Exercice 103 (Nicolas)

$$u_n = \int_0^{\frac{\pi}{2}} \cos\left(\frac{\pi}{2}\sin(x)\right)^n dx.$$

Pour $\alpha > 0$, nature de $\sum u_n^\alpha$?

Idée : ça fait penser à Wallis.

$x \mapsto \frac{\pi}{2}\sin(x)$ réalise une bijection C^1 de $[0;\frac{\pi}{2}]$ sur $[0;\frac{\pi}{2}]$

On pose $u = \frac{\pi}{2}\sin(x)$.

$$x = \arcsin\left(\frac{2}{\pi}u\right)$$

$$dx = \frac{\frac{2}{\pi}\,du}{\sqrt{1-\left(\frac{2}{\pi}u\right)^2}}$$

Donc $u_n = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{\cos^n(u)}{\sqrt{1-\left(\frac{2u}{\pi}\right)^2}}\,du$

$\forall u \in [0;\frac{\pi}{2}]$, $\cos^n u \geq 0$ et $\frac{1}{\sqrt{1-\left(\frac{2u}{\pi}\right)^2}} \geq 1$

Donc $u_n \geq W_n \times \frac{2}{\pi}$.

$$u_n = \frac{2}{\pi}\left[\underbrace{\int_0^{\frac{\pi}{4}} \frac{(\cos u)^n}{\sqrt{1-\left(\frac{2u}{\pi}\right)^2}}\,du}_{A_n} + \underbrace{\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{(\cos u)^n}{\sqrt{1-\left(\frac{2u}{\pi}\right)^2}}\,du}_{B_n}\right]$$

$\cdot B_n \leq \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos(\frac{\pi}{4})^n}{\sqrt{1-(\frac{2u}{\pi})^2}} du = \underbrace{\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{\sqrt{1-(\frac{2u}{\pi})^2}}}_{K = cte} \left(\frac{\sqrt{2}}{2}\right)^n$

$= K \times \left(\frac{\sqrt{2}}{2}\right)^n.$

$\cdot A_n \leq \int_0^{\frac{\pi}{4}} \frac{(\cos u)^n}{\sqrt{1-(\frac{2}{\pi}\times\frac{\pi}{4})^2}} \leq \frac{2}{\sqrt{3}} \int_0^{\pi/4} (\cos u)^n du \leq \frac{2}{\sqrt{3}} W_n.$

Ainsi, $\underline{B_n = K \times \left(\frac{1}{\sqrt{2}}\right)^n}$ et $\underline{A_n \sim \frac{2}{\sqrt{3}} \sqrt{\frac{\pi}{2n}}}$

D'où $B_n = o(A_n)$ et $\begin{cases} 0 \leq B_n & \forall n \in \mathbb{N} \\ 0 \leq A_n. \end{cases}$

D'où APCR, $B_n \leq A_n$ i.e. $A_n + B_n \leq 2A_n$.

D'où APCR, $\boxed{\frac{4}{\sqrt{3}} \times \frac{2}{\pi} W_n \geq u_n \geq W_n \geq 0}$

Or $W_n \sim \sqrt{\frac{\pi}{2n}}$

$\boxed{\sum u_n^\alpha \; CV \iff \alpha > 2}.$