Exe 817 :

Soit $n \in \mathbb{N}$. $\int_0^1 f(t)\, g(nt)\, dt \underset{\uparrow}{=} \int_0^n f\left(\frac{u}{n}\right) g(u) \frac{du}{n}$

on pose $u = nt \Rightarrow du = n\,dt$

$$\int_0^n f\left(\frac{u}{n}\right) g(u) \frac{du}{n} = \frac{1}{n}\sum_{K=0}^{n-1} \left(\int_0^1 f\left(\frac{K+t}{n}\right) g(t)\, dt \right)$$

Pour $K \in [\![0, n-1]\!]$ et $t \in [0,1]$: $\left| f\left(\frac{K+t}{n}\right) - f\left(\frac{K}{n}\right) \right| \leq \frac{\|f'\|_\infty^{[0,1]}}{n}$

$f\left(\frac{K+t}{n}\right) = f\left(\frac{K}{n}\right) + R_n(t)$.

$$\int_0^1 f(t)\, g(nt)\, dt = \frac{1}{n}\sum_{K=0}^{n-1} \int_0^1 \left(f\left(\frac{K}{n}\right) + R_n(t) \right) g(t)\, dt = \frac{1}{n}\sum_{K=0}^{n-1} \left(\int_0^1 g(t)\, dt \right) f\left(\frac{K}{n}\right) + \frac{1}{n}\sum_{K=0}^{n-1} \int_0^1 R_n(t)\, g(t)\, dt$$

$$\frac{1}{n}\sum_{K=0}^{n-1} \left(f\left(\frac{K}{n}\right) \int_0^1 g(t)\, dt \right) \longrightarrow \int_0^1 f(t)\, dt \times \int_0^1 g(t)\, dt$$

$$\frac{1}{n}\left| \sum_{K=0}^{n-1} \int_0^1 R_n(t)\, g(t)\, dt \right| \underset{IT}{\leq} \frac{1}{n}\sum_{K=0}^{n-1} \int_0^1 |R_n(t)|\, g(t)\, dt \leq \frac{1}{n^2} \|f'\|_\infty \left(\sum_{K=0}^{n-1} \int_0^1 g(t)\, dt \right) \underset{n \to +\infty}{\longrightarrow} 0$$

$$\Rightarrow \boxed{\int_0^1 f(t)\, g(nt)\, dt \longrightarrow \int_0^1 f(t)\, dt \int_0^1 g(t)\, dt}$$

Exe 717 :

$$\prod_{i=1}^{K} (1 + x_i^K) \geq \left(1 + \prod_{i=1}^{K} x_i\right)^K \Leftrightarrow \ln\left(\prod_{i=1}^{K} (1 + x_i^K) \right) \geq K \ln\left(1 + \prod_{i=1}^{K} x_i\right)$$

$$\Leftrightarrow \sum_{i=1}^{K} \ln(1 + x_i^K) \geq K \ln\left(1 + \exp\left(\sum_{i=1}^{K} \ln(x_i) \right)\right)$$

$$\Leftrightarrow \frac{1}{K}\sum_{i=1}^{K} \ln(1 + \exp[K \ln(x_i)]) \geq \ln\left(1 + \exp\left[K \times \frac{1}{K} \sum_{i=1}^{K} \ln(x_i)\right]\right)$$

$f \left| \begin{array}{l} \mathbb{R} \to \mathbb{R} \\ x \to \ln(1 + e^{Kx}) \end{array} \right.$ $\qquad \Leftrightarrow \frac{1}{K}\sum_{i=1}^{K} f(\ln(x_i)) \geq f\left(\frac{1}{K} \sum_{i=1}^{K} \ln(x_i) \right)$ $\leftarrow$ Jensen

$\forall x \in \mathbb{R}$, $f''(x) = \frac{K^2 e^{Kx}}{1 + e^{Kx}} \geq 0 \Rightarrow f$ convexe.