$\boxed{\text{exo 513}}$ Arthur

a) $n(A) = \mathrm{tr}(A^T A) = \sum_{i=1}^{n} \sum_{j=1}^{n} A^T_{ij} A_{ji}$

$$= \sum_{1 \leq i,j \leq n} (A_{ji})^2$$

$$n(AB) = \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\sum_{k=1}^{n} A_{jk} B_{ki} \right)^2$$

Or par Cauchy-Schwarz $\left(\sum_{k=1}^{n} A_{jk} B_{ki}\right)^2 \leq \sum_{k=1}^{n} A_{jk}^2 \sum_{\ell=1}^{n} B_{j\ell}^2$

$$n(AB) \leq \sum_{1 \leq i,j \leq n} \left(\underbrace{\sum_{k=1}^{n} A_{jk}^2}_{\leq A_{ji}^2} \underbrace{\sum_{k=1}^{n} B_{jk}^2}_{\leq n(B)} \right) \leq n(A)\, n(B)$$

b) $n(A) = \mathrm{tr}(A^2) = \sum_{i=1}^{n} \lambda_i^2$

$$(\lambda_n - \lambda_1)^2 \leq 2(\lambda_1^2 + \dots + \lambda_n^2)$$

$$\iff 0 \leq -\lambda_n^2 - \lambda_1^2 + 2\lambda_1\lambda_n + 2\lambda_1^2 + \dots + 2\lambda_n^2$$

$$\iff 0 \leq (\lambda_1 + \lambda_n)^2 + 2\lambda_2^2 + \dots + 2\lambda_{n-1}^2$$ ce qui est vrai

$\boxed{\text{exo 605}}$ Martin A. puis M. Fagebaume

$\cancel{(A, B, B-A, B+A) \in (S_n^+(\mathbb{R}))^4}$

La réponse est oui

Cas particulier : $B = I_n \in S_n^+(\mathbb{R})$

$I_n - A \in S_n^+(\mathbb{R}) \iff sp(I_n - A) \subset \mathbb{R}^+$

$\lambda \in sp(A) \Rightarrow 0 \leq \lambda \leq 1$

$\Rightarrow 0 \leq \lambda^2 \leq 1$

$sp(I_n - A^2) \subset \mathbb{R}^+ \Rightarrow I_n - A^2 \in S_n^+(\mathbb{R})$