)).$$

$$\int_1^n f(x)\,dx = \sum_{k=2}^{n} f(k) - \frac{1}{2}(f(n) - f(1)) + \int_1^n (\{x\} - \frac{1}{2}) f'(x)\,dx$$

b) $u_n = \frac{1}{n}\sum_{k=1}^{n} e^{2i\pi \ln(k)}$

$= \frac{1}{n}\sum_{k=1}^{n} f(k)$ où $\boxed{f(x) = e^{2i\pi \ln(x)}}$

$$\frac{1}{n}\sum_{k=1}^{n} e^{2i\pi \ln(k)} = \frac{1}{n}\int_1^n e^{2i\pi \ln(x)}\,dx + \frac{1}{2n}\left(e^{2i\pi \times 0} - 2i\pi n\right)$$
①

$$+\frac{1}{n}\int_1^n \left(\{x\} - \frac{1}{2}\right)\frac{2i\pi}{x} e^{2i\pi \ln(x)}\,dx$$
②

$e^{2i\pi \ln(x)} = x^{2i\pi}$, $F(x) = \frac{x^{2i\pi+1}}{2i\pi+1}$

$$|②| \leq \frac{1}{n}\int_1^n \left|\{x\} - \frac{1}{2}\right| \frac{2\pi}{x}\,dx \leq \frac{\pi}{n}\int_1^n \frac{dx}{x} \leq \frac{\pi \ln(n)}{n}$$

① $\frac{1}{n}\int_1^n x^{2i\pi} = \frac{1}{n}\left[\frac{x^{2i\pi+1}}{2i\pi+1}\right]_1^n$

$= \frac{1}{(2i\pi+1)n}\left[n^{2i\pi+1} - 1\right]$

$$\frac{1}{n}\sum_{k=1}^{n} e^{2i\pi \ln(k)} = \frac{e^{2i\pi \ln(n)}}{2i\pi+1} - \frac{1}{n(2i\pi+1)}$$

donc ne converge pas.