Exe 800 :

a) On ~~pose~~ $\cancel{g(x) = f(x+1) \Rightarrow g^K(x) = f(x+K)}$

$$\cancel{\sum_{K=0}^{n} \binom{n}{K} (-1)^{n-K} f(x+K) = \sum_{K=0}^{n} \binom{n}{K} (-1)^{n-K} g^K(x) = \left(\sum_{K=0}^{n} \binom{n}{K} (-1)^{n-K} g^K\right)(x) = (g - id)^n (x)}$$

1) $u : \begin{array}{l} \mathcal{E}^\infty \to \mathcal{E}^\infty \\ f(x) \mapsto f(x+1) \end{array}$. $\displaystyle\sum_{K=0}^{n} \binom{n}{K} (-1)^{n-K} \underbrace{f(x+K)}_{= u^K(f)(x)} = \left(\sum_{K=0}^{n} \binom{n}{K} (-1)^{n-K} u^K\right)(f)(x)$

$$= (u - id)^n (f)(x)$$

récurrence sur K :

$K = 1$: $(u - id)(f)(x) = f(x+1) - f(x) = f'(c) \times 1$ où $c \in]x ; x+1[$ (TAF)

$K \in \mathbb{N}$: $(u - id)^{K+1}(f)(x) = (u - id)\Big(\underbrace{(u - id)^K (f)(x)}_{f^K(c_K) \text{ où } c_K \in]x ; x+K[\text{ (H.R)}}\Big)$

$= f^K(c_K + 1) - f^K(c_K)$

$= f^{K+1}(c_{K+1})$ où $c_{K+1} \in]c_K ; c_K + 1[$ (TAF)

D'où $\displaystyle\sum_{K=0}^{n} \binom{n}{K} (-1)^{n-K} f(x+K) = f^{(n)}(c)$

$\forall x \geq 0, \exists c \in]x ; x+n[$

2) $f(x) = x^\lambda$. $x = m$: $\displaystyle\underbrace{\sum_{K=0}^{n} \binom{n}{K} (-1)^{n-K} \underbrace{(m+K)^\lambda}_{\in \mathbb{N}}}_{\in \mathbb{Z}} = \lambda(\lambda - 1) \cdots (\lambda - n + 1) c^{\lambda - n} = \Delta$

où $c \in]m ; m+n[$

On prend $\lambda < n$: $\Delta = \dfrac{\lambda(\lambda-1) \cdots (\lambda - n + 1)}{c^{n-\lambda}}$ $\quad u_m = m^{n-\lambda} \xrightarrow[m \to +\infty]{} +\infty$