Exercice 617 (Martin)

$A, B \in \mathcal{S}_n^+(\mathbb{R})$

th spectral $A = P \underbrace{D}_{\in D_n^+(\mathbb{R})} P^T$

$\lambda_1, — \lambda_n$ valeurs propres de A

$$tr(AB) = tr\, PDP^T B$$
$$= tr(D\underbrace{P^T B P}_{= B'})$$
$$= \sum \underbrace{\lambda_i}_{\geq 0} \underbrace{b'_{ii}}_{\geq 0}$$
$$\leq \underbrace{\sum \lambda_i}_{\substack{tr D \\ = tr(A)}} \underbrace{\sum b'_{ii}}_{\substack{tr B' \\ = tr B}}$$

$$\underbrace{\langle BE_{ii}, E_{ii} \rangle}_{\geq 0} = b_{ii}$$

$$\langle BPE_{ii}, PE_{ii} \rangle \geq 0$$

Exercice 634 (Anaïs)

$a \in \mathbb{R}_+^*$ $\begin{bmatrix} 1 & \frac{a}{n} \\ -\frac{a}{n} & 1 \end{bmatrix}^n \xrightarrow[n\infty]{} \; ?$

$$A = \begin{bmatrix} 1 & a/n \\ -a/n & 1 \end{bmatrix}$$

$$\chi_A = x^2 - 2x + 1 + \frac{a^2}{n^2}$$

$$\Delta = -\frac{a^2}{n^2} < 0$$

A est diagonalisable dans $\mathbb{C}$

$$\chi_A = \left(X - \left(1 + i\frac{a}{n}\right)\right)\left(X - \left(1 - i\frac{a}{n}\right)\right)$$

$$A = P\begin{bmatrix} \overbrace{1+i\frac{a}{n}}^{\lambda_1} & 0 \\ 0 & \underbrace{1 - i\frac{a}{n}}_{\lambda_2} \end{bmatrix} P^T$$

$$E_{\lambda_1}(A) = \operatorname{Ker}\left(A - \lambda_1 I_n\right)$$

$$= \operatorname{Ker}\begin{bmatrix} -i\frac{a}{n} & \frac{a}{n} \\ -\frac{a}{n} & -i\frac{a}{n} \end{bmatrix}$$

$$= \operatorname{Vect}\left(\begin{pmatrix} 1 \\ i \end{pmatrix}\right) \qquad \left(C_1 + iC_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right)$$

$$E_{\lambda_2}(A) = \operatorname{Vect}\left(\begin{pmatrix} 1 \\ -i \end{pmatrix}\right)$$

$$P = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}$$

$$A^n = PD^nP^{-1} = P\begin{bmatrix} \left(1 + i\frac{a}{n}\right)^n & 0 \\ 0 & \left(1 - i\frac{a}{n}\right)^n \end{bmatrix} P^{-1}$$

$$z = \rho e^{i\vartheta}$$

$$1 + i\frac{a}{n} = \sqrt{1 + \frac{a^2}{n^2}}\; e^{i\arctan\left(\frac{a}{n}\right)}$$

$$1 - i\frac{a}{n} = \sqrt{1 + \frac{a^2}{n^2}}\; e^{-i\arctan\left(\frac{a}{n}\right)}$$

$$D^n = \begin{pmatrix} e^{\frac{n}{2}\ln\left(1+\frac{a^2}{n^2}\right)} e^{in\arctan\left(\frac{a}{n}\right)} & 0 \\ 0 & e^{\frac{n}{2}\ln\left(1+\frac{a^2}{n^2}\right)} e^{in\arctan\left(\frac{a}{n}\right)} \end{pmatrix}$$

Exercice 634 (suite)

$$D^n \xrightarrow[n\infty]{} \begin{pmatrix} e^{ia} & 0 \\ 0 & e^{-ia} \end{pmatrix}$$

$$A^n \xrightarrow[n\infty]{} \begin{pmatrix} \cos a & -\sin a \\ \sin a & \cos a \end{pmatrix}$$

Exercice 658 (William)

Sur $\mathbb{R}^n$, $\|x\|_1 = \sum_{k=1}^{n} |x_k|$

Mq Soient $(x,y) \in \mathbb{R}^n$, $\|x+y\|_1 + \|x-y\|_1 = 2\left[\|x\|_1 + \|y\|_1\right]$

$\iff \forall k \in [\![1,n]\!]$, $x_k y_k = 0$

$\boxed{\Rightarrow}$ $\sum_{k=1}^{n} |x_k + y_k| + \sum_{k=1}^{n} |x_k - y_k| = 2 \sum_{k=1}^{n} |x_k| + |y_k|$

Or $\forall k \in [\![1,n]\!]$, $|x_k + y_k| + |x_k - y_k| \leqslant 2(|x_k| + |y_k|)$

$$\sum_{k=1}^{n} \underbrace{2(|x_k| + |y_k|) - |x_k + y_k| - |x_k - y_k|}_{\geqslant 0} = 0$$

$\Rightarrow \forall k \in [\![1,n]\!]$, $2(|x_k| + |y_k|) = |x_k + y_k| + |x_k - y_k|$

$\Rightarrow x_k, y_k, -y_k$ de même signe $\forall k \in [\![1,n]\!]$.

$\Rightarrow x_k y_k = 0 \ \forall k \in [\![1,n]\!]$