# TD : 03/11

William : exercice 346

$$A = \left(\left(\frac{1}{a_i + b_j}\right)\right) \in M_n(\mathbb{R}) = \begin{pmatrix} \frac{1}{a_1+b_1} & \cdots & \frac{1}{a_1+b_n} \\ \frac{1}{a_2+b_1} & & \vdots \\ \vdots & & \vdots \\ \frac{1}{a_n+b_1} & & \frac{1}{a_n+b_n} \end{pmatrix}$$

$$\det A = \begin{vmatrix} \frac{1}{a_1+b_1} & \cdots & \frac{1}{a_1+b_n} \\ \vdots & & \vdots \\ \frac{1}{a_n+b_1} & & \frac{1}{a_n+b_n} \end{vmatrix}$$

$$= \frac{1}{\prod_{j=1}^{n}(a_n+b_j)} \begin{vmatrix} \frac{a_n+b_1}{a_1+b_1} & \frac{a_n+b_2}{a_1+b_2} & \frac{a_n+b_n}{a_1+b_n} \\ \frac{a_n+b_1}{a_2+b_1} & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ \frac{a_n+b_1}{a_{n-1}+b_1} & \frac{a_n+b_2}{a_{n-1}+b_2} & \frac{a_n+b_n}{a_{n-1}+b_n} \\ 1 & \cdots & 1 \end{vmatrix}$$

$$\det A \underset{\substack{C_i \leftarrow C_i - C_n \\ \forall i \in [\![1, n-1]\!]}}{=} \frac{1}{\prod_{j=1}^{n}(a_n+b_j)} \begin{vmatrix} \frac{(b_n-b_1)(a_n-a_1)}{(a_1+b_1)(a_1+b_n)} & \frac{(b_n-b_2)(a_n-a_2)}{(a_1+b_2)(a_1+b_n)} & \cdots & \frac{a_n+}{a_1+} \\ \vdots & \vdots & & \cdots \\ \frac{(b_n-b_1)(a_n-a_1)}{(a_n+b_1)(a_{n-1}+b_n)} & \frac{(b_n-b_2)(a_n-a_2)}{(a_n+b_2)(a_{n-1}+b_n)} & \cdots & \frac{a_n+b_n}{a_{n-1}+b_n} \\ 0 & 0 & \cdots & 1 \end{vmatrix}$$

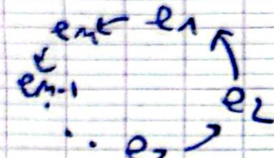
$$= \frac{\prod_{j=1}^{n-1}(b_n-b_j)(a_n-a_j)}{\prod_{j=1}^{n}(a_n+b_j)\prod_{i=1}^{n-1}(a_i+b_n)} \begin{vmatrix} \frac{1}{a_1+b_1} & \frac{1}{a_1+b_2} & \cdots & \frac{1}{a_1+b_{n-1}} \\ \vdots & & & \\ \frac{1}{a_{n-1}+b_1} & & \cdots & \frac{1}{a_{n-1}+b_{n-1}} \end{vmatrix}$$

$$= \frac{V(b_1, \dots, b_n)\, V(a_1, \dots, a_n)}{\prod_{i \neq j} (a_i + b_j)}$$

Exercice 352 : Ulysse

$$H = \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & & 1 \\ 1 & & & 0 \end{pmatrix} \quad M = P(H) \text{ où } P(X) = a_0 + \dots + a_{n-1} X^{n-1}$$

Soit $u$ canoniquement associé à $H$



$\forall i \in [\![1, n]\!]$, $u^n(e_i) = e_i$

$q(X) = X^n - 1$ annulateur donc A

$$q(X) = \prod_{k=0}^{n-1} (X - \omega^k)$$

donc $H$ diagonalisable.

$\underline{Sp(H) \subset \mathbb{U}_n}$

Soit $\lambda \in Sp(H)$, $\lambda I_n - H = \begin{pmatrix} \lambda & -1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & -1 \\ -1 & & & \lambda \end{pmatrix}$

$$rg(\lambda I_n - H) \geq n - 1$$

th du rg $\Downarrow$ $\dim(\operatorname{Ker}(\lambda I_n - H)) \leq 1$

Or, $H$ diagonalisable : $\bigoplus_{\lambda \in Sp(H)} E_\lambda = E$

donc $\operatorname{card}(Sp(H)) = n$

donc $Sp(H) = \mathbb{U}_n$

$$H = P^{-1} \begin{pmatrix} \omega^0 & & \\ & \ddots & \\ & & \omega^{n-1} \end{pmatrix} P$$