(117) (Nsam) Soit $\begin{cases} (\lambda,\mu) \in (\mathbb{C}^*)^2 \text{ tq } \lambda \neq \mu, \\ (M, A, B) \in M_n(\mathbb{C}) \text{ tq:} \end{cases}$

$$\left| \begin{array}{l} I_n = A + B, \\ M = \lambda A + \mu B \\ M^2 = \lambda^2 A + \mu^2 B \end{array} \right.$$

a) $M^2 = \lambda^2 A + \mu^2 (I_n - A) = (\lambda^2 - \mu^2) A + \mu^2 I_n$

$M = (\lambda - \mu) A + \mu I_n$.

d'où $A = \dfrac{(M - \mu I_n)}{\lambda - \mu}$

$$M^2 = (\lambda^2 - \mu^2) \cdot \frac{(M - \mu I_n)}{(\lambda - \mu)} + \mu^2 I_n$$

$$\iff M^2 + \frac{\mu^2 - \lambda^2}{\lambda - \mu} M = \left( - \frac{(\lambda^2 - \mu^2)}{\lambda - \mu} \mu + \mu^2 \right) I_n$$

$$= \lambda\mu \frac{(\mu - \lambda)}{\lambda - \mu} I_n = -\lambda\mu$$

d'où $\underbrace{M \left( M + \dfrac{\mu^2 - \lambda^2}{\lambda - \mu} \right) \cdot \dfrac{1}{(-\lambda\mu)}}_{M^{-1}} = I_n$

b) $\Leftrightarrow$ A et B projecteurs $I_n = A + B$

$\Rightarrow A^2 = A - AB = A - BA$

d'où $\boxed{AB = BA}$

$M^2 = (\lambda A + \mu B)(\lambda A + \mu B) = \lambda^2 A^2 + \mu^2 B^2 + 2\lambda\mu AB$

$= \lambda^2 (A - AB) + \mu^2 (B - AB) + 2\lambda\mu AB$

$= \underbrace{\lambda^2 A + \mu^2 B}_{M^2} + \underbrace{(2\lambda\mu - \lambda^2 - \mu^2)}_{?} AB$

or $\underset{\neq 0}{2\lambda\mu - \lambda^2 - \mu^2 = -(\lambda - \mu)^2}$

d'où $\underline{AB = 0}$

d'où $\begin{cases} A^2 = A \\ B^2 = B \end{cases}$

c)