

Exercice 613

(Markin)

$$A, B \in \mathcal{Y}_m^+(\mathbb{R}) \quad \text{Mq} \quad \det(A+B) \geq \det A + \det B$$

Cas ① $B = I_m$

$$\det(A + I_m)$$

$$A = P \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_m \end{bmatrix} P^T \quad (\text{th spectral}) \quad \text{sp}(A) \subset \mathbb{R}_+$$

$$\begin{aligned} \det(A + I_m) &= \det(D + I_m) = \prod_{i=1}^m (\lambda_i + 1) \\ &= \prod_{i=1}^m \lambda_i + 1 + \widehat{R} \end{aligned} \quad \text{reste} \geq 0$$

$$\geq \det D + \det I_m$$

Cas ② $B \in \mathcal{Y}_m^{++}(\mathbb{R})$

$$\exists C \in \mathcal{Y}_m^{++}(\mathbb{R}) : B = C^2 \quad \text{inversible.}$$

$$\det(A+B) = \det(A+C^2) \quad \text{cf page suivante}$$

$$= \det(C^2) \times \det(C^{-1}AC^{-1} + I_m)$$

$$\geq \det(C^2) \times (\det(C^{-1}AC^{-1}) + 1)$$

$$= \det A + \det C^2$$

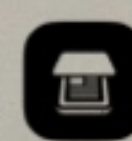
$$= \det A + \det B$$

Cas ③ : $B \in \mathcal{Y}_m^+(\mathbb{R})$

$$B_\epsilon = B + \epsilon I_m \in \mathcal{Y}_m^{++}(\mathbb{R})$$

$$\det(A+B_\epsilon) \geq \det A + \det B_\epsilon$$

$$\xrightarrow{\epsilon \rightarrow 0} \det(A+B) \geq \det A + \det B$$



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$$(*) \quad C^{-1} A C^{-1} \in \mathcal{S}_n(\mathbb{R})$$

$$X \in \mathcal{M}_{m,n}(\mathbb{R}) \text{ tq } X^T A X \geq 0$$

$$Y = C^{-1} X \quad \text{---} C^{-1}$$

$$Y^T = X^T (C^T)^{-1}$$

$$Y^T A Y \geq 0 \Rightarrow X^T \underbrace{C^{-1} A C^{-1}}_{\in \mathcal{S}_n^+(\mathbb{R})} X \geq 0$$

Exercice 552 (Amaïs)

$M = (\langle x^i, x^j \rangle)_{(i,j) \in [1,m]}$ est inversible (cf exo 526)

$M \in \mathcal{S}_n(\mathbb{R})$ donc diagonalisable.

λ valeur propre de M , A vecteur propre associé à λ

$$A^T M A = \underbrace{\lambda}_{\in \mathbb{R}_+^*} \underbrace{A^T A}_{\langle x^i, x^j \rangle}$$

$$\text{et } A^T M A = \sum_{i=0}^m \sum_{j=0}^m a_i \overbrace{M[i,j]}^{\langle x^i, x^j \rangle} a_j$$

$$= \sum_{i=0}^m \sum_{j=0}^m \langle a_i x^i, a_j x^j \rangle$$

$$= \left\langle \sum_{i=0}^m a_i x^i, \sum_{j=0}^m a_j x^j \right\rangle$$

$$= \|x\|^2$$

$$\text{avec } x = \sum_{i=0}^m a_i x^i$$