Exercice sur les séries :

Exercice 98 :

a) Mq $\sum \arctan(1/2n^2)$ cv.

on a $\frac{1}{1+x^2} \underset{x\to 0}{=} 1 - x^2 + o(x^2) \xrightarrow{\text{primitivate}} \arctan(x) = x - \frac{x^3}{3} + o(x^3)$

$\arctan\left(\frac{1}{2n^2}\right) \underset{n\infty}{=} \frac{1}{2n^2} + o\left(\frac{1}{n^2}\right) \sim \underbrace{\frac{1}{2n^2}}_{\text{tg d'une série cv}} \geq 0$

donc $\boxed{\sum \arctan(1/2n^2) \text{ cv}}$

b) $\forall x, y \in \mathbb{R}$. Montrer l'existence de $k \in \{-1; 0; 1\}$ tq
$\arctan(x) - \arctan(y) = \arctan\left(\frac{x-y}{1+xy}\right) + k\pi$

$\forall x \in \mathbb{R}^*$, $\arctan(x) + \arctan(1/x) = \pm \pi/2$
↑ signe de x

donc $\forall x, y \in \mathbb{R}^*$

$\arctan(x) - \arctan(y) = -\arctan(1/x) + \arctan(1/y) + \alpha$

$$\alpha = \begin{cases} 0 & \text{si } x, y \text{ de m̂ signes} \\ +\pi & \text{si } x > 0 \quad y < 0 \\ -\pi & \text{si } x < 0 \quad y > 0 \end{cases}$$

on $\tan(a+b) = \frac{\tan(a) + \tan(b)}{1 - \tan(a)\tan(b)}$ pour $\begin{cases} a, b \neq \pi/2 \, [\pi] \\ \tan(a)\tan(b) \neq 1 \end{cases}$

- $y \neq -1/x$

$x \mapsto \arctan(x) \in \mathcal{C}^\infty(\mathbb{R},]-\pi/2 ; \pi/2[)$

$\tan(-\arctan(1/x) + \arctan(1/y)) = \frac{-1/x + 1/y}{1 + \frac{1}{xy}} = \frac{x-y}{xy+1}$

on a bien $\arctan(x) - \arctan(y)$
$= \arctan\left(\frac{x-y}{xy+1}\right)$

- $y = -1/x$: $\arctan(x) - \arctan(y) = \pm \pi/2$
- si $x = 0$: on a bien $-\arctan(y) = -\arctan(y)$

idem $y = 0$

c) Valeur de $\sum_{n=1}^{\infty} \arctan(1/2n^2)$?

$x = 2(n+1)$ et $y = 2n$

Soit $N \in \mathbb{N}$

$\sum_{n=0}^{N} \arctan\left(\frac{x-y}{1+xy}\right) = \sum_{n=0}^{N} \arctan\left(\frac{2}{1+4n^2+4n}\right)$

• $\displaystyle\sum_{n=0}^{N} \arctan\left(\frac{1}{1/2 + 2n^2 + 2n}\right) = \sum_{n=0}^{N} \arctan(2(n-1)) - \arctan(2n)$

$\cancel{\displaystyle\sum_{n\infty} \arctan\left(\frac{1}{2n^2}\right)}$ NON

$= \arctan(2(N+1)) - \arctan(0)$

$= \arctan(2(N+1)) \xrightarrow[N\infty]{} \pi/2$

/ bon abandon