Exercice 867

$$f(x) = \int_x^{2x} \frac{dt}{\sqrt{t^2+t^4}}$$

a) On pose $ux = t \Rightarrow du \cdot x = dt$

$$f(x) = \int_1^2 \frac{x\,du}{\sqrt{x^2u^2+x^4u^4}} = \int_1^2 \frac{du}{\sqrt{u^2+x^2u^4}}$$

$$\sqrt{2+16x^2} \geq \sqrt{u^2+x^2u^4} \geq \sqrt{1+x^2}$$

$$\underbrace{\int \frac{1}{\sqrt{2+16x^2}}}_{\to 0} \leq \underbrace{\int \frac{1}{\sqrt{u^2+x^2u^4}}}_{\text{donc } \to 0} \leq \underbrace{\int \frac{1}{\sqrt{1+x^2}}}_{\to 0}$$

b) $$f'(x) = \frac{2}{\sqrt{4x^2+(4x^2)^2}} - \frac{1}{\sqrt{x^2+x^4}} = \frac{1}{\sqrt{4x^2+(4x^2)^2}\sqrt{x^2+x^4}}\underbrace{\left(2x(\sqrt{x^2+1}-\sqrt{1-4x^2})\right)}_{<0}$$

Donc f est strictement décroissante

c) Équivalent en $0^+$

$$f(u) = \int_1^2 \frac{du}{u\sqrt{1+u^2x^2}}$$

Or $$\frac{du}{u\sqrt{1+u^2x^2}} \underset{0^+}{=} \frac{du}{u}\left(1-\frac{u^2x^2}{2}+O(u^4x^4)\right)$$

$$f(u) = \int_1^2 \frac{du}{u} - \int_1^2 \frac{x^2}{2} u\,du + \underbrace{O(u^4x^4)}_{=c(u)}$$

$$c(u) \leq Mu^4x^4 = Cx^4 = O(x^4)$$

Donc $f(x) - \ln 2x \sim \frac{3}{4}x^2$