Exercice 69 (Quentin)

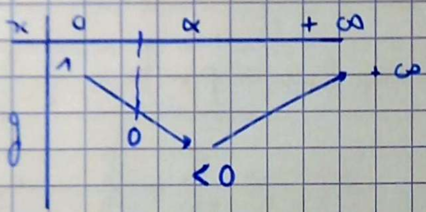
On pose : $\begin{cases} u_{n+1} = \sqrt{2}^{\,u_n} \\ u_0 = \sqrt{2} \end{cases}$

Soit $f : x \longmapsto \sqrt{2}^{\,x}$ et $g : x \longmapsto f(x) - x$

$$g'(x) = \frac{\ln(2)}{2}\sqrt{2}^{\,x} - 1$$

On a : $g'(x) = 0 \iff x = 2\left[1 - \underbrace{\frac{\ln(\ln(2))}{\ln(2)}}_{<0}\right] > 2$

$= \alpha$



Récurrence : $1 \leq u_n \leq \alpha$ $\quad I = [1, 2] \subset f(I)$

$\forall x \in [1, 2], \; f(x) \geq x \implies u_{n+1} - u_n$

TLM $(u_n)_n \nearrow$ & majorée. $u_n \longrightarrow l$.

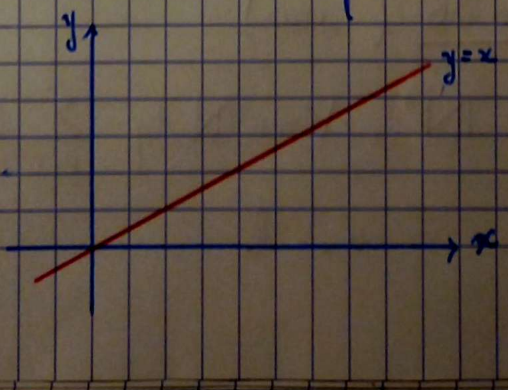
$u_n \longrightarrow 2$ seul point fixe de f sur $[0, 2]$

Donc $\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\cdots}}}} = 2$.

Exercice 58 (Lucas)

$\forall n, \quad u_{n+1} = |u_n - n| = \begin{cases} u_n - n & \text{si } u_n \geq n \\ n - u_n & \text{si } u_n \leq n \end{cases}$

$u_1 = |u_0|$



On note p le premier indice tq $u_p < p$

$$\cancel{u_{p+1} = u_p - p < p}$$

$$u_{p+1} = p - u_p < p$$

$$u_{p+2} = p+1-(p-u_p) = 1+u_p < p+1$$

$\dots$

Soit $k \in \mathbb{N}$, supposons $u_{p+k} = \begin{cases} \frac{k}{2} + u_p & \text{si } k \text{ pair} \\ p + \frac{k-1}{2} - u_p & \text{si } k \text{ impair} \end{cases}$

dans les 2 cas : $u_{p+k} < p+k$

- k pair : $u_{p+k+1} = p+k-\left(\frac{k}{2}+u_p\right)$

$$= p + \frac{k}{2} - u_p$$

- k impair :

$$u_{p+k+1} = p+k-\left(p+\frac{k-1}{2}-u_p\right)$$

$$= \frac{k+1}{2} + u_p.$$

Pour $k \in \mathbb{N}$, on pose $n = k+p$

- k pair : $u_n = \frac{n}{2} - \frac{p}{2} + u_p \sim \frac{n}{2}$
- k impair : $u_n = \frac{n}{2} - \frac{p+1}{2} + p - u_p \sim \frac{n}{2}$

$$\Longrightarrow \quad u_n \sim \frac{n}{2}.$$

Exercice 56 (Lucas)

Pour $k \in \mathbb{N}^*$, $\forall n \in \mathbb{N}$ tq $\sum_{l=1}^{k} l - k + 1 \leq n \leq \sum_{l=1}^{k} l$

$$u_n = k$$