Exercice 529 (Antoine)

$A \in A_n(\mathbb{R})$, $S \in S_n^+(\mathbb{R})$

a. $sp(A) \subset i\mathbb{R}$, cf compléments

b. dans $M_n(\mathbb{C})$, A est trigonalisable

$\exists P \in GL_n(\mathbb{C})$, $T \in T_n^+(\mathbb{C})$

$A = PTP^{-1}$ où

car $A_n(\mathbb{R})$

$$T = \begin{pmatrix} \lambda_1 & & & & & & & \times \\ & \ddots & & & & & & \\ & & \lambda_1 & & & & & \\ & & & \bar{\lambda}_1 & & & & \\ & & & & \ddots & & & \\ & & & & & \bar{\lambda}_1 & & \\ & & & & & & \ddots & \\ & & 0 & & & & & \lambda_p \cdots \lambda_p \; \bar{\lambda}_p \cdots \bar{\lambda}_p \end{pmatrix}$$

$m_1, \dots, m_p$ les multiplicités de $\lambda_1, \dots, \lambda_p$.

$$\det(A + I_n) = \det(T + I_n)$$
$$= \prod_{i=1}^{p} \underbrace{\underbrace{(1+\lambda_i)}_{z_i}{}^{m_i} \underbrace{(1+\bar{\lambda}_i)}_{\bar{z}_i}{}^{m_i}}_{\geq 0} \geq 0$$

$$= \prod_{i=1}^{p} \underbrace{\left(|1+\lambda_i|^2\right)}_{\geq 1}{}^{m_i}$$

$$\geq 1$$

c. 1er cas où $S \in S_n^{++}(\mathbb{R})$

$\exists P \in O_n(\mathbb{R})$, $D = \mathrm{diag}(\mu_1, \dots, \mu_n)$ $\mu_i \in \mathbb{R}_+^*$

tq $S = PDP^T$

$\Delta = \mathrm{diag}(\sqrt{\mu_1}, \dots, \sqrt{\mu_n})$

$S = P\Delta\Delta P^{-1}$

$= P\Delta (P\Delta)^T$

$$\det(A+S) = \det\left(A+(P\Delta)(P\Delta)^T\right) = \det(P\Delta)\det\left((P\Delta)^{-1}A(P\Delta^{-1})^T + I_n\right) \times \det\left((P\Delta)^T\right)$$

$\begin{cases} \Delta \in GL_n(\mathbb{R}), P\Delta \in GL_n(\mathbb{R}) \\ P \in GL_n(\mathbb{R}) \end{cases}$

$$= \underset{\substack{\in S_n^+ \\ >0}}{\det S} \times \underbrace{\det\left(\underbrace{(P\Delta)^{-1}}_{=B} A(P\Delta^{-1})^T + I_n\right)}_{\text{car } \ell : \geq 1}$$

$(BAB^T)^T = BA^TB^T = -BAB^T$

donc $BAB^T \in A_n(\mathbb{R})$

$\det(A+S) \geq 1 \det S.$

<u>2e cas</u>: Si $S \in S_n^+(\mathbb{R})$

$\varepsilon > 0 \quad \det(A + \underbrace{S + \varepsilon I_n}_{\in S_n^{++}(\mathbb{R})}) \underset{1^{er}\text{cas}}{\geq} \det(S + \varepsilon I_n)$

$\varepsilon \to 0 \downarrow \qquad \downarrow \varepsilon \to 0$

$\det(A+S) \geq \det S$

par C^0té du det.

<u>Exercice 437</u> (Nicolas)

$f \in \mathcal{L}(E)$ diagonalisable

$g \in \mathcal{L}(E)$

$u_1, \dots, u_n \in \mathcal{L}(E)$

$\begin{cases} \forall i \in [\![1;n]\!], \quad u_i \circ g = f \circ u_i \\ \bigcap_{i=1}^{n} \ker u_i = \{0\}. \end{cases}$

Soit $i \in [\![1;n]\!]$.

Par récurrence, mq $\forall k \in \mathbb{N}, \quad u_i \circ g^k = f^k \circ u_i$

cas $k=0$ ok.