Exercice 132 (Ulysse)

$f : \mathbb{R} \to \mathbb{R}$, $\mathcal{C}^0$, 1-périodique.

$$\forall n \in \mathbb{N}^*,\quad u_n = \int_n^{n+1} \frac{\overset{\swarrow F(t)}{f(t)}}{\underset{\searrow -\frac{1}{t^2}}{t}}\, dt.$$

IPP :

$$u_n = \left[\frac{F(t)}{t} \right]_n^{n+1} + \int_n^{n+1} \frac{F(t)}{t^2}\, dt.$$

$$= \frac{F(n+1)}{n+1} - \frac{F(n)}{n} + \int_n^{n+1} \frac{F(t)}{t^2}\, dt$$

D'où par télescopage :

$$\sum_{k=1}^{n-1} u_k = \frac{F(n)}{n} - F(1) + \int_1^n \frac{F(t)}{t^2}\, dt.$$

$$F(x) = \int_0^x f(t)\,dt = \int_0^{\lfloor x \rfloor} f(t)\,dt + \int_{\lfloor x \rfloor}^x f(t)\,dt.$$

$$= \lfloor x \rfloor \underbrace{\int_0^1 f(t)\,dt}_{f_{moy}} + \underbrace{\int_{\lfloor x \rfloor}^x f(t)\,dt}_{M}$$

$$|M| \leq \int_{\lfloor x \rfloor}^x |f(t)|\,dt$$

$$\leq \sup |f|$$

• Si $f_{moy} \neq 0$, $F(x) \sim f_{moy}\, x$.

Donc $\frac{F(n)}{n} \longrightarrow f_{moy}$

$$\int_1^n \frac{F(t)}{t}\,dt = \int_1^n \left(\frac{f_{moy}}{t} + o\left(\frac{1}{t}\right)\right)dt$$

$$\sim f_{moy} \ln(t)$$

Donc $\boxed{\sum u_k \quad DV}$.

• si $f_{moy} = 0$, $|F(x)| \leq \sup(|f|)$.

$\frac{F(n)}{n} \longrightarrow 0$; $\int_1^n \frac{F(t)}{t^2}dt \leq \sup|f| \int_1^n \frac{1}{t^2}dt$.

$\longrightarrow$ on ne peut pas conclure.

Fagebaume 2ème méthode : $u_n = \int_n^{n+1} \frac{f(t)}{t}dt$

changement de variable : $u = t - n$.

$$u_n = \int_0^1 \frac{f(u+n)}{u+n}\,du = \int_0^1 \frac{f(u)}{u+n}\,du$$

On pose $\alpha = \int_0^1 f(t)\,dt$.

$$u_n - \frac{\alpha}{n} = \int_0^1 \left[\frac{f(u)}{u+n} - \frac{f(u)}{n}\right]du$$

$$= \int_0^1 - \frac{f(u)\,u\,du}{n(n+u)}$$

f $\mathcal{C}^0$ sur $[0,1] \Rightarrow f$ bornée.

On note M tq $|f(x)| \leq M \ \forall x \in [0,1]$
$|xf(x)| \leq M \ \forall x \in [0,1]$

$$|u_n| = \left|\int_0^1 - \frac{f(u)\,u\,du}{n(n+u)}\right| \leq \frac{M}{n^2} \Rightarrow u_n = \frac{\alpha}{n} + O\left(\frac{1}{n^2}\right)$$

Ainsi, $\boxed{\sum u_n \ DV \Leftrightarrow \alpha = 0}$