Exercice 874

$f \in \mathcal{C}^1([1,+\infty[, \mathbb{R})$ et $(f')^2$ intégrable

a) $f(t) = f(1) + \int_1^t f'(u)\,du$

$$f^2(t) = f^2(1) + \left(\int_1^t f'(u)\,du\right)^2 + \underbrace{2f(1)\int_1^t f'(u)\,du}_{\leq f^2(1) + \left(\int_1^t f'(u)du\right)^2}$$

do

$$\frac{f^2(t)}{t^2} \leq \frac{2f^2(1)}{t^2} + \frac{2\left(\int_1^t f'(u)\,du\right)^2}{t^2}$$

$h(t) = \int_1^t f'(u)\,du$ Mq $\frac{h^2(t)}{t^2}$ est intégrable

$$\left(\frac{h(t)^2}{t^2}\right)' = \frac{2h(t)f'(t)}{t} - \frac{h(t)^2}{t^2}$$

$$\int_1^{x_0} \frac{h^2(t)}{t^2}\,dt = 2\int_1^{x_0} \frac{h(t)f'(t)}{t}\,dt - \overbrace{\frac{h^2(x_0)}{x_0}}^{\geq 0} + \overbrace{h^2(1)}^{=0}$$

$$\underset{CS}{\leq} 2\sqrt{\int_1^{x_0} \frac{h^2(t)}{t^2}\,dt}\;\underbrace{\sqrt{\int_1^{x_0} f'(t)^2\,dt}}_{\leq M}$$

$$\int_1^{x_0} \frac{h^2(t)}{t^2}\,dt \leq 2M\sqrt{\int_1^{x_0} \frac{h^2(t)}{t^2}\,dt} \Rightarrow \int_1^{x_0} \frac{h^2(t)}{t^2}\,dt \leq 4M^2$$

Donc $\int_1^{+\infty} \frac{h^2(t)}{t^2}\,dt$ converge

b) $\dfrac{f(t)}{\sqrt{t}} = \dfrac{f(1)}{\sqrt{t}} + \dfrac{\int_1^t f'(u)\,du}{\sqrt{t}}$

$\int_1^{+\infty} (f')^2$ converge donc $\int_x^{+\infty} (f')^2 \underset{x \to +\infty}{\longrightarrow} 0$

$\int_x^{+\infty} (f')^2 \longrightarrow 0$ donc $\forall \varepsilon > 0,\ \exists A \geq 1,\ \forall x \geq A,\ \int_A^{+\infty} f'(t)^2\,dt \leq \varepsilon$

$$\int_1^t f'(u)\,du = \int_1^A f'(u)\,du + \underbrace{\int_A^t f'(u)\,du}_{\underset{CS}{\leq} \underbrace{\sqrt{t-A}}_{\leq \sqrt{t}} \underbrace{\sqrt{\int_A^t f'(u)^2\,du}}_{\leq \varepsilon}}$$

Donc $\dfrac{\int_1^t f'(u)\,du}{\sqrt{t}} \leq \varepsilon + \underbrace{\dfrac{k}{\sqrt{t}}}_{\longrightarrow 0 \text{ donc } \leq \varepsilon} \leq 2\varepsilon$

$\dfrac{f(1)}{\sqrt{t}} \underset{t \to +\infty}{\longrightarrow} 0$ donc $\leq \varepsilon$

Donc $\dfrac{f(t)}{\sqrt{t}} \leq 3\varepsilon$ et enfin $f(t) = o(\sqrt{t})$