Exercice 55 :

$$u_n = \sum_{k=1}^{n} \underbrace{\frac{1}{\sqrt{n^2+k}}}_{\le 1/n} \le 1$$

$$\cancel{u_n = \sum_{k=1}^{n} \frac{1}{n\sqrt{1+\frac{k}{n^2}}} = \frac{1}{n} \sum_{k=1}^{n}}$$

on peut encadrer (u_n) :

$$\frac{n}{\sqrt{n^2+n}} \le u_n \le \frac{n}{\sqrt{n^2+1}}$$

or $\sqrt{n^2+n} \sim n$

$\sqrt{n^2+1} \sim n$

donc par encadrement : $\boxed{u_n \longrightarrow 1}$

$$v_n = \sum_{k=1}^{n} \frac{1}{\sqrt{k^2+n^2}} = \frac{1}{n} \sum_{k=1}^{n} \frac{1}{\sqrt{1+\frac{k^2}{n^2}}}$$

$$= \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right)$$

donc $v_n \longrightarrow \int_0^1 f(t)\,dt$

$$\left(\int_0^1 f(t)\,dt = \int_0^1 \frac{1}{\sqrt{1+t^2}}\,dt = \big[\operatorname{argsh}(t)\big]_0^1 = \ln\left(1+\sqrt{2}\right) \right)$$

Calcul de $\int_0^1 \frac{1}{\sqrt{1+t^2}} dt$:

$t \mapsto sh(t)$ est C^0 et bijective :

$t = sh(u)$ $\qquad dt = ch(u)\, du$

$t = 0 \iff u = 0$

$t = 1 \iff \frac{e^u - e^{-u}}{2} = 1$

donc $e^{2u} - 2e^u - 1 = 0$

d'où e^u racine de $X^2 - 2X - 1$

donc $e^u = 1 + \sqrt{2}$

Ainsi : $\int_0^1 \frac{1}{\sqrt{1+t^2}} dt = \int_0^{\ln(1+\sqrt{2})} \frac{ch\,u\, du}{\sqrt{1+sh^2(u)}}$

$ch\,u^2 - sh\,u^2 = 1 \nearrow$ $= \int_0^{\ln(1+\sqrt{2})} \frac{ch\,u}{ch\,u} du$

$= [u]_0^{\ln(1+\sqrt{2})}$

donc : $\boxed{v_n \longrightarrow \ln(1+\sqrt{2})}$

$$\omega_n = \sum_{k=1}^{n} \frac{1}{\sqrt{n^2+k^3}} = \sum_{k=1}^{\lfloor n^{3/4} \rfloor} \frac{1}{\sqrt{n^2+k^3}} + \sum_{k=\lfloor n^{3/4}\rfloor+1}^{n} \frac{1}{\sqrt{n^2+k^3}}$$

$$\leq \underbrace{\sum_{k=1}^{\lfloor n^{3/4} \rfloor} \frac{1}{\sqrt{n^2}}} + \underbrace{\sum_{k=\lfloor n^{3/4}\rfloor+1}^{n} \frac{1}{\sqrt{n^2+k^3}}}$$

$$\sim \frac{n^{3/4}}{n} \longrightarrow 0$$

$$\leq \sum_{k=\lfloor n^{3/4}\rfloor+1}^{n} \frac{1}{\sqrt{n^2+n^{3/4}}} \quad \sim n^{3/8}$$

$$\leq \frac{n}{n^{3/8}} \longrightarrow 0$$

d'où : $\underline{\omega_n \longrightarrow 0}$