Exercice 851

On pose $f : t \longrightarrow e^{-t}\left(\ln t - \frac{1}{t} + \frac{1}{1-e^{-t}}\right) \in \mathcal{C}^0(]0,+\infty[)$

En 0 : $f(t) = e^{-t}\ln t - \frac{e^{-t}}{t} + \frac{e^{-t}}{1-e^{-t}}$

$\underset{t\to 0}{=} e^{-t}\ln t - \frac{(1-t+\frac{t^2}{2}+o(t^2))}{t} + \frac{e^{-t}}{1-e^{-t}}$

$\underset{t\to 0}{=} e^{-t}\ln t - \frac{1}{t} + 1 - \frac{t}{2} + o(t) + \frac{e^{-t}}{1-e^{-t}}$

$\underset{t\to 0}{=} (1-t+\frac{t^2}{2}+o(t^2))\ln t + 1 - \frac{1}{t} - \frac{t}{2} + \frac{1-t+\frac{t^2}{2}+o(t)^2}{t-\frac{t^2}{2}+o(t^2)} + o(t)$

$f(t) \underset{0}{=} \ln t - t\ln t + \frac{t^2\ln t}{2} + 1 - \frac{1}{t} + \frac{t}{2} + \frac{1-t+\frac{t^2}{2}+o(t^2)}{t-\frac{t^2}{2}+o(t^2)}$

$\underset{0}{=} \ln t - \frac{1}{t} + \frac{1-t+\frac{t^2}{2}+o(t^2)}{t(1-\frac{t}{2}+o(t))} + u(t)$

Où $u(t) = t\ln t + \frac{t^2\ln t}{2} + 1 + \frac{t}{2}$

$f(t) \underset{0}{=} \ln t - \frac{1}{t} + \left(\frac{1}{t} - 1 + \frac{t}{2} + o(t)\right)\left(1+\frac{t}{2}+o(t)\right) + u(t)$

$\underset{0}{=} \ln t - 1 + \frac{t}{2} + u(t) + o(t)$

Donc $f(t) \underset{0}{\sim} \ln t$ or $\int_0^A \ln t \, dt$ converge pour $A>0$

Donc par comparaison $\int_0^A f$ converge

En $+\infty$: $e^{-t}\left(\ln t - \frac{1}{t} + \frac{1}{1-e^{-t}}\right) \underset{+\infty}{\sim} e^{-t}\ln t \underset{t\to+\infty}{=} o\left(\frac{1}{t^2}\right)$

Donc par comparaison $\int_\varepsilon^{+\infty} f$ converge

Soit $\varepsilon > 0$ et $x > 0$

$\int_\varepsilon^x e^{-t}\ln t - \frac{e^{-t}}{t} + \frac{e^{-t}}{1-e^{-t}}\,dt = \underbrace{\int_\varepsilon^x e^{-t}\ln t - \frac{e^{-t}}{t}\,dt}_{=c} + \underbrace{\int_\varepsilon^x \frac{e^{-t}}{1-e^{-t}}\,dt}_{=[\ln(1-e^{-t})]_\varepsilon^x}$

$c = \int_\varepsilon^x e^{-t}\ln t - \int_\varepsilon^x \frac{e^{-t}}{t}\,dt$

$= \left[-e^{-t}\ln t\right]_\varepsilon^x + \int_\varepsilon^x \frac{e^{-t}}{t}\,dt - \int_\varepsilon^x \frac{e^{-t}}{t}\,dt = -e^{-x}\ln x + e^{-\varepsilon}\ln\varepsilon$

$$\int_\varepsilon^x f = -e^{-x}\ln x + e^{-\varepsilon}\ln\varepsilon + \ln(1-e^{-x}) - \ln(1-e^{-\varepsilon})$$

$$= \underbrace{e^{-\varepsilon}\ln\varepsilon - \ln(1-e^{-\varepsilon})}_{D} \underbrace{- e^{-x}\ln x + \ln(1-e^{-x})}_{E}$$

$$D \underset{0}{=} (1-\varepsilon+o(\varepsilon))\ln\varepsilon - \ln(1-1+\varepsilon+o(\varepsilon))$$

$$\underset{0}{=} \ln\varepsilon - \varepsilon\ln\varepsilon - \ln(\varepsilon+o(\varepsilon)) + o(\varepsilon\ln\varepsilon)$$

$$\underset{0}{=} \ln\varepsilon - \ln\varepsilon - \ln(1+o(1)) - \varepsilon\ln\varepsilon + o(\varepsilon\ln\varepsilon)$$

$$\xrightarrow[\varepsilon\to 0]{} 0$$

$$E = \underbrace{e^{-x}\ln x}_{\xrightarrow[x\to+\infty]{} 0} + \underbrace{\ln(1-e^{-x})}_{\xrightarrow[x\to+\infty]{} 0}$$

Donc $\int_0^{+\infty} f = 0$