Caso práctico. (Ceuta 2016) Resuelva la siguiente ecuación en $\mathbb{R}$, conocido $\alpha \in \mathbb{R}$:

$$\left(\frac{1+ix}{1-ix}\right)^4 = \frac{1+i\tan(\alpha/2)}{1-i\tan(\alpha/2)}$$

Teniendo en cuenta que $\sin\alpha = \dfrac{e^{i\alpha}-e^{-i\alpha}}{2i}$; $\cos\alpha = \dfrac{e^{i\alpha}+e^{-i\alpha}}{2}$ y que

$$\tan\frac{\alpha}{2} = \sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}} = \sqrt{\frac{2-e^{i\alpha}-e^{-i\alpha}}{2+e^{i\alpha}+e^{-i\alpha}}} = \sqrt{\frac{2-e^{i\alpha}-\frac{1}{e^{i\alpha}}}{2+e^{i\alpha}+\frac{1}{e^{i\alpha}}}} = \sqrt{\frac{\dfrac{2e^{i\alpha}-e^{2i\alpha}-1}{e^{i\alpha}}}{\dfrac{2e^{i\alpha}+e^{2i\alpha}+1}{e^{i\alpha}}}} = \sqrt{\frac{-e^{2i\alpha}+2e^{i\alpha}-1}{e^{2i\alpha}+2e^{i\alpha}+1}} =$$

$$= \sqrt{\frac{-(1-e^{i\alpha})^2}{(e^{i\alpha}+1)^2}} = \frac{(1-e^{i\alpha})\cdot i}{e^{i\alpha}+1},$$

por lo que la expresión $\dfrac{1+i\tan(\alpha/2)}{1-i\tan(\alpha/2)}$ resulta:

$$\frac{1+i\tan(\alpha/2)}{1-i\tan(\alpha/2)} = \frac{1+i\cdot\dfrac{(1-e^{i\alpha})\cdot i}{e^{i\alpha}+1}}{1-i\cdot\dfrac{(1-e^{i\alpha})\cdot i}{e^{i\alpha}+1}} = \frac{(e^{i\alpha}+1)-(1-e^{i\alpha})}{(e^{i\alpha}+1)+(1-e^{i\alpha})} = \frac{2e^{i\alpha}}{2} = e^{i\alpha} = \cos\alpha + i\sin\alpha = 1_\alpha$$

esta expresión es muy sencilla y más cómoda para trabajar. Si ahora resolvemos la ecuación:

$$\left(\frac{1+ix}{1-ix}\right)^4 = 1_\alpha \Rightarrow \frac{1+ix}{1-ix} = \sqrt[4]{1_\alpha} \Rightarrow \frac{1+ix}{1-ix} = 1_{\frac{\alpha+2n\pi}{4}},$$

por lo que despejando x, tenemos que:

$$x = \frac{1_{\frac{\alpha+2n\pi}{4}}-1}{\left(1+1_{\frac{\alpha+2n\pi}{4}}\right)\cdot i} = \frac{\left(1_{\frac{\alpha+2n\pi}{4}}-1\right)\cdot i}{-1-1_{\frac{\alpha+2n\pi}{4}}} = \frac{1-1_{\frac{\alpha+2n\pi}{4}}}{1+1_{\frac{\alpha+2n\pi}{4}}}\cdot i$$

Veamos ahora las cuatro soluciones:

$$x_1 = \frac{1-1_{\frac{\alpha+2n\pi}{4}}}{1+1_{\frac{\alpha+2n\pi}{4}}}\cdot i = \frac{1-\cos\alpha/4 - i\sin\alpha/4}{1+\cos\alpha/4+i\sin\alpha/4}\cdot i = \frac{(1-\cos\alpha/4-i\sin\alpha/4)(1+\cos\alpha/4-i\sin\alpha/4)}{(1+\cos\alpha/4+i\sin\alpha/4)(1+\cos\alpha/4-i\sin\alpha/4)}\cdot i =$$

$$= \frac{(1-i\sin\alpha/4)^2-\cos^2\alpha/4}{(1+\cos\alpha/4)^2+\sin^2\alpha/4}\cdot i = \frac{1-2i\sin\alpha/4-\sin^2\alpha/4-\cos^2\alpha/4}{1+2\cos\alpha/4+\cos^2\alpha/4+\sin^2\alpha/4}\cdot i = \frac{-2i\sin\alpha/4}{2+2\cos\alpha/4}\cdot i =$$

$$= \frac{\sin\alpha/4}{1+\cos\alpha/4} = \frac{2\sin\alpha/8\cos\alpha/8}{1+\cos^2\alpha/8-\sin^2\alpha/8} = \frac{2\sin\alpha/8\cos\alpha/8}{\sin^2\alpha/8+\cos^2\alpha/8+\cos^2\alpha/8-\sin^2\alpha/8} = \frac{2\sin\alpha/8\cos\alpha/8}{2\cos^2\alpha/8} =$$

$$= \frac{\sin\alpha/8}{\cos\alpha/8} = \tan\alpha/8$$

$$x_2 = \tan\frac{\alpha+2\pi}{8}; \quad x_3 = \tan\frac{\alpha+4\pi}{8}; x_4 = \tan\frac{\alpha+6\pi}{8}$$

A tener en cuenta que: $\dfrac{\alpha+2n\pi}{8} \neq \dfrac{\pi}{2}+k\pi \Rightarrow \dfrac{\alpha+2n\pi}{8} \neq \dfrac{4\pi+8k\pi}{8} \Rightarrow \boxed{\alpha \neq (2+4k-n)\cdot 2\pi}$