d'où $\sum_{k=1}^{n} x_k^2 = 0 \Rightarrow x_k = 0 \quad \forall k \in [\![1, n]\!]$

CQFD

b) On suppose $\|e_k - f_k\| \leq \frac{1}{\sqrt{n}}$

hyp $\sum_{k=1}^{n} f_k = 0$ — on pose $\forall k \in [\![1, n]\!]$ $f_k = e_k + u$

$\Leftrightarrow \sum_{k=1}^{n} e_k = -nu$

$\Leftrightarrow \boxed{u = -\frac{1}{n} \sum_{k=1}^{n} e_k}$

donc $\underbrace{\left(e_k - \frac{1}{n} \sum_{i=1}^{n} e_i\right)}_{f_k}{}_{1 \leq k \leq n}$ bte

et $\|e_k - f_k\|^2 = \frac{1}{n^2} \sum_{i=1}^{n} \|e_i\| = \frac{1}{n}$

d'où $\boxed{\|e_k - f_k\| \leq \frac{1}{\sqrt{n}}}$

504 (Ambroise) $\begin{cases} E = \mathbb{R}^4, \text{ muni du psc} \\ \mathcal{B}_c = (e_1, \dots, e_4) \end{cases}$

Soit $G = \left\{ (x_1, x_2, x_3, x_4) \in E^4, \begin{cases} x_1 + x_2 = 0 \\ \text{et} \\ x_3 + x_4 = 0 \end{cases} \right\}$

$G = \mathrm{vect}(u, v)$

où $u = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$, $v = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}$

orthonormalisation $\langle u, v \rangle = 0$.

$\|u\| = 2$ ; $\|v\| = 2$

d'où $\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$ bon de $G$

$p_G$ la proj° orthogonale sur $G$,

$M = \mathrm{mat}_B(p_G) = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 0 & 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix}$

$= \frac{1}{2} \begin{pmatrix} 1 & -1 & & \\ -1 & 1 & & \\ & & 1 & -1 \\ & & -1 & 1 \end{pmatrix}$

2) Soit $x = (a\ b\ c\ d)^T$

$d(x, G) = \|x - p_G(x)\|$

$p_G(x) = Mx = \frac{1}{2} \begin{pmatrix} 1 & -1 & & 0 \\ -1 & 1 & & \\ & & 1 & -1 \\ 0 & & -1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$

$= \frac{1}{2} \begin{pmatrix} a-b \\ -a+b \\ c-d \\ -c+d \end{pmatrix}$ $x - p_G(x) = \frac{1}{2} \begin{pmatrix} a+b \\ a+b \\ c+d \\ c+d \end{pmatrix}$

d'où $d(x, F) = \frac{1}{2} \sqrt{(a+b)^2 + (c+d)^2}$