Ex 661 - Nathanael

$A \in S_n^{++}(\mathbb{R})$. $\lim\limits_{k \to +\infty} \dfrac{\langle A^{k+1}X, X\rangle}{\langle A^k X, X\rangle}$?

$A = PDP^{-1}$ $\quad D = \mathrm{diag}(\lambda_1, \text{—}, \lambda_n)$

avec $0 < \lambda_1 \leq \ldots \leq \lambda_n$

$P = \left(\theta_1 \middle| \cdots \middle| \theta_n \right)$ $\quad (\theta_1, \ldots, \theta_n)$ BON de $\mathbb{R}^n$

$X = \sum\limits_{i=1}^{n} x_i \theta_i$

Soit $k \in \mathbb{N}$, $\langle A^k X, X\rangle = (A^k X)^T X$

$= (PD^k P^T X)^T X$

$= X^T P D^k P^T X$

$X^T P = \left(\sum x_i \theta_i^T\right)\left(\theta_1 \middle| \cdots \middle| \theta_n\right) = (x_1 \text{ —— } x_n)$

$P^T X = \begin{pmatrix} \theta_1^T \\ \vdots \\ \theta_n^T \end{pmatrix} \left(\sum x_i \theta_i\right) = \begin{pmatrix} x_1 \\ | \\ x_n \end{pmatrix}$

Donc $\boxed{\langle A^k X, X\rangle = \sum\limits_{i=1}^{n} x_i^2 \lambda_i^k}$

Donc $\dfrac{\langle A^{k+1}X, X\rangle}{\langle A^k X, X\rangle} = \dfrac{\sum_{i=1}^{n} x_i^2 \lambda_i^{k+1}}{\sum_{i=1}^{n} x_i^2 \lambda_i^{k}}$

$\exists\, n_0 \leq n$ tq $x_{n_0} \neq 0$ et $\forall k > n_0,\ x_k = 0$

$\exists\, p \leq n_0$ tq $\lambda_p = \dots = \lambda_{n_0}$

$$= \frac{\lambda_{n_0}^{k+1}}{\lambda_{n_0}^{k}} \frac{\sum_{i=1}^{n} x_i^2 \left(\frac{\lambda_i}{\lambda_{n_0}}\right)^{k+1}}{\sum_{i=1}^{n} x_i^2 \left(\frac{\lambda_i}{\lambda_{n_0}}\right)^{k}}$$

$$= \lambda_{n_0} \frac{\sum_{i=1}^{p-1} x_i^2 \left(\frac{\lambda_i}{\lambda_{n_0}}\right)^{k+1}}{\sum_{i=1}^{n} x_i^2 \left(\frac{\lambda_i}{\lambda_{n_0}}\right)^{k}} + \lambda_{n_0} \frac{\sum_{i=p}^{n} x_i^2}{\underbrace{\sum_{i=1}^{p-1} x_i^2 \left(\frac{\lambda_i}{\lambda_{n_0}}\right)^{k}}_{\longrightarrow 0} + \sum_{i=p}^{n} x_i^2}$$

$$\longrightarrow \lambda_{n_0} \in sp(A)$$