Exercice 11

(Nathanaël)

$u_0 = a$, $v_0 = b$ $\quad 0 \leq a < b$

$u_n = \dots \qquad v_n = \dots$

1) $P(n)$: $u_n \leq u_{n+1} \leq v_{n+1} \leq v_n$

I : $P(0)$: $b > u_1 = \frac{a+b}{2} > a$

soit $u_0 < u_1 < v_0$

$v_1 = \sqrt{u_1 b} > u_1$ car $b > u_1$.

Donc $u_0 \leq u_1 \leq v_1 \leq v_0$

H : Supposons $P(n)$

$u_{n+2} \geq u_{n+1}$ car $u_{n+1} \leq v_{n+1}$ (H.R)

$u_{n+2} \leq v_{n+2} \leq v_{n+1}$ car $u_{n+2} \leq v_{n+1}$

donc $u_{n+1} \leq u_{n+2} \leq v_{n+2} \leq v_{n+1}$

CCL : $(u_n) \nearrow$

$(v_n) \searrow$

$\forall n \in \mathbb{N}, \begin{cases} u_n \leq v_n \leq v_0 \\ v_n \geq u_0 \end{cases}$

TLM : $u_n \to l$

$v_n \to l'$

donc $l = \frac{1}{2}(l + l')$

$l' = \sqrt{l l'}$ $\quad$ donc $l = l'$.

2) $a = b\cos\varphi$ $\qquad$ On utilise $\cos\theta + 1 = 2\cos^2\left(\frac{\theta}{2}\right)$

On voit que $u_1 = \frac{1}{2}(b\cos\varphi + b)$

$= b\cos^2\left(\frac{\varphi}{2}\right)$

$v_1 = \sqrt{b \times b\cos^2\frac{\varphi}{2}} = b\cos\frac{\varphi}{2}$

On suppose alors :

$\begin{cases} u_n = b\prod_{k=1}^{n}\cos\left(\frac{\varphi}{2^k}\right) \times \cos\left(\frac{\varphi}{2^n}\right) \\ v_n = b\prod_{k=1}^{n}\cos\left(\frac{\varphi}{2^k}\right) \end{cases}$

Alors : $u_{n+1} = \frac{1}{2} b \prod_{k=1}^{n}\cos\left(\frac{\varphi}{2^k}\right)\left(1 + \cos\frac{\varphi}{2^n}\right)$

$= b\prod_{k=1}^{n+1}\cos\left(\frac{\varphi}{2^k}\right) \times \cos\left(\frac{\varphi}{2^{n+1}}\right)$

et $o_{n+1} = \sqrt{\left(b \times \prod_{k=1}^{n} \cos\left(\frac{\varphi}{2^k}\right)\right)^2 \cos^2\left(\frac{\varphi}{2^{n+1}}\right)}$

$$= b \prod_{k=1}^{n+1} \cos\left(\frac{\varphi}{2^k}\right)$$

Donc on a bien nos expressions de v_n et o_n.
On cherche désormais leur limite.

On note $P_n = \prod_{k=1}^{n} \cos\left(\frac{\varphi}{2^k}\right)$

$\sin(2\theta) = 2\sin\theta \times \cos\theta$

$\forall k,\ \sin\left(\frac{\varphi}{2^{k-1}}\right) = 2 \underbrace{\sin\left(\frac{\varphi}{2^k}\right)}_{\neq 0} \cos\left(\frac{\varphi}{2^k}\right)$

$$\text{donc } \cos\left(\frac{\varphi}{2^k}\right) = \frac{1}{2} \frac{\sin\left(\frac{\varphi}{2^{k-1}}\right)}{\sin\left(\frac{\varphi}{2^k}\right)}$$

Donc $P_n = \prod_{k=1}^{n} \frac{\sin\left(\frac{\varphi}{2^{k-1}}\right)}{2\sin\left(\frac{\varphi}{2^k}\right)} = \frac{1}{2^n} \frac{\sin\varphi}{\sin\left(\frac{\varphi}{2^n}\right)}$

$$\underset{n\infty}{\sim} \frac{\sin\varphi}{\cancel{2^n}\frac{\varphi}{\cancel{2^n}}} = \frac{\sin\varphi}{\varphi}$$