CHAPTER 1 LINEAR RELATIONS

Table of Contents

Chapter 1.1 Reviewing Algebra

Key Concepts in Algebra:

1. **Term:**
   - A term is a number or a product of numbers and variables (pronumerals). Examples include $5x, 7x, 2y$, and 7 (a constant term).

2. **Coefficient:**
   - The coefficient is the number multiplying the variable in a term. For example, in $7 - 3x^2$, the coefficient of x^2 is -3.

3. **Expression:**
   - An expression is a combination of terms that may involve addition, subtraction, multiplication, or division. Examples: $7x, 3x + 2xy, x + 3$.

4. **Equation:**
   - An equation is a mathematical statement that shows the equality of two expressions. Examples: $x = 5, 7x - 1 = 2, x^2 + 2x = -4$.

Evaluating Expressions:

To evaluate an expression, substitute the given values for each variable. Follow the order of operations:

1. Brackets
2. Indices (powers)
3. Multiplication and Division (from left to right)
4. Addition and Subtraction (from left to right)

For example, if $x = -2$ and $y = 4$:

$$3x - 2y = 3(-2) - 2(4) = -6 - 8 = -14$$

Like Terms:

- Like terms have the same variable part. You can add or subtract like terms. For example:
  3x+7x=10x,2a+2a=4a3x + 7x = 10x, \quad 2a + 2a = 4a3x+7x=10x,2a+2a=4a

Multiplication and Division:

- The multiplication (×) and division (÷) symbols are often omitted. For example:

$$7x \times y = 7xy, \qquad \frac{a}{b} = ab$$

Distributive Law:

The distributive law is used to expand expressions within brackets.

- $a(b + c) = ab + ac$
- Example: $2(x + 7) = 2x + 14$

Factorization:

- Factorization involves writing expressions as a product of factors.

- For example:

$$3x - 12 = 3(x - 4)$$
$$9x^2y - 6xy + 3x = 3x(3xy - 2y + 1)$$

General Properties:

1. **Associative Property:**
   - $a \times (b \times c) = (a \times b) \times c$ or $a + (b + c) = (a + b) + c$
2. **Commutative Property:**
   - $ab = ba$ or $a + b = b + a$
   - (Note: $a \div b \neq b \div a$ and $a - b \neq b - a$)
3. **Identity Property:**
   - $a \times 1 = a$ or $a + 0 = a$
4. **Inverse Property:**
   - $a \times \frac{1}{a} = 1$ or $a + (-a) = 0$

These properties help simplify algebraic expressions and solve equations efficiently.

Practice Questions

1. Simplify by collecting like terms.

   a. $5y - 5y$

   b. $2xy + 3xy$

c. $7b - b + 3b$

d. $3st^2 - 4st^2$

e. $4gh + 5 - 2gh$

f. $7xy + 5xy - 3y$

g. $4a + 5b - a + 2b$

h. $3jk - 4j + 5jk - 3j$

i. $2ab^2 + 5a^2b - ab^2 + 5ba^2$

j. $3mn - 7m^2n + 6nm^2 - mn$

k. $4st + 3ts^2 + st - 4s^2t$

l. $7x^3y^4 - 3xy^2 - 4y4x^3 + 5y^2x$

Chapter 1.2 Algebraic Functions

Practice Questions

1. Simplify and write the answer in simplest form.

a. $\frac{12}{10} \div 3\frac{1}{5}$

b. $\left(\frac{3}{7}\right)^2 \times 4\frac{1}{5} \times 3\frac{8}{9}$

2. Simplify by cancelling common factors.

a. $\frac{-14x^2y}{7xy}$

b. $\frac{-36ab^2}{4ab}$

c. $\frac{8xy^3}{-4xy^2}$

d. $\frac{-20s}{45s^2t}$

e. $\frac{-48x^2}{16xy}$

f. $\frac{120ab^2}{140ab}$

3. Simplify by cancelling common factors.

a. $\frac{6x-18}{2}$

b. $\frac{5-15y}{5}$

c. $\frac{9t-27}{-9}$

d. $\frac{44-11x}{-11}$

e. $\frac{a^2-a}{a}$

f. $\frac{7a+14a^2}{21a}$

4. Simplify the following.

a. $\frac{x+4}{10} \times \frac{2}{x}$

b. $\frac{-8a}{7} \times \frac{7}{2a}$

c. $\frac{y-7}{y} \times \frac{5y}{y-7}$

d. $\frac{10a^2}{a+6} \times \frac{a+6}{4a}$

e. $\frac{6-18x}{2} \times \frac{5}{1-3x}$

f. $\frac{b-1}{10} \times \frac{5}{1-b}$

5. Simplify the following.

a. $\frac{x+4}{2} \div \frac{x+4}{6}$

b. $\frac{6x-12}{5} \div \frac{x-2}{3}$

c. $\frac{2}{a-1} \div \frac{3}{2a-2}$

d. $\frac{2}{10x-5} \div \frac{10}{2x-1}$

e. $\frac{2x-6}{5x-20} \div \frac{x-3}{x-4}$

f. $\frac{t-1}{9} \div \frac{1-t}{3}$

6. Simplify the following.

a. $\frac{3}{10} - \frac{3b}{2}$

b. $\frac{2}{5} + \frac{4x}{15}$

c. $\frac{7}{9} - \frac{3}{a}$

d. $\frac{4}{b} - \frac{3}{4}$

e. $\frac{-4}{x} - \frac{2}{3}$

f. $\frac{-9}{2x} - \frac{1}{3}$

7. Simplify the following algebraic expressions.

a. $\frac{x+2}{3} + \frac{x+1}{4}$

b. $\frac{x-3}{4} - \frac{x+2}{2}$

c. $\frac{2x+1}{2} - \frac{x-2}{3}$

d. $\frac{3x+1}{5} + \frac{2x+1}{10}$

e. $\frac{5x+3}{10} + \frac{2x-2}{4}$

f. $\frac{3-x}{14} - \frac{x-1}{7}$

8. Simplify the following algebraic expressions.

a. $\frac{4}{x-7}+\frac{3}{x+2}$

b. $\frac{1}{x-3}+\frac{2}{x+5}$

c. $\frac{6}{2x-1}-\frac{3}{x-4}$

d. $\frac{4}{x-5}+\frac{2}{3x-4}$

e. $\frac{2}{x-3} - \frac{3}{3x+4}$

f. $\frac{8}{3x-2} - \frac{3}{1-x}$

9. Now simplify these expressions.

a. $\frac{a+1}{a} - \frac{4}{a^2}$

b. $\frac{7}{2x^2} + \frac{3}{4x}$

Chapter 1.3 Solving Linear Equations

Practice Questions

1. Solve the following equations and check your solution by substitution.

a. $-x + 4 = 7$

b. $-x - 5 = -9$

c. $3x - 3 = -4$

d. $6x + 5 = -6$

e. $-4x - 9 = 9$

f. $-3x - 7 = -3$

g. $6 - 5x = 16$

h. $4 - 9x = -7$

2. Solve the following equations and check your solution by substitution.

a. $2(x - 3) = 12$

b. $3(1 - 2x) = 8$

c. $2(4x - 5) = -7x$

d. $2(2x - 3) + 3(4x - 1) = 23$

e. $5(2x + 1) - 3(x - 3) = 63$

f. $4(2x + 5) = 3(x + 15)$

g. $3(4x - 1) = 7(2x - 7)$

3. Solve the following equations and check your solution by substitution.

a. $\frac{x+4}{3} = -6$

b. $\frac{3x-2}{4} = 4$

c. $\frac{2x}{3} - 2 = -8$

d. $5 - \frac{4x}{7} = 9$

e. $4 + \frac{x-5}{2} = -3$

4. Solve the following equations that involve algebraic fractions.

a. $\frac{5x-4}{4} = \frac{x-5}{5}$

b. $\frac{3x-5}{4} = \frac{2x-8}{3}$

c. $\frac{6-2x}{3} = \frac{5x-1}{4}$

d. $\frac{10-x}{2} = \frac{x+1}{3}$

e. $\frac{-2(x-1)}{3} = \frac{2-x}{4}$

f. $\frac{3(6-x)}{2} = \frac{-2(x+1)}{5}$

5. A service technician charges $\$30$ up front and $\$46$ for each hour he works.

   a. What will a 4 hour job cost?

   b. If the technician works on a job for 2 days and averages 6 hours per day, what will be the overall cost?

c. Find how many hours he worked if the cost is:

i $\$76$

ii $\$513$

iii $\$1000$ (round to the nearest half hour)

6. The capacity of a petrol tank is 80 litres. If it initially contains 5 litres and a petrol pump fills it at 3 litres per 10 seconds, find:

   a. the amount of fuel in the tank after 2 minutes

   b. how long it will take to fill the tank to 32 litres

   c. how long it will take to fill the tank.

Chapter 1.4 Inequalities

Key Concepts on Inequalities:

1. **Inequality Signs:**
   - $\boldsymbol{x > a}$**:** This means x is greater than a.
   - $\boldsymbol{x \geq a}$**:** This means x is greater than or equal to a.
   - $\boldsymbol{x < a}$**:** This means x is less than a.
   - $\boldsymbol{x \leq a}$**:** This means x is less than or equal to a.

2. For example:
   - $a < x \leq b$ means x is between a and b, including b but not a.

3. **Solving Linear Inequalities:** Solving linear inequalities follows the same rules as solving linear equations, except with these key differences:
   - **Reversing the inequality sign when multiplying or dividing by a negative number:**
     - For example, if $-5 < -3$, then multiplying both sides by -1 will reverse the inequality sign to give $5 > 3$.
     - If $-2x < 4$, dividing both sides by -2 will reverse the inequality sign to give $x > -2$.

   - **Reversing the inequality sign when switching the sides:**
     - For example, if $2 \geq x$, switching the sides results in $x \leq 2$.

4. **Graphical Representation:** Inequalities can also be represented on a number line:
    - For $x > a$, represent with an open circle at a and an arrow pointing to the right.
    - For $x \geq a$, represent with a closed circle at a and an arrow pointing to the right.
    - For $x < a$, represent with an open circle at a and an arrow pointing to the left.
    - For $x \leq a$, represent with a closed circle at a and an arrow pointing to the left.

These rules are essential for solving and graphing inequalities accurately.

Practice Questions

1. Solve the following inequalities and graph their solutions on a number line.

a. $4x - 7 \geq 9$

b. $\frac{x}{5} \leq 2$

c. $\frac{2x+3}{5} > 3$

d. $\frac{x}{3} + 4 \leq 6$

e. $3(3x - 1) \leq 7$

f. $2(4x + 4) < 5$

2. Solve the following inequalities. Remember: if you multiply or divide by a negative number you must reverse the inequality sign.

a. $-5x \; - \; 7 \geq 18$

b. $\frac{3-x}{2} \geq 5$

c. $3 - \frac{x}{2} \leq 8$

d. $-\frac{x}{3} - 5 > 2$

3. Solve the following inequalities.

a. $5x + 2 \geq 8x - 4$

b. $7 - x > 2 + x$

c. $7(1 - x) \geq 3(2 + 3x)$

d. $-(2 - 3x) < 5(4 - x)$

4. Solve these inequalities by firstly multiplying by the LCD.

a. $\frac{2x-3}{2} \geq \frac{x+1}{3}$

b. $\frac{3-2x}{5} \leq \frac{5x-1}{2}$

c. $\frac{5x}{3} \geq \frac{3(3-x)}{4}$

d. $\frac{2(4-3x)}{5} > \frac{2(1+x)}{3}$

Chapter 1.5 Graphing Straight Lines

Key Concepts on Gradients and Intercepts:

1. **Gradient (m):**
   - The gradient, denoted as mmm, describes the slope or steepness of a line.
   - It is calculated as the ratio of the **rise** (change in y) over the **run** (change in x):

$$m = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1}$$

   - The gradient is also referred to as the **rate of change of y** with respect to x.
   - It indicates how much y changes for every 1 unit change in x.

2. **Intercepts:**
   - **Y-Intercept:** The point where the line crosses the y-axis, occurring when $x = 0$.
   - **X-Intercept:** The point where the line crosses the x-axis, occurring when $y = 0$.

3. **Linear Equation Form:**
   - The **gradient-intercept form** of a straight line is given by:

$$y = mx + c$$

     where:
     - m is the gradient,
     - c is the **y-intercept** (the value of y when $x = 0$).

4. **Special Lines:**
   - **Horizontal Lines:** These have the equation $y = c$, where c is a constant. The gradient is 0.
   - **Vertical Lines:** These have the equation $x = b$, where b is a constant. The gradient is undefined.

- **Lines Passing Through the Origin:** These have the equation $y = mx$, where mmm is the gradient, and the line passes through the point $(0,0)$.

5. **Deciding if a Point is on a Line:**
   - To determine if a point (x, y) is on a given line, substitute the values of x and y into the equation of the line. If the equation holds true, then the point lies on the line.

6. **Example:**
   - **For the line $\boldsymbol{y = -3x + 1}$:**
     - Given point: $(-2,7)$
     - Substituting $x = -2$ and $y = 7$ into the equation:

$$7 = -3(-2) + 1 \implies 7 = 7(True)$$

     - Therefore, $(-2,7)$ is on the line.

   - **For the line $\boldsymbol{2x + 2y = 1}$:**
     - Given point: $(-2,7)$
     - Substituting $x = -2$ and $y = 7$ into the equation:

$$2(-2) + 2(7) = 1 \implies -4 + 14 = 1 \implies 10 = 1(False)$$

     - Therefore, $(-2,7)$ is not on the line.

Understanding gradients and intercepts is fundamental to interpreting and graphing linear equations.

Practice Questions

1. Find the gradient and y-intercept for these linear relations and sketch a graph.

a. $y = -2x - 1$

b. $y = -x + 2$

c. $y = \frac{4}{3}x - 2$

d. $y = -\frac{7}{2}x + 6$

e. $y = 3 + \frac{2}{3}x$

f. $y = 0.4 - 0.2x$

2. Find the gradient and y-intercept for these linear relations and sketch each graph.

a. $x - y = 7$

b. $3x - 3y = 6$

c. $-y - 4x = 8$

d. $2y + x = \frac{1}{2}$

3. Sketch the following by finding x- and y-intercepts.

a. $y = 4x + 10$

b. $y = 3x - 4$

c. $3x + 2y = 12$

d. $2x + 5y = 10$

e. $3x + 4y = 7$

f. $5y - 2x = 12$

4. Sketch these special lines.

a. $x = 2$

b. $x = -\frac{5}{2}$

c. $y = 4x$

d. $y = -3x$

e. $x + y = 0$

f. $4 - y = 0$

5. Sam is earning some money picking apples. She gets paid $\$10$ plus $\$2$ per kilogram of apples that she picks. If Sam earns $\$C$ for n kg of apples picked, complete the following.

   a. Write a rule for C in terms of n.

   b. Sketch a graph for $0 \leq n \leq 10$, labelling the endpoints.

c. Use your rule to find:

i. the amount she earned after picking 9 kg of apples

ii. the number of kilograms of apples she picked if she earned $57.

6. A 90 L tank full of water begins to leak at a rate of 1.5 litres per hour. If V litres is the volume of water in the tank after t hours, complete the following.

   a. Write a rule for V in terms of t.

   b. Sketch a graph for $0 \leq t \leq 60$, labelling the endpoints.

c. Use your rule to find:

i. the volume of water after 5 hours

ii. the time taken to completely empty the tank.

7. It costs Jack $\$1600$ to maintain and drive his car for $32\ 000\ \text{km}$.

   a. Write the rate of cost in $ per km.

   b. Write a formula for the cost $\$C$ of driving Jack's car for k kilometres.

   c. If Jack also pays a total of $\$1200$ for registration and insurance, write the new formula for the cost to Jack of owning and driving his car for k kilometres.

8. $D = 25t + 30$ is an equation for calculating the distance D km from home that a cyclist has travelled after t hours.

a. What is the gradient of the graph of the given equation?

b. What could the 30 represent?

c. If a graph of D against t was drawn, what would be the intercept on the D-axis?

Chapter 1.6 Finding a Rule for a Linear Graph

Key Points on Equations of Lines:

1. **Horizontal Lines:**
   - The equation of a horizontal line is $y = c$, where c is the y-intercept.
   - The gradient of a horizontal line is 0 because there is no change in the y-values.

2. **Vertical Lines:**
   - The equation of a vertical line is $x = k$, where k is the x-intercept.
   - The gradient of a vertical line is undefined because there is no change in the x-values.

3. **Equation of a Line with Given Gradient and $\boldsymbol{Y}$-Intercept:**
   - If the gradient (m) and the y-intercept (c) are known, the equation of the line is written as:

$$y = mx + c$$

   - Here, m is the gradient and c is the y-intercept.

4. **Equation of a Line Given Two Points:**
   - To find the equation of a line from two points (x_1, y_1) and (x_2, y_2), follow these steps:

     1. **Find the Gradient ($\boldsymbol{m}$):** The gradient m is given by the formula:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

2. **Find the Y-Intercept (c):**
   - Substitute the gradient mmm and one of the points (e.g., (x_1, y_1)) into the equation $y = mx + c$ to solve for c.

     $$y_1 = mx_1 + c$$

     Solve for c.

3. **Write the Equation:** The equation of the line is then:

$$y = mx + c$$

Alternatively, you can use the **Point-Slope Formula** to write the equation of the line:

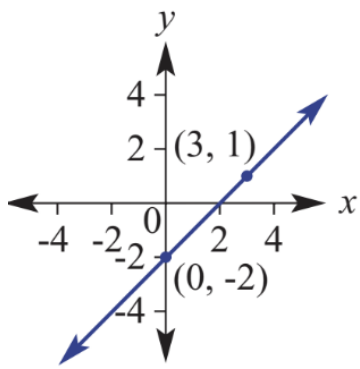
$$y - y_1 = m(x - x_1)$$

- Here, m is the gradient, and (x_1, y_1) is a point on the line. You can use either of the given points for (x_1, y_1).

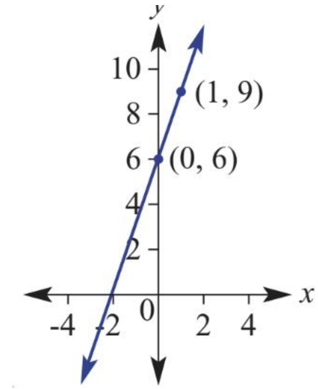
Practice Questions

1. Find the equation of the straight lines with the given y-intercepts.

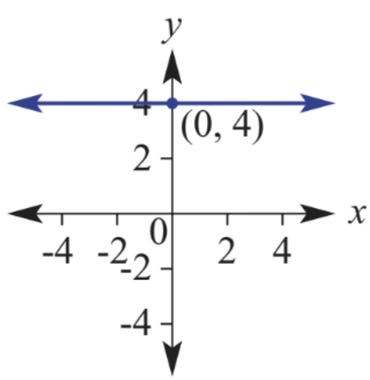
a.



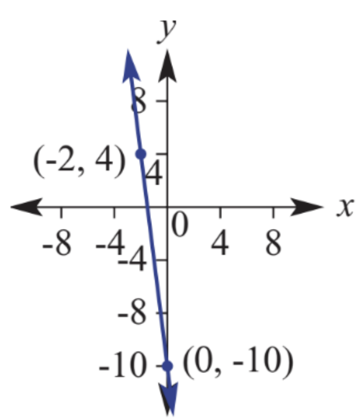
b.



c.

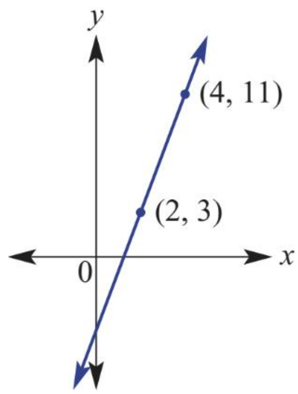


d.

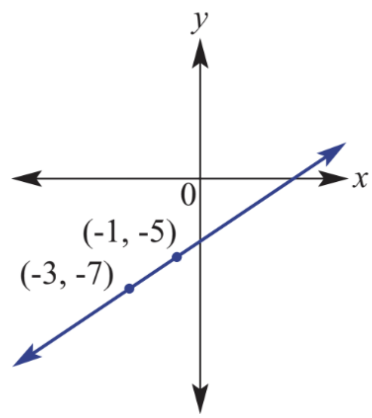


2. Find the equation of the straight lines with the given points.

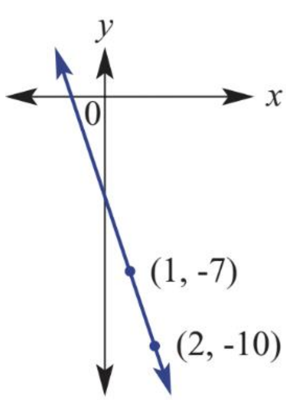
a.



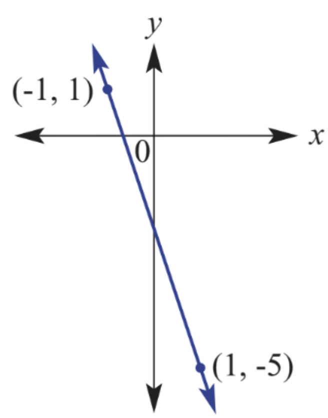
b.



c.



d.



3. Randy invests some money in a simple savings fund and the amount increases at a constant rate over time. He hopes to buy a boat when the investment amount reaches $\$20\ 000$. After 3 years the amount is $\$16\ 500$ and after 6 years the amount is $\$18\ 000$.

   a. Find a rule linking the investment amount ($\$A$) and time ($t$ years).

   b. How much did Randy invest initially (i.e. when $t = 0$)?

c. How long does Randy have to wait before he buys his boat?

d. What would be the value of the investment after $12\frac{1}{2}$ years?

4. The cost of hiring a painter involves an up-front fee plus an hourly rate. Three hours of hire costs $\$50$ and 7 hours costs $\$90$.

   a. Sketch a graph of cost $\$C$ for t hours of hire using the information given above.

   b. Find a rule linking the cost $\$C$ in terms of t hours.

c. i State the cost per hour.

ii State the up-front fee.

5. a. The following information applies to the filling of a flask with water at a constant rate. In each case, find a rule for the volume V litres in terms of t minutes.

   i. Initially (at $t = 0$) the flask is empty ($V = 0$) and after 1 minute it contains 4 litres of water.

   ii. Initially (at $t = 0$) the flask is empty ($V = 0$) and after 3 minutes it contains 9 litres of water.

iii. After 1 and 2 minutes, the flask has 2 and 3 litres of water respectively.

iv. After 1 and 2 minutes the flask has 3.5 and 5 litres of water respectively.

b. For parts iii and iv above, find how much water was in the flask initially.

c. Write your own information that would give the rule $V = -t + b$.

Chapter 1.7 Length and Midpoint of a Line Segment

Key Formulas for Line Segments:

1. **Distance Between Two Points:**

   The length of a line segment (or distance) between two points (x_1, y_1) and (x_2, y_2) is given by the **distance formula**:

   $$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

   This formula uses the Pythagorean theorem to find the distance between the points by considering the horizontal and vertical distances.

2. **Midpoint of a Line Segment:**

   The midpoint M of a line segment between the points (x_1, y_1) and (x_2, y_2) is the point that is equidistant from both endpoints. The coordinates of the midpoint are given by:

   $$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

   This formula finds the average (mean) of the x-coordinates and y-coordinates of the two points.

Example:

For the points $(1,2)$ and $(4,6)$, let's calculate the distance and midpoint.

1. **Distance:**

$$d = \sqrt{(4-1)^2 + (6-2)^2} = \sqrt{3^2 + 4^2} = \sqrt{9+16} = \sqrt{25} = 5$$

2. **Midpoint:**

$$M = \left(\frac{1+4}{2}, \frac{2+6}{2}\right) = \left(\frac{5}{2}, \frac{8}{2}\right) = (2.5, 4)$$

So, the distance between the points is 5 units, and the midpoint is $(2.5, 4)$.

Practice Questions

1. Find the exact distance between these pairs of points.

   a. $(-1, 4)$ and $(0, -2)$

   b. $(-8, 9)$ and $(1, -3)$

c. $(-10, 11)$ and $(-4, 10)$

2. Find the midpoint of the line segment joining the given points in Question **4**.

3. A line segment has endpoints $(-2, 3)$ and $(1, -1)$.

a. Find the midpoint using $(x_1, y_1) = (-2, 3)$ and $(x_2, y_2) = (1, -1)$.

b. Find the midpoint using $(x_1, y_1) = (1, -1)$ and $(x_2, y_2) = (-2, 3)$.

c. Give a reason why the answers to parts **a** and **b** are the same.

d. Find the length of the segment using $(x_1, y_1) = (-2, 3)$ and $(x_2, y_2) = (1, -1)$.

e. Find the length of the segment using $(x_1, y_1) = (1, -1)$ and $(x_2, y_2) = (-2, 3)$.

f. What do you notice about your answers to parts **d** and **e**? Give an explanation for this.

Chapter 1.8 Perpendicular and Parallel Lines

Key Concepts for Parallel and Perpendicular Lines:

1. **Parallel Lines:**
   - Two lines are parallel if they have the **same gradient**.
   - For example, consider the lines $y = 3x - 1$ and $y = 3x + 8$. Both lines have the same gradient of 3, meaning they are parallel to each other.

   General Rule: If two lines are parallel, their gradients are equal:

   $$m_1 = m_2$$

   where m_1 and m_2 are the gradients of the two lines.

2. **Perpendicular Lines:**
   - Two lines are **perpendicular** if the product of their gradients is equal to -1.
   - This means the gradients m_1 and m_2 of two perpendicular lines satisfy the equation:

   $$m_1 \times m_2 = -1$$

   or equivalently:

   $$m_2 = -\frac{1}{m_1}$$

   In other words, the gradient of one line is the **negative reciprocal** of the gradient of the other line.

   Example: If $m_1 = 2$, then $m_2 = -\frac{1}{2}$.

3. **Finding the Equations of Parallel or Perpendicular Lines:**
   - **Step 1:** Find the gradient m of the line using the given information (e.g., through points or slope).
   - **Step 2:** Use the point-slope form $y - y_1 = m(x - x_1)$ or the gradient-intercept form $y = mx + c$ to find the equation of the line by substituting a known point (x_1, y_1) and solving for c.

Example 1: Parallel Lines

Given the line $y = 3x - 1$, find the equation of a parallel line passing through the point $(2,5)$.

- The gradient of the given line is $m = 3$.
- The parallel line will have the same gradient: $m = 3$.
- Use the point $(2,5)$ and the gradient-intercept form $y = mx + c$:

$$5 = 3(2) + c$$

$$5 = 6 + c$$

$$c = -1$$

So, the equation of the parallel line is:

$$y = 3x - 1$$

Example 2: Perpendicular Lines

Given the line $y = 2x + 4$, find the equation of a perpendicular line passing through the point $(1,3)$.

- The gradient of the given line is $m_1 = 2$.
- The gradient of the perpendicular line is the negative reciprocal:

$$m_2 = -\frac{1}{2}$$

- Use the point $(1,3)$ and the gradient-intercept form $y = mx + c$:

$$3 = -\frac{1}{2}(1) + c$$

$$3 = -\frac{1}{2}(1) + c$$

$$c = 3 + \frac{1}{2} = \frac{7}{2}$$

So, the equation of the perpendicular line is:

$$y = -\frac{1}{2}x + \frac{7}{2}$$

These steps can be applied to both parallel and perpendicular lines to determine their equations.

Practice Questions

1. Decide if the line graphs of each pair of rules will be parallel, perpendicular or neither.

   a. $y = \frac{1}{2}x - 6$ and $y = \frac{1}{2}x - 4$

   b. $y = -4x - 2$ and $y = x - 7$

c. $y = -8x + 4$ and $y = \frac{1}{8}x - 2$

d. $x - y = 4$ and $y = x + \frac{1}{2}$

e. $3x - y = 2$ and $x + 3y = 5$

2. Find the equation of the line that is:

a. parallel to $y = -4x - 1$ and passes through $(-1, 3)$

b. parallel to $y = \frac{2}{3}x + 1$ and passes through $(3, -4)$

c. perpendicular to $y = -4x + 1$ and passes through $(-4, -3)$

d. perpendicular to $y = \frac{2}{3}x - 4$ and passes through $(4, -1)$

3. Find the equation of the line that is parallel to these equations and passes through the given points.

a. $y = \frac{3-5x}{7}$, $(1, 7)$

b. $7x - y = -1, \ (-3, -1)$

4. A line with equation $3x - 2y = 12$ intersects a second line at the point where $x = 2$. If the second line is perpendicular to the first line, find where the second line cuts the x-axis.

5. Find the value of a if $y = \frac{2a+1}{3}x + c$ is parallel to $y = -x - 3$.

6. Find the value of a if $y = \left(\frac{1-a}{2}\right)x + c$ is perpendicular to $y = \frac{1}{2}x - \frac{3}{5}$.

Chapter 1.9 Simultaneous Equations - Substitution

Solving Simultaneous Equations

1. Types of Solutions

- **Unique Solution:** If two lines are **not parallel**, they intersect at a **single point**.
- **No Solution:** If the lines are **parallel** (same gradient but different intercepts), they **never intersect**.
- **Infinite Solutions:** If the two equations represent the **same line**, they have **infinitely many** points in common.

2. Substitution Method

Use this method when **one equation is already solved for one variable** (e.g., $y = 3x + 2$).

Example:

Solve the system of equations:

$$x + y = 8 \ \ \text{(Equation 1)}$$
$$y = 3x + 4 \ \ \text{(Equation 2)}$$

Step 1: Substitute Equation (2) into Equation (1):

$$x + (3x + 4) = 8$$

Step 2: Solve for x:

$$4x + 4 = 8$$
$$4x = 4$$
$$x = 1$$

Step 3: Substitute $x = 1$ back into Equation (2):

$$y = 3(1) + 4 = 7$$

Solution: $(1,7)$

3. Elimination Method

Use this method when **both equations are in standard form** (e.g., $ax + by = c$).

Example:

Solve:

$$2x + 3y = 12 \ \ \text{(Equation 1)}$$
$$4x - y = 5 \ \ \text{(Equation 2)}$$

Step 1: Multiply Equation (2) by **3** to align y-coefficients:

$$2x + 3y = 12$$
$$12x - 3y = 15$$

Step 2: Add both equations:

$$(2x + 3y) + (12x - 3y) = 12 + 15$$
$$14x = 27$$
$$x = \frac{27}{14}$$

Step 3: Substitute $x = \frac{27}{14}$ into Equation (2):

$$4\left(\frac{27}{14}\right) - y = 5$$
$$\frac{108}{14} - y = 5$$
$$y = \frac{108}{14} - \frac{70}{14} = \frac{38}{14} = \frac{19}{7}$$

Solution: $\left(\frac{27}{14}, \frac{19}{7}\right)$

Practice Questions

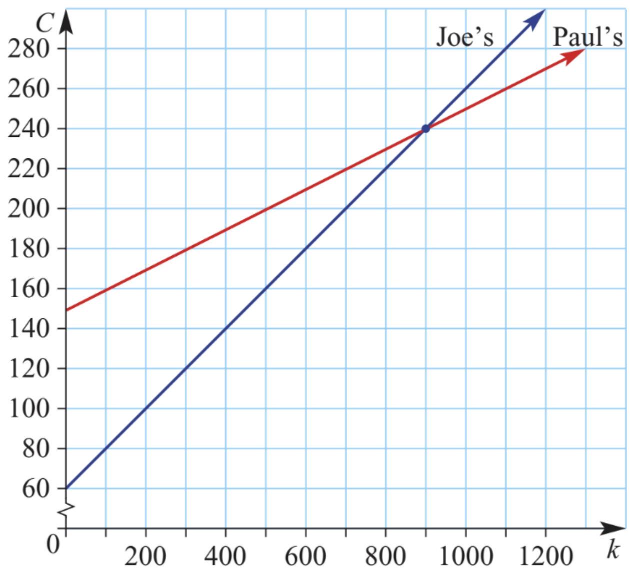
1. Determine the point of intersection of each pair of equations by plotting accurate graphs.

   a. $x = 2$
   $y = 4$

   b. $y = 2x - 6$
   $3x - y = 7$

   c. $y = -2x + 3$
   $y = 3x + 4$

2. This graph represents the rental cost $\$C$ after k kilometres of a new car from two car rental firms called Paul's Motor Mart and Joe's Car Rental.



a. i. Determine the initial rental cost from each company.

ii. Find the cost per kilometre when renting from each company.

iii. Find the linear equations for the total rental cost from each company.

iv. Determine the number of kilometres for which the cost is the same from both rental firms.

b. If you had to travel 300 km, which company would you choose?

c. If you had $\$260$ to spend on travel, which firm would give you the most kilometres?

3. Solve the following pairs of simultaneous equations by using the method of substitution.

   a. $y = -2x - 3$ and $y = -x - 4$

   b. $x = -7y - 1$ and $x = -y + 11$

c. $y = 5 - 2x$ and $y = \frac{3}{2}x - 2$

d. $y = 8x - 5$ and $y = \frac{5x+13}{6}$

4. Solve the following pairs of simultaneous equations by using the method of substitution.

a. $y = x + 3$ and $6x + y = 17$

b. $y = x - 1$ and $3x + 2y = 8$

c. $y = x$ and $7x + 3y = 10$

d. $x = 3y - 2$ and $7y - 2x = 8$

e. $x = 4y + 1$ and $2y - 3x = -23$

5. The value of two cars is depreciating (decreasing) at a constant rate according to the information in this table.

Car	Initial value	Annual depreciation
Luxury sports coupe	\$62 000	\$5000
Family sedan	\$40 000	\$3000

a. Write rules for the value $\$V$ after t years for:
i. the luxury sports coupe ii the family sedan

b. Solve your two simultaneous equations from part **a**.

c. i. State the time taken for the cars to have the same value.

ii. State the value of the cars when they have the same value.

6. For what value of k will these pairs of simultaneous equations have no solution?

a. $y = -4x - 7$ and $y = kx + 2$

b. $y = kx + 4$ and $3x - 2y = 5$

c. $kx - 3y = k$ and $y = 4x + 1$

1. Solve these simple equations for x and y. Your solution should contain the pronumeral k.

   a. $x - y = k$ and $y = -x$

   b. $y - 4x = 2k$ and $x = y + 1$

Chapter 1.10 Simultaneous Equations - Elimination

Practice Questions

1. Solve the following pairs of simultaneous equations using the elimination method.

   a. $x + y = 5$ and $3x - y = 3$

   b. $x - y = 0$ and $4x + y = 10$

c. $x + 3y = 5$ and $4x + 3y = 11$

d. $4x + y = 10$ and $4x + 4y = 16$

e. $3x + 2y = 8$ and $3x - y = 5$

f. $-2x + 3y = 8$ and $-4x - 3y = -2$

2. Solve the following pairs of simultaneous equations using the elimination method.

a. $2x + y = 10$ and $3x - 2y = 8$

b. $3x - 4y = 24$ and $x - 2y = 10$

c. $7x - 2y = -\frac{5}{2}$ and $3x + y = -2$

3. Solve the following pairs of simultaneous equations using the elimination method.

a. $3x + 2y = 5$ and $2x + 3y = 5$

b. $2x + 3y = 10$ and $3x - 4y = -2$

c. $-7x + 3y = 22$ and $3x - 6y = -11$

Chapter 1.11 Further Applications of Simultaneous Equations

Practice Questions

1. Let x and y be two numbers that satisfy the following statements. Set up two linear equations according to the information and then solve them simultaneously to determine the numbers in each case.

   a. They sum to 16 but their difference is 2.

   b. They sum to 30 but their difference is 10.

c. They sum to 7 and twice the larger number plus the smaller number is 12.

d. The sum of twice the first plus three times the second is 11 and the difference between four times the first and three times the second is 13.

2. The sum of the ages of two children is 24 and the difference between their ages is 8. If Nikki is the older sister of Travis, determine their ages by setting up and solving a pair of simultaneous equations.

3. It costs $\$3$ for children and $\$7$ for adults to attend a basketball game. If 5000 people attended the game and the total takings at the door was $\$25\ 000$, determine the number of children and the number of adults that attended the game.

4. A paddock contains both ducks and sheep. There are a total of 42 heads and 96 feet in the paddock. How many ducks and how many sheep are in the paddock?

5. Connie the fruiterer sells two fruit packs.
   Pack 1: 10 apples and 5 mangoes ($\$12$)
   Pack 2: 15 apples and 4 mangoes ($\$14.15$)
   Determine the cost of 1 apple and 5 mangoes.

6. Five years ago I was 5 times older than my son. In 8 years' time I will be 3 times older than my son. How old am I today?

7. Two ancient armies are 1 km apart and begin walking toward each other. The Vikons walk at a pace of $3\ \mathrm{km/h}$ and the Mohicas walk at a pace of $4\ \mathrm{km/h}$. How long will they walk for before the battle begins?

Chapter 1.12 Half Planes

Graphing Linear Inequalities in Two Variables

A **linear inequality** in two variables (e.g., $y > 2x + 3$) represents a **region** of the coordinate plane, not just a line. The boundary line divides the plane into two **half-planes**, and shading shows the solution region.

1. Graphing Steps

Step 1: Convert to Equation Form

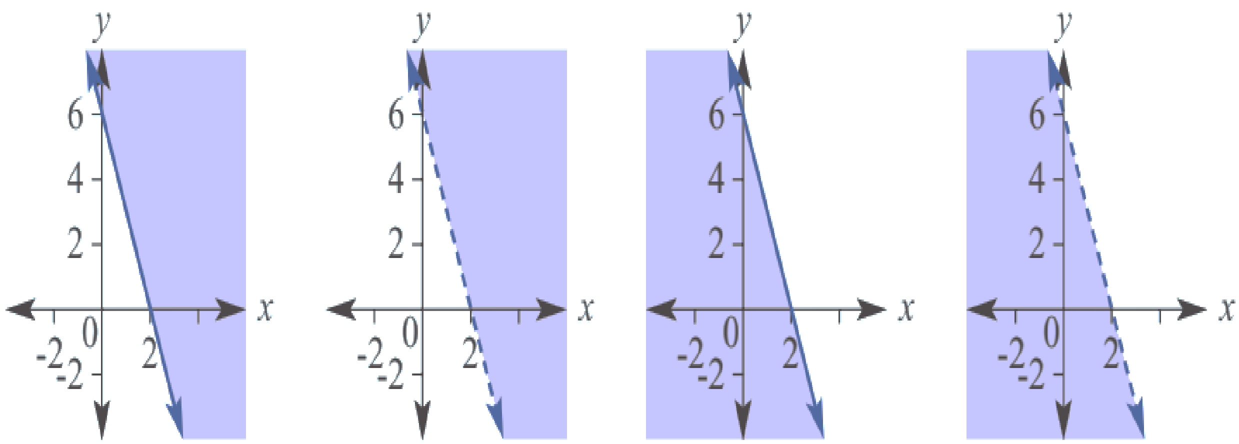
Rewrite the inequality as an equation (replace $<, \leq, >, \geq$ with $=$).

Step 2: Draw the Boundary Line

- Solid Line ($\leq$ or $\geq$) → The boundary is included in the solution.
- Dashed Line ($<$ or $>$) → The boundary is not included.

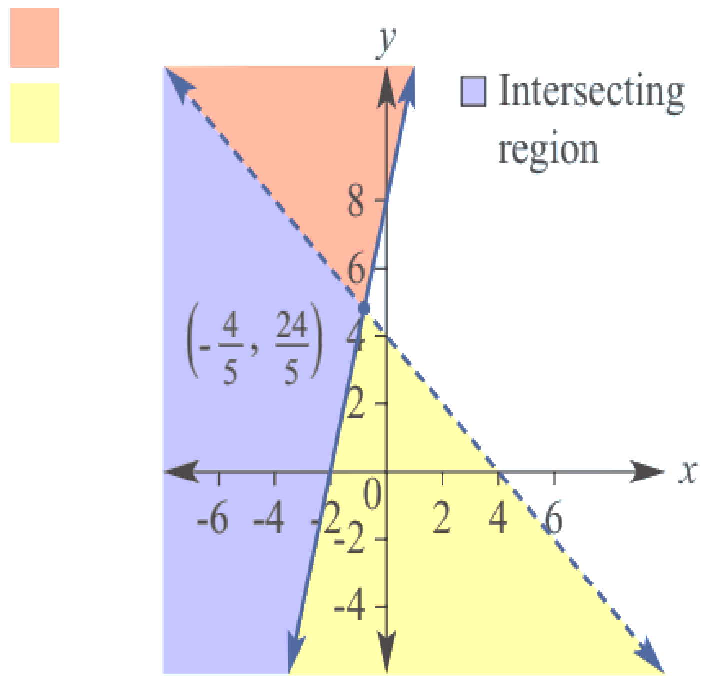
Step 3: Shade the Correct Region

- $y \geq mx + c$ → Shade above the line.
- $y > mx + c$ → Shade above the dashed line.
- $y \leq mx + c$ → Shade below the line.
- $y < mx + c$ → Shade below the dashed line.



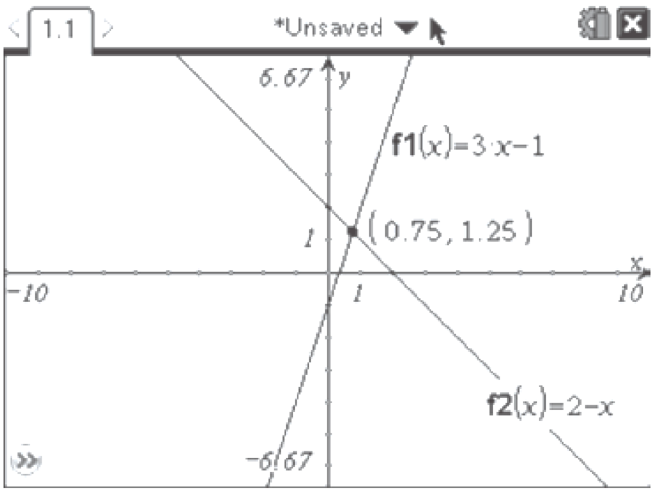
$y \geq -3x + 6$ $y > -3x + 6$ $y \leq -3x + 6$ $y < -3x + 6$

For example, $y \geq 4x + 8$
$y < -x + 4$

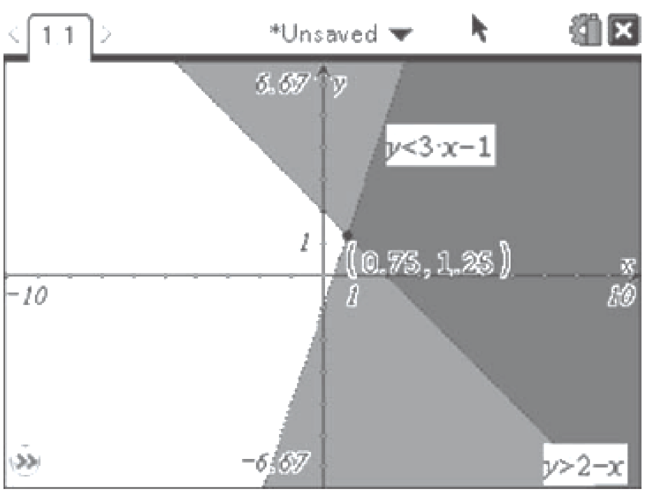


Using the TI-Nspire:

1 In a graphs and geometry page enter the rules $f1(x) = 3x - 1$ and $f2(x) = 2 - x$. Select **menu**, **Analyze Graph**, **Intersection**. Then select a lower and upper bound containing the intersection point and press **enter**.

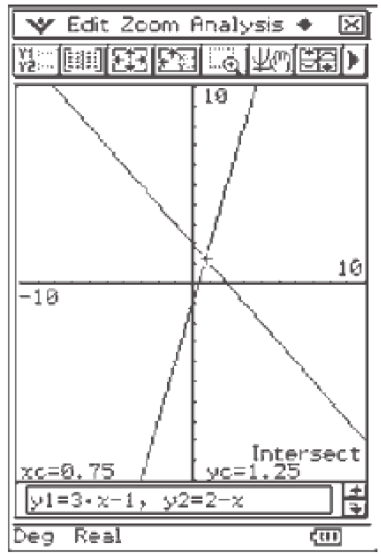


2 Show the entry line and edit $f1(x)$ to $y < 3x - 1$ and $f2(x)$ to $y > 2 - x$.

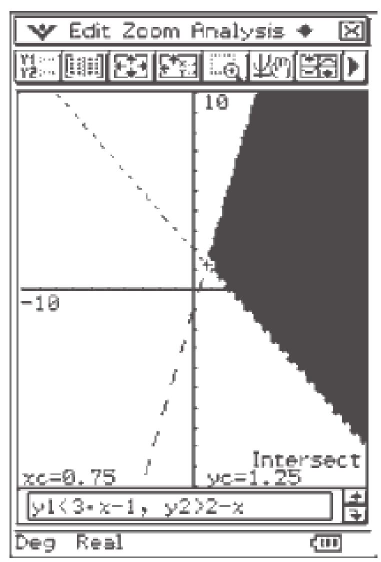


Using the ClassPad:

1 In the **Graph&Table** application enter the rules $y1 = 3x - 1$ and $y2 = 2 - x$ then tap . Tap **Analysis**, **G-Solve**, **Intersect**.



2 Tap and clear all functions. With the cursor in y1 tap , select , enter the rule $3x - 1$ and press **EXE**. With the cursor in y2 tap , select , enter the rule $2 - x$ and press **EXE**. Tap .



Practice Questions

1. Sketch the half planes for the following linear inequalities.

   a. $y > 2x - 8$

   b. $y < 4x$

   c. $x < -2$

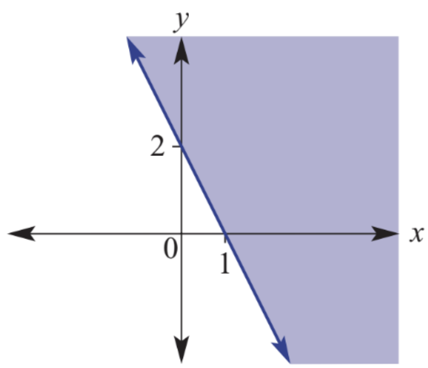
2. Sketch the half planes for the following linear inequalities.

a. $2x - 3y > 18$

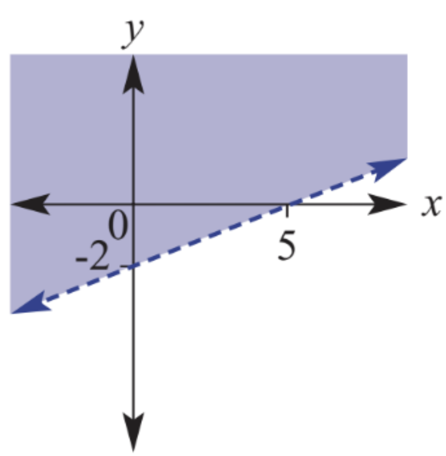
b. $4x + 9y < -36$

3. Write down the inequalities that give these half planes.

a.



b.



4. Sketch both inequalities on the same set of axes, shade the region of intersection and find the point of intersection of the two lines.

a. $3x - 5y \leq 15$
$y - 3x > -3$

b. $2y \geq 5 + x$
$y < 6 - 3x$

5. Sketch the following systems of inequalities on the same axes. Show the intersecting region and label the points of intersection. The result should be a triangle in each case.

a. $x \geq 0$
$5x + 2y \leq 30$
$4y - x \geq 16$

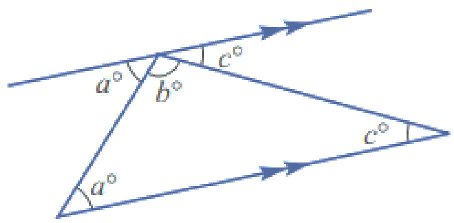
b. $x + y \leq 9$
$2y - x \geq 6$
$3x + y \geq -2$

CHAPTER 2 GEOMETRY

Table of Contents

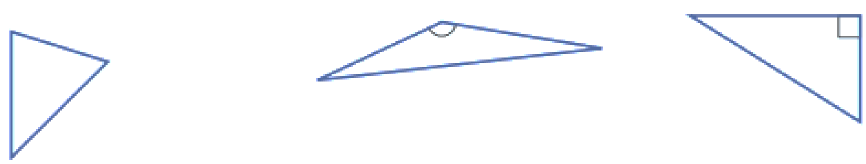
Chapter 2.1 Geometry Review

- Angles at a point
  - **Complementary** (sum to 90°)
  - **Supplementary** (sum to 180°)
  - **Revolution** (360°)
  - **Vertically opposite angles** (equal)
- **Triangles**
  - Angle sum is 180°.
    To prove this, draw a line parallel to a base then mark the alternate angles in parallel lines. Note that angles on a straight line are supplementary.



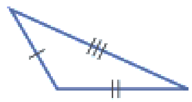
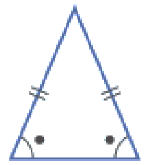
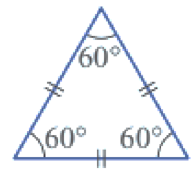
$a + b + c = 180$

  - Triangles classified by angles.
    Acute: all angles acute   Obtuse: one angle obtuse   Right: one right angle



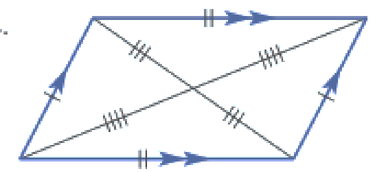
- Triangles classified by side lengths.

Scalene	Isosceles	Equilateral
(3 different sides)	(2 equal sides)	(3 equal sides)

Quadrilaterals

- **Parallelograms** are quadrilaterals with two pairs of parallel sides.
- **Rectangles** are parallelograms with all angles 90°.
- **Squares** are rectangles with equal length sides.
- **Rhombuses** are parallelograms with equal length sides.
- **Kites** are quadrilaterals with two pairs of equal adjacent sides.
- **Trapeziums** are quadrilaterals with one pair of parallel sides.


Parallelogram

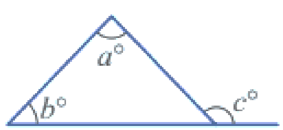
- **Polygons** have an angle sum given by $S = 180(n - 2)$ where n is the number of sides.
  - **Regular polygons** have equal sides and angles.

    A single interior angle $= \dfrac{180(n-2)}{n}$

n	Name
3	triangle
4	quadrilateral
5	pentagon
6	hexagon
7	heptagon
8	octagon
9	nonagon
10	decagon

- An **exterior angle** is supplementary to an interior angle.
  - For a triangle, the **exterior angle theorem** states that the exterior angle is equal to the sum of the two opposite interior angles.

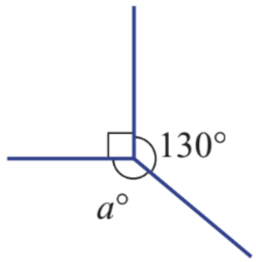
$c = a + b$



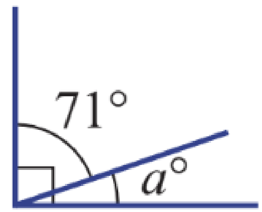
Practice Questions

1. Find the values of the pronumerals giving reasons.

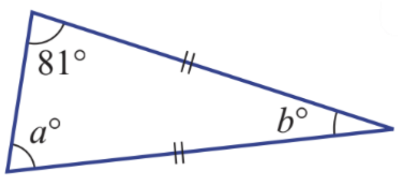
a.



b.

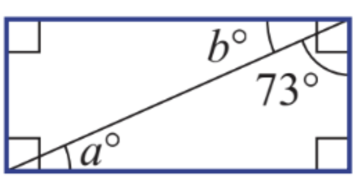


c.

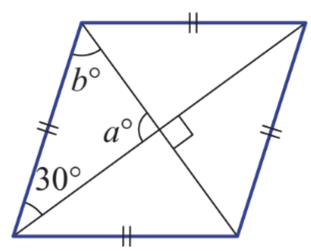


d.

e.

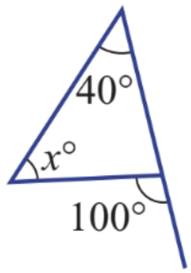


f.

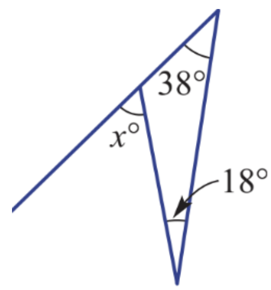


2. Using the exterior angle theorem, find the value of the pronumeral.

a.

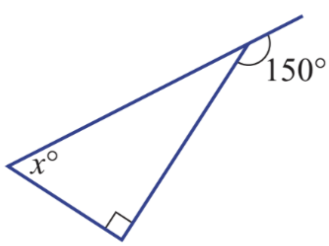


b.

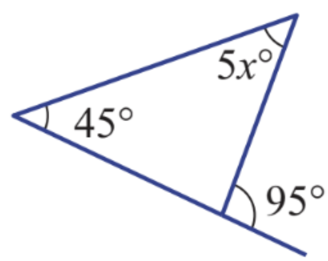


3. Find the value of the pronumeral, giving reasons.

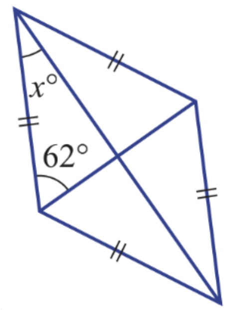
a.



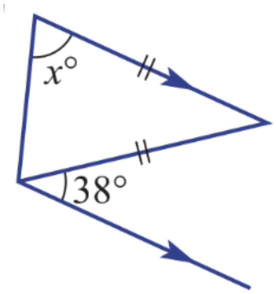
b.



c.

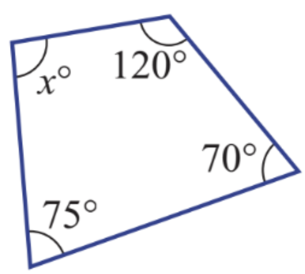


d.

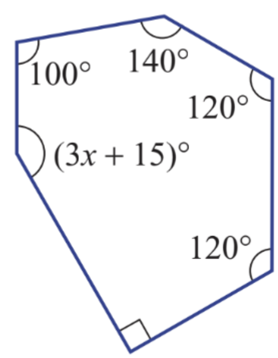


4. Find the value of x in the following, giving reasons.

a.

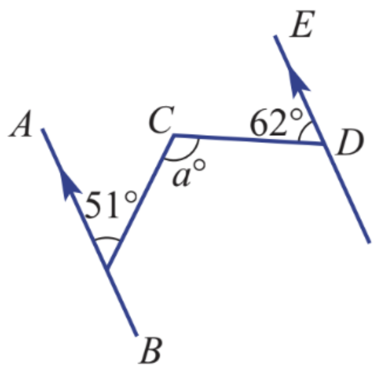


b.

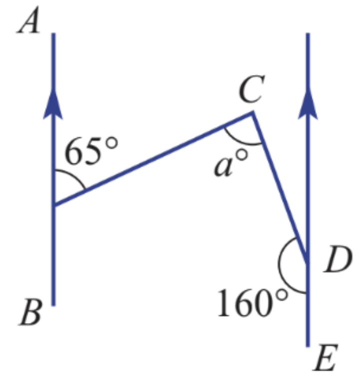


5. Find the value of the pronumeral a, giving reasons.

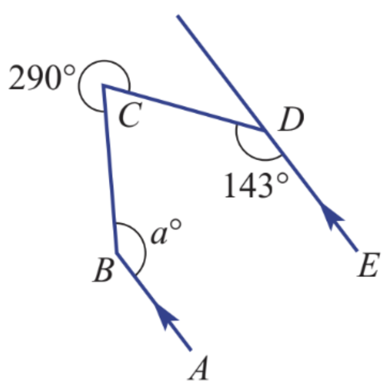
a.



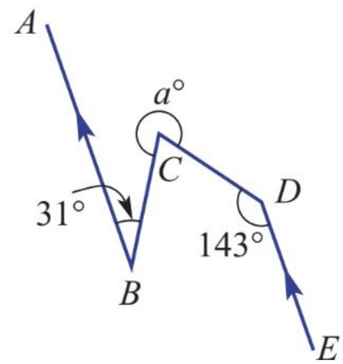
b.



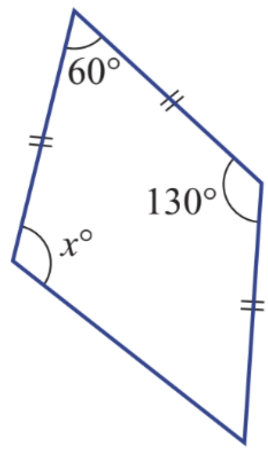
c.



d.

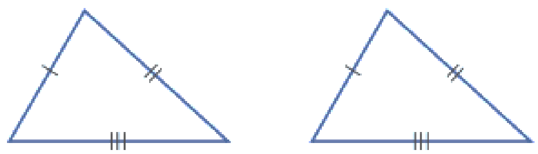


6. Find the value of x in this diagram giving reasons.
   Hint: Form isosceles and/or equilateral triangles.

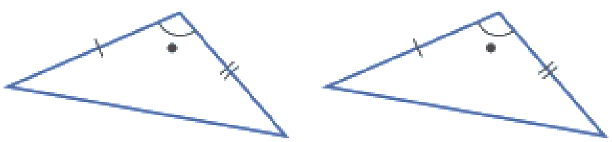


Chapter 2.2 Congruent Triangles

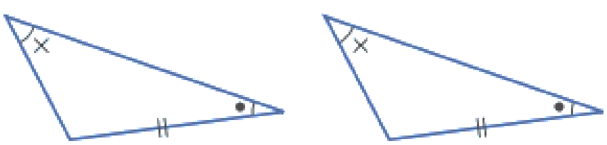
- Two objects are said to be **congruent** if they are exactly the same size and shape. For two congruent triangles ΔABC and ΔDEF, we write $\Delta ABC \equiv \Delta DEF$.
  - When comparing two triangles, corresponding sides are equal in length and corresponding angles are equal.
  - When we prove congruence in triangles, we usually write vertices in matching order.
- Two triangles can be tested for **congruence** by using the following conditions.
  - Corresponding sides are equal (SSS).



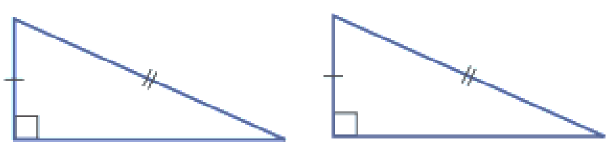
  - Two corresponding sides and the included angle are equal (SAS).



  - Two corresponding angles and a side are equal (AAS).



  - A right angle, the hypotenuse and one other pair of corresponding sides are equal (RHS).

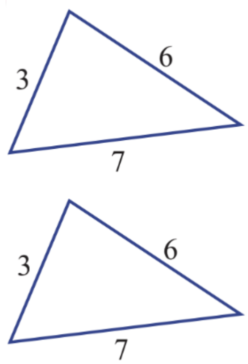


- $AB \parallel CD$ means AB is parallel to CD.
- $AB \perp CD$ means AB is perpendicular to CD.

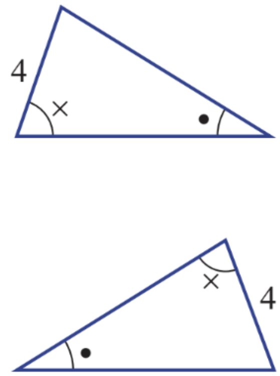
Practice Questions

1. Which of the tests (SSS, SAS, AAS or RHS) would be used to decide whether the following pairs of triangles are congruent?

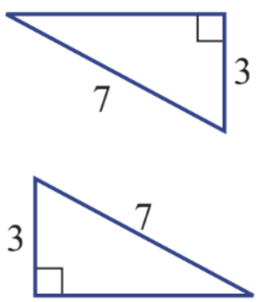
a.



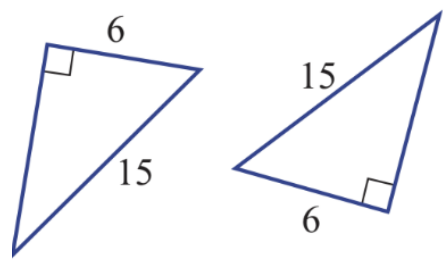
b.



c.

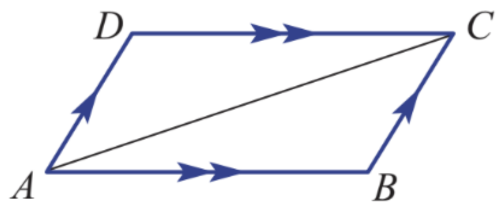


d.

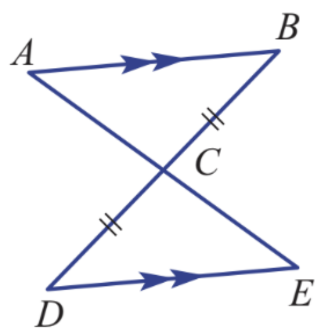


2. Prove that each pair of triangles is congruent, giving reasons. Write the vertices in matching order.

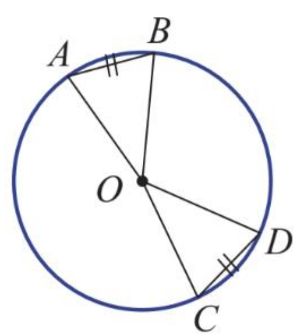
a.



b.

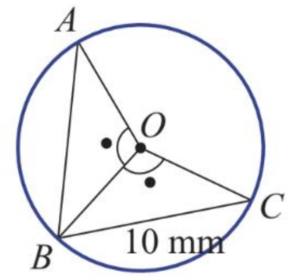


c.



3. In this diagram O is the centre of the circle and $\angle AOB = \angle COB$.

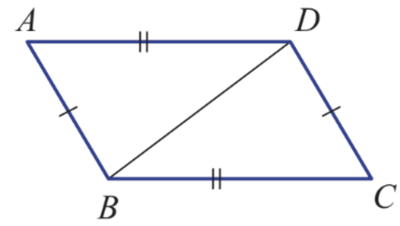
a. Prove $\Delta\, AOB \equiv \Delta\, COB$.



b. Prove $AB = BC$.

c. State the length AB.

4. In this diagram $AB = CD$ and $AD = CB$.



a. Prove $\Delta\, ABD \equiv \Delta\, CDB$.

b. Prove $\angle DBC = \angle BDA$.

c. Prove $AD \,||\, CB$.

Chapter 2.3 Investigating Parallelograms Using Congruence

Quadrilateral Properties and Tests

Key Vocabulary & Symbols

- **Parallel Lines:** If AB is parallel to CD, we write $AB \parallel CD$.
- **Perpendicular Lines:** If AB is perpendicular to CD, we write $AB \perp CD$.
- **Bisect:** To bisect means to cut into two equal parts.

Properties & Tests of Special Quadrilaterals

1. Parallelogram

A **parallelogram** is a **quadrilateral with opposite sides parallel**.

Properties:

- Opposite sides are **equal**.
- Opposite angles are **equal**.
- Diagonals **bisect** each other.

Tests (A quadrilateral is a parallelogram if):

- Opposite sides are **equal**.
- Opposite angles are **equal**.
- One pair of opposite sides is **both equal and parallel**.
- Diagonals **bisect** each other.

2. **Rhombus**

   A **rhombus** is a parallelogram where **all sides are equal**.

   Properties:

   - Opposite sides are **equal**.
   - Opposite angles are **equal**.
   - **Diagonals** bisect each other **at right angles**.
   - **Diagonals bisect the interior angles**.

   Tests (A parallelogram is a rhombus if):

   - All sides are **equal**.
   - Diagonals **bisect** each other at **right angles**.

3. **Rectangle**

   A **rectangle** is a **parallelogram where all angles are 90°.**

   Properties:

   - Opposite sides are **equal**.
   - **All angles are 90°.**
   - **Diagonals are equal** and bisect each other.

   Tests (A parallelogram is a rectangle if):

   - **All angles are 90°.**
   - Diagonals are **equal** and bisect each other.

4. **Square**

A **square** is both **a rectangle and a rhombus**.

Properties:

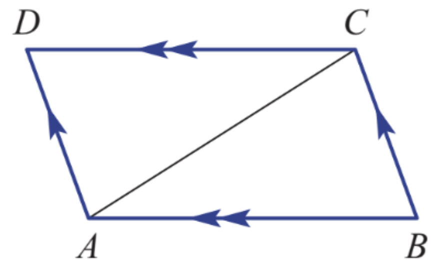
- Opposite sides are **equal**.
- **All angles are 90°.**
- Diagonals are **equal** and bisect each other **at right angles**.
- **Diagonals bisect the interior angles.**

Tests (A quadrilateral is a square if):

- **All sides are equal** and **at least one angle is 90°.**
- **Diagonals are equal** and **bisect each other at right angles**.

Practice Questions

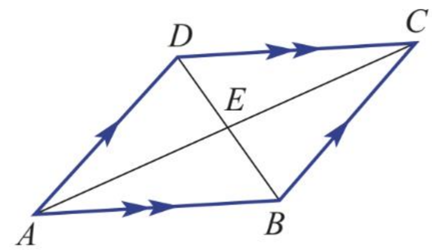
1. Complete these steps to prove that a parallelogram (with opposite parallel sides) has equal opposite sides.



a. Prove $\Delta ABC \equiv \Delta CDA$.

b. Hence, prove opposite sides are equal.

2. Complete these steps to prove that a parallelogram (with opposite parallel sides) has diagonals that bisect each other.

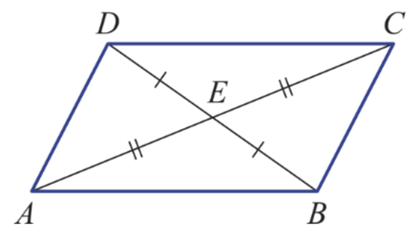


a. Prove $\Delta\, ABE \equiv \Delta\, CDE$.

b. Hence, prove $AE = CE$ and $BE = DE$.

3. Complete these steps to prove that if the diagonals in a quadrilateral bisect each other then it is a parallelogram.

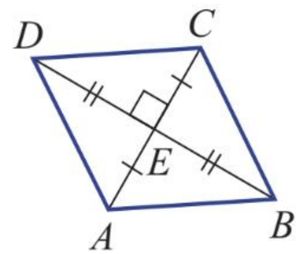
a. Prove $\Delta\, ABE \equiv \Delta\, CDE$.



b. Hence, prove $AB \,||\, DC$ and $AD \,||\, BC$.

4. Complete these steps to prove that if the diagonals of a quadrilateral bisect each other at right angles then it is a rhombus.

   a. Give a brief reason why $\Delta\, ABE \equiv \Delta\, CBE \equiv \Delta\, ADE \equiv \Delta\, CDE$.



   b. Hence, prove $ABCD$ is a rhombus.

Chapter 2.4 Similar Figures

Similarity in Geometry

Definition of Similar Figures

Two figures are **similar** if they have the **same shape but different sizes**.

Key Properties of Similar Figures:

- **Corresponding angles are equal**.
- **Corresponding sides are proportional** (i.e., they have the same ratio).

Scale Factor

$$\text{Scale factor} = \frac{\text{Image length}}{\text{Original length}}$$

- If the **scale factor > 1**, the image is **enlarged**.
- If the **scale factor < 1**, the image is **reduced**.

Similarity Symbols

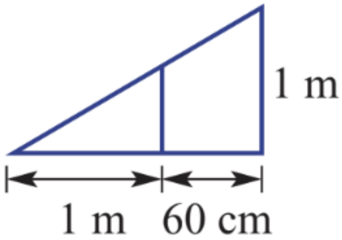
- The symbols ||| **or** $\sim$ indicate similarity.
- To express similarity between triangles, we write:

$$\triangle ABC \;|||\; \triangle DEF \quad \text{or} \quad \triangle ABC \sim \triangle DEF$$

Practice Questions

1. A 50 m tall structure casts a shadow 30 m in length.
   At the same time a person casts a shadow of 1.02 m. Estimate the height of the person.
   Hint: draw a diagram of two triangles.

2. A BMX ramp has two vertical supports as shown.

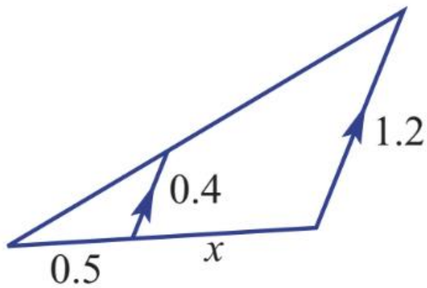


a. Find the scale factor for the two triangles in the diagram.

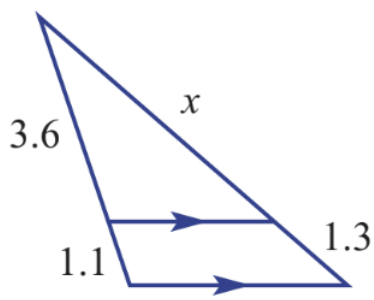
b. Find the length of the inner support.

3. Find the value of the pronumeral if the pairs of triangles are similar. Round to one decimal place in part **d**.

a.



b.



Chapter 2.5 Proving Similar Triangles

Similarity Conditions for Triangles

Two triangles are **similar** if they have the **same shape but different sizes**. We denote similarity as:

$$\triangle ABC \ ||| \triangle DEF \ \text{ or } \ \triangle ABC \sim \triangle DEF$$

Tests for Triangle Similarity:

1. **Side-Side-Side (SSS) Similarity**

   - If all **corresponding sides** are in the **same ratio**, the triangles are similar.

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}$$

2. **Side-Angle-Side (SAS) Similarity**

   - If **two pairs of corresponding sides** are in the **same ratio** and the **included angle** is equal, the triangles are similar.

$$ABDE = BCEF \ \text{ and } \ \angle B = \angle E$$

3. **Angle-Angle-Angle (AAA) Similarity**

   - If **two pairs of corresponding angles** are equal, the third pair must also be equal.

$$\angle A = \angle D, \angle B = \angle E, \angle C = \angle F$$

4. Right-Angle Hypotenuse-Side (RHS) Similarity

- If two **right-angled triangles** have their **hypotenuses and another pair of corresponding sides** in the **same ratio**, they are similar.

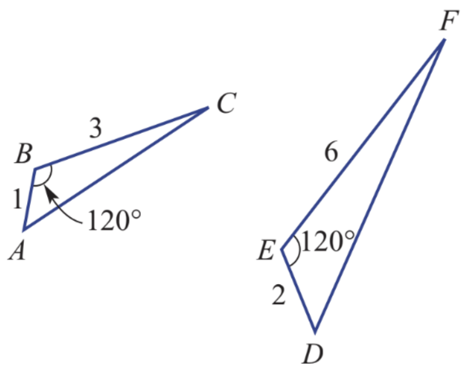
$$\frac{\text{Hypotenuse of } \triangle ABC}{\text{Hypotenuse } of \triangle DEF} = \frac{\text{Another pair of sides}}{\text{Corresponding sides}}$$

Key Takeaway: If two triangles satisfy any one of these conditions, they are **similar**!

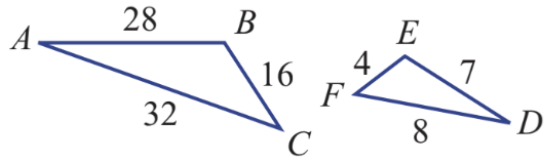
Practice Questions

1. Prove that the following pairs of triangles are similar.

   a.

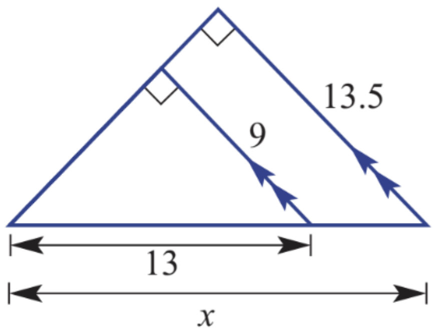


   b.

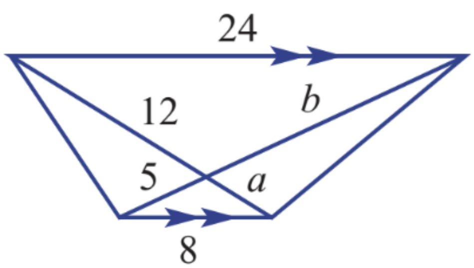


2. Find the value of the pronumerals in these pairs of similar triangles.

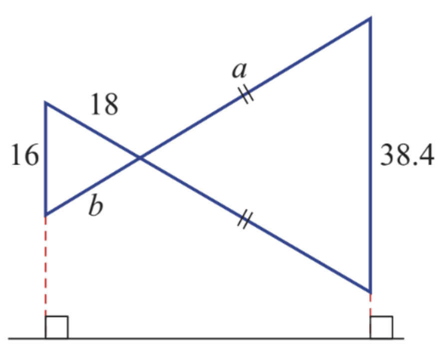
a.



b.

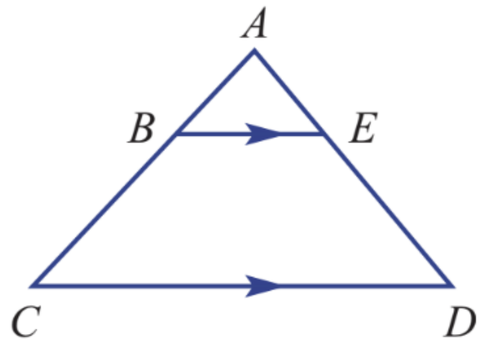


c.

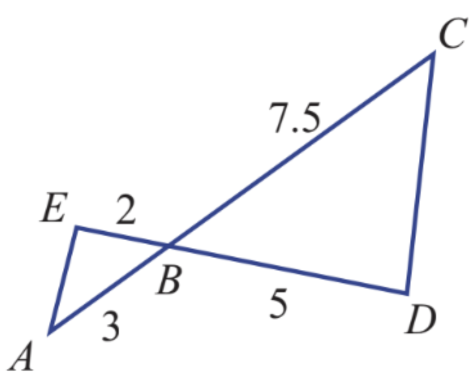


3. For the following proofs, give reasons at each step.

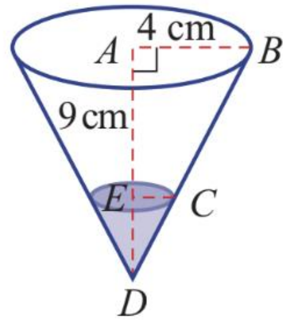
a. Prove $\Delta\, ABE \,|||\, \Delta\, ACD$.



b. Prove $\Delta\, AEB \,|||\, \Delta\, CDB$.



4. A right cone with radius 4 cm has a total height of 9 cm. It contains an amount of water as shown.

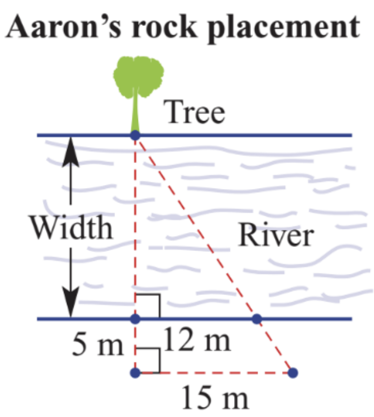


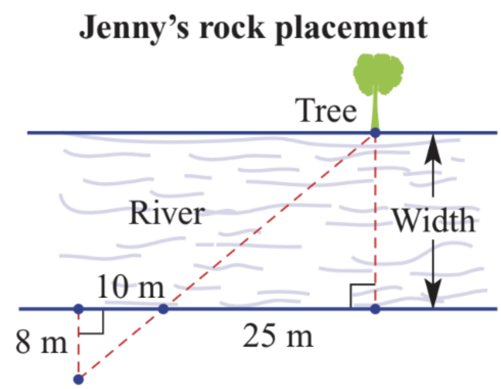
a. Prove $\Delta\, EDC \;|||\; \Delta\, ADB$.

b. If the depth of water in the cone is 3 cm, find the radius of the water surface in the cone.

5. At a particular time in the day, Aaron casts a shadow 1.3 m long while Jack who is 1.75 m tall casts a shadow 1.2 m long. Find Aaron's height, correct to two decimal places.

6. Aaron and Jenny come to a river and notice a tree on the opposite bank. Separately they decide to place rocks (indicated with dots) on their side of the river to try to calculate the river's width. They then measure the distances between some pairs of rocks as shown.





a. Have both Aaron and Jenny constructed a pair of similar triangles? Give reasons.

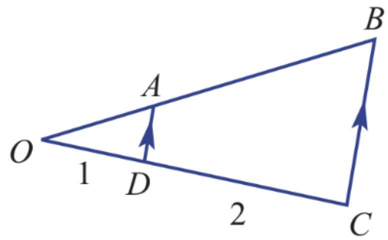
b. Using Jenny's triangles calculate the width of the river.

c. Using Aaron’s triangles calculate the width of the river.

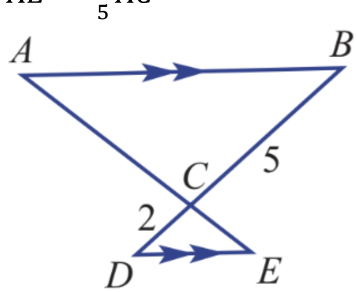
d. Which pair of triangles did you prefer to use? Give reasons.

7. Prove the following, giving reasons.

a. $OB = 3OA$

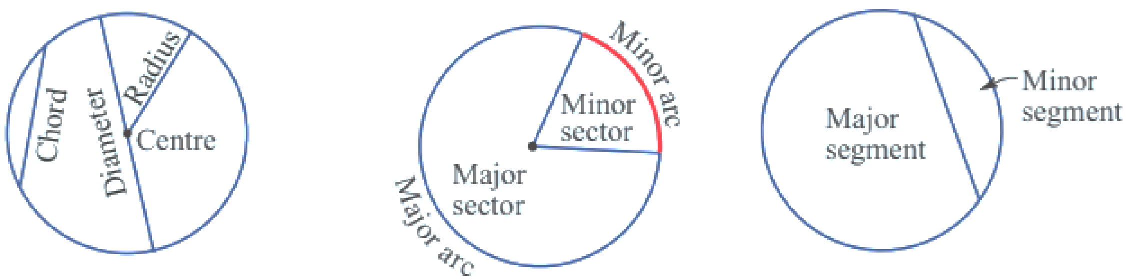


b. $AE = \frac{7}{5}AC$

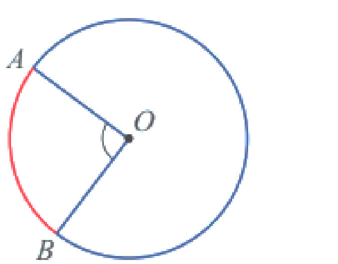


Chapter 2.6 Circles and Chord Properties

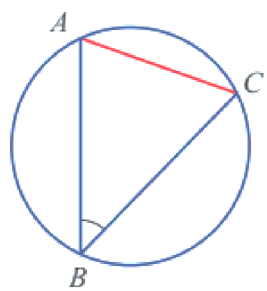
- Circle language



- An angle is **subtended** by an arc/chord if the arms of the angle meet the endpoints of the arc/chord.

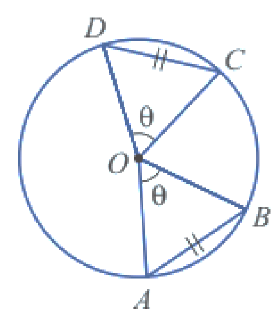


$\angle AOB$ is subtended at the centre by the minor are AB.

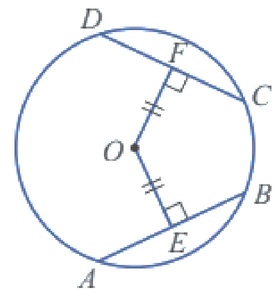


$\angle ABC$ is subtended at the circumference by the chord AC.

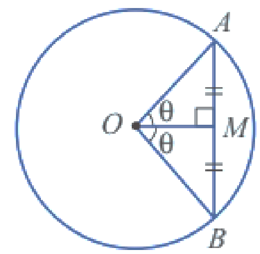
- **Chord theorem 1:** Chords of equal length subtend equal angles at the centre of the circle.
  - If $AB = CD$ then $\angle AOB = \angle COD$.
  - Conversely, if chords subtend equal angles at the centre of the circle then the chords are of equal length.



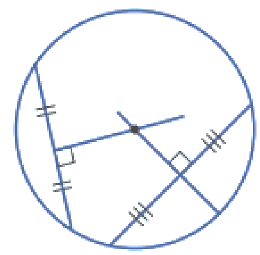
- **Chord theorem 2:** Chords of equal length are equidistant (of equal distance) from the centre of the circle.
  - If $AB = CD$ then $OE = OF$.
  - Conversely, if chords are equidistant from the centre of the circle then the chords are of equal length.



- **Chord theorem 3:** The perpendicular from the centre of the circle to the chord, bisects the chord and the angle at the centre subtended by the chord.
  - If $OM \perp AB$ then $AM = BM$ and $\angle AOM = \angle BOM$.
  - Conversely, if a radius bisects the chord (or angle at the centre subtended by the chord) then the radius is perpendicular to the chord.



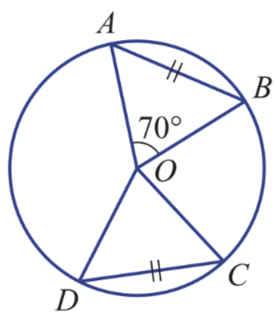
- **Chord theorem 4:** The perpendicular bisectors of every chord of a circle intersect at the centre of the circle.
  - Constructing perpendicular bisectors of two chords will therefore locate the centre of a circle.



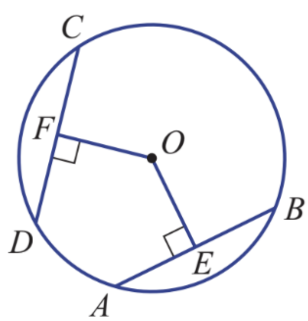
Practice Questions

1. For each part, use the information given and state which chord theorem is used.

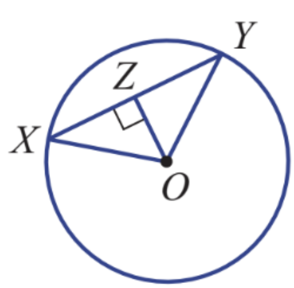
   a. Given $AB = CD$ and $\angle AOB = 70°$ find $\angle DOC$.



   b. Given $AB = CD$ and $OF = 7.2$ cm find OE.

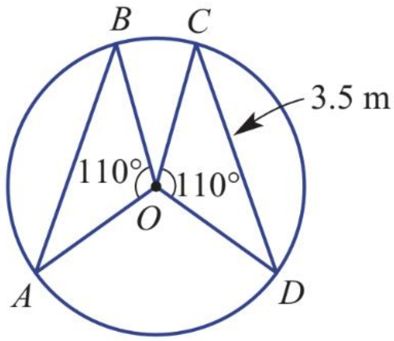


   c. Given $OZ \perp XY$, $XY = 8$ cm and $\angle XOY = 102°$, find XZ and $\angle XOZ$.

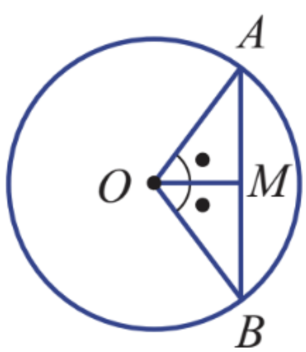


2. Use the information given to answer the following.

a. Given $\angle AOB = \angle COD$ and $CD = 3.5$ m, find AB.

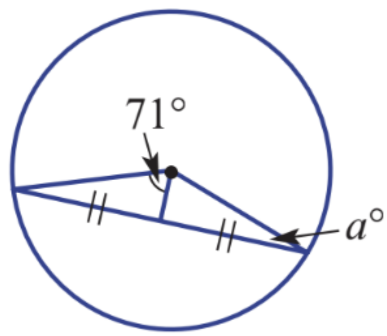


b. Given $\angle AOM = \angle BOM$, find $\angle OMB$.

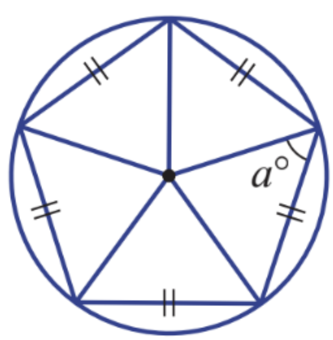


3. Find the size of each unknown angle a.

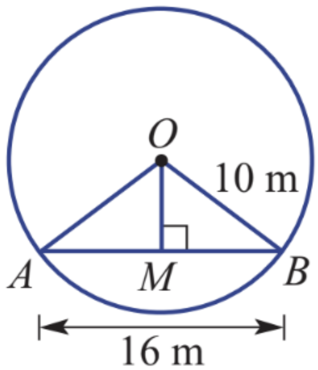
a.



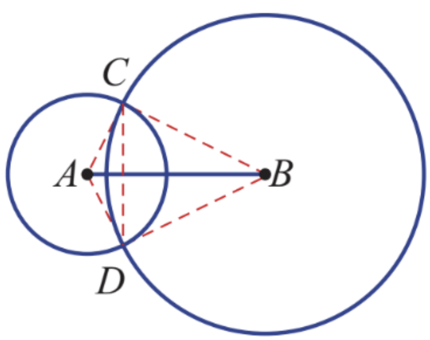
b.



4. Find the length OM. Hint: Use Pythagoras' theorem.

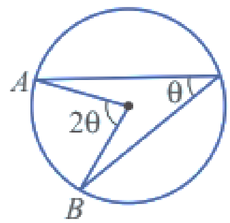


5. In this diagram, radius $AD = 5\text{ mm}$, radius $BD = 12\text{ mm}$ and chord $CD = 8\text{ mm}$. Find the exact length of AB in surd form.



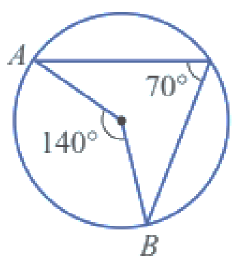
Chapter 2.7 Angle Properties of Circles - Theorems 1 and 2

- Theorem 1: Angles at the centre and circumference
  - The angle at the centre of a circle is twice the angle at a point on the circle subtended by the same arc.

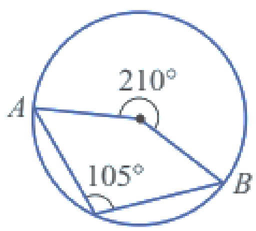


For example:

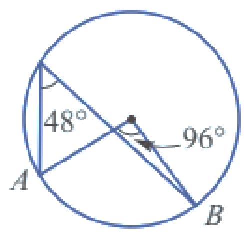
1



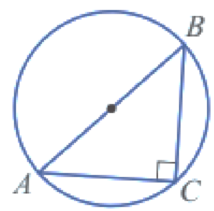
2



3



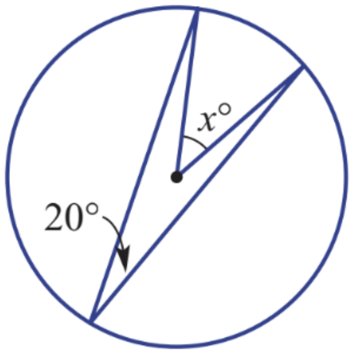
- Theorem 2: Angle in a semicircle
  - The angle in a semicircle is 90°.
    This is a specific case of Theorem 1 and $\angle ACB$ is known as the angle in a semicircle.



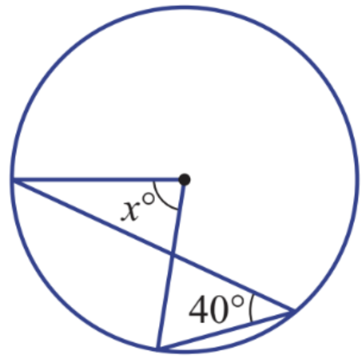
Practice Questions

1. Find the value of x in these circles.

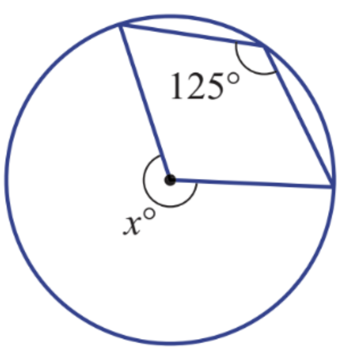
a.



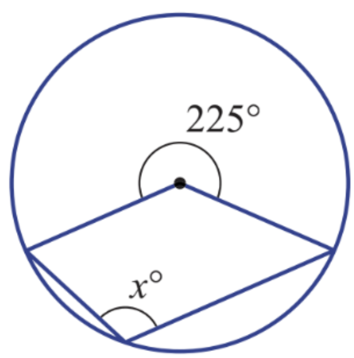
b.



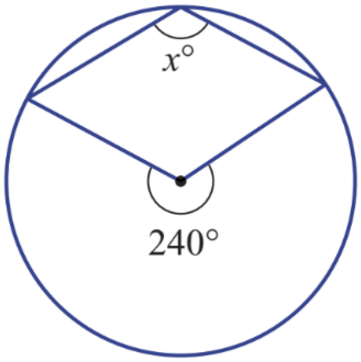
c.



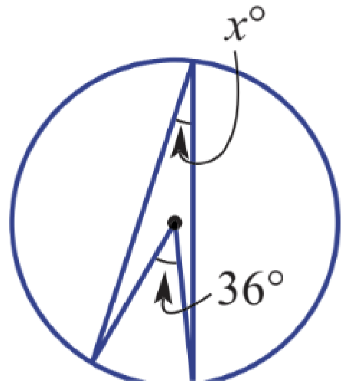
d.



e.

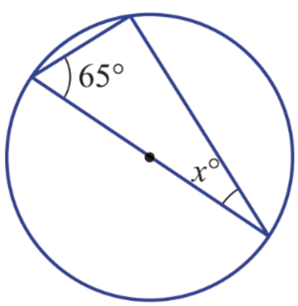


f.

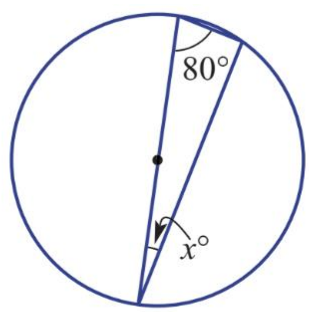


2. Find the value of x in these circles.

a.

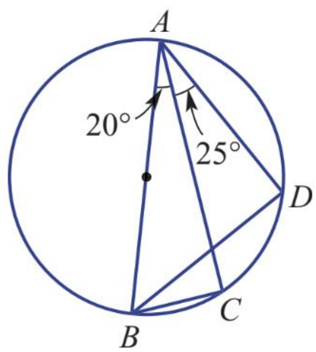


b.

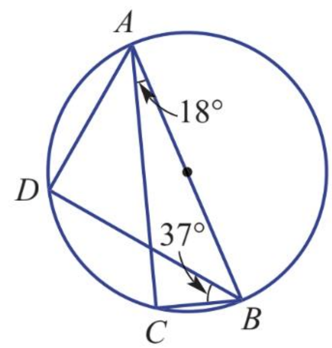


3. Find the size of both $\angle ABC$ and $\angle ABD$.

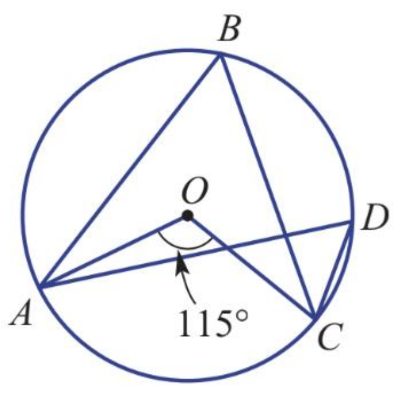
a.



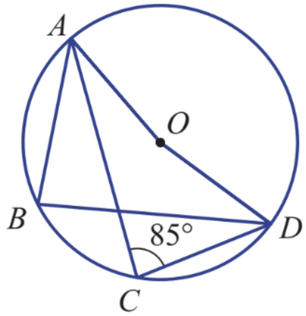
b.



4. Find $\angle ABC$ and $\angle ADC$.

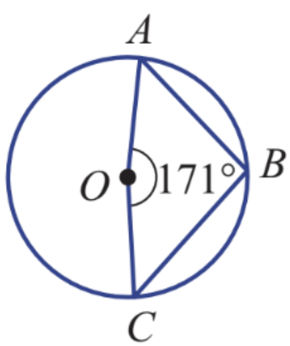


5. Find $\angle AOD$ and $\angle ABD$.

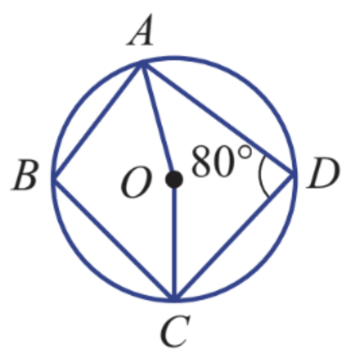


6. Find $\angle ABC$.

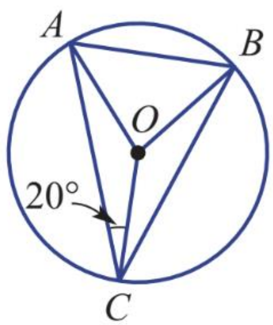
a.



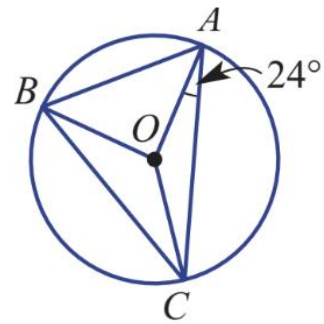
b.



c.

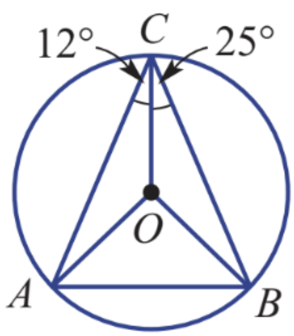


d.

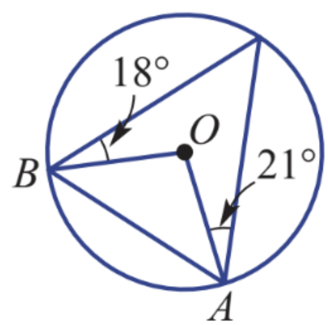


7. Find $\angle OAB$.

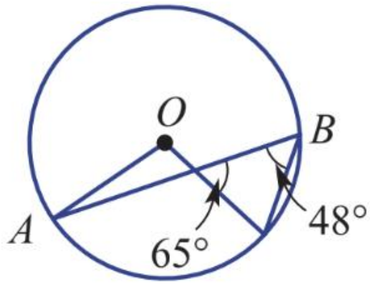
a.



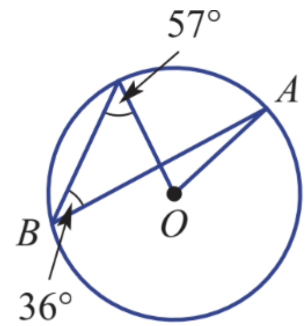
b.



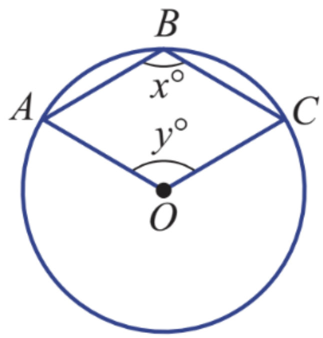
c.



d.



8. Consider this circle.

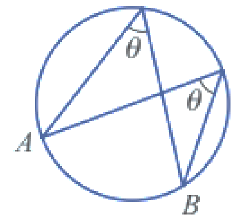


a. Write reflex $\angle AOC$ in terms of x.

b. Write y in terms of x.

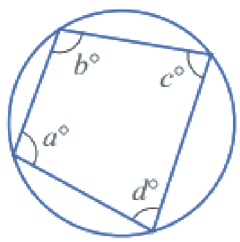
Chapter 2.8 Angle Properties of Circles - Theorems 3 and 4

- Theorem 3: Angles at the circumference
  - Angles at the circumference of a circle subtended by the same arc are equal.



- Theorem 4: Opposite angles in cyclic quadrilaterals
  - Opposite angles in a **cyclic quadrilateral** are supplementary (sum to 180°). A cyclic quadrilateral has all four vertices sitting on the same circle.

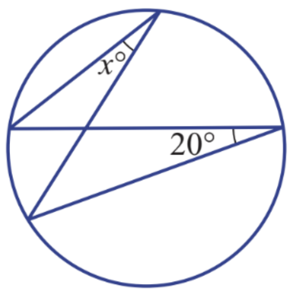
$$a + c = 180$$
$$b + d = 180$$



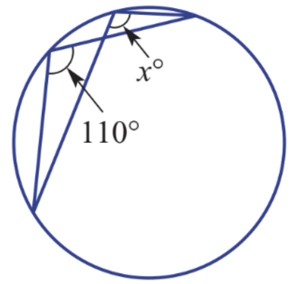
Practice Questions

1. Find the value of x in these circles.

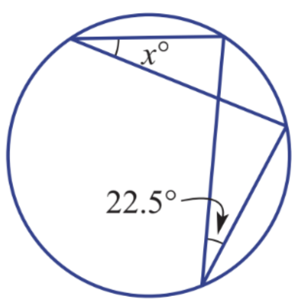
a.



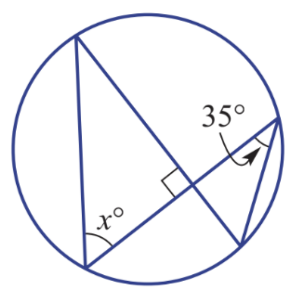
b.



c.

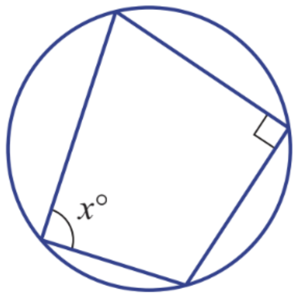


d.

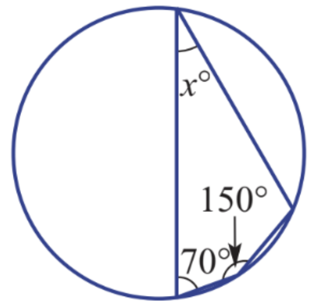


2. Find the value of the pronumerals in these circles.

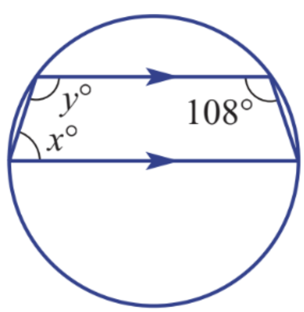
a.



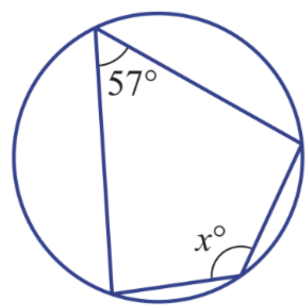
b.



c.

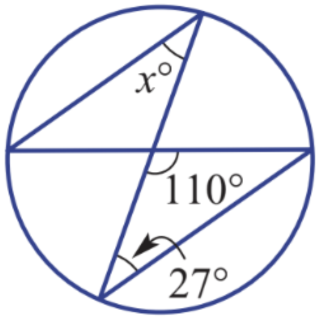


d.

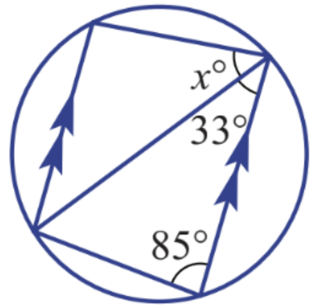


3. Find the value of x.

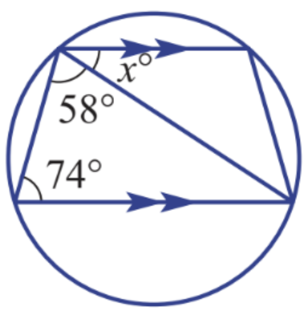
a.



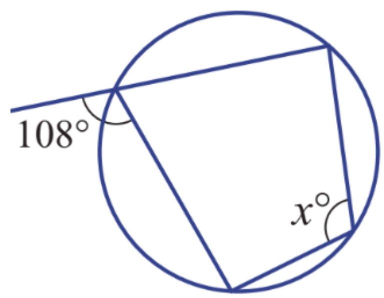
b.



c.

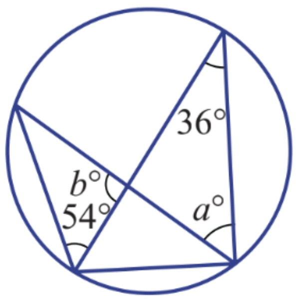


d.

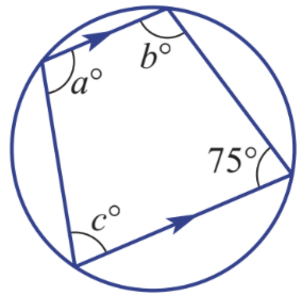


4. Find the values of the pronumerals in these circles.

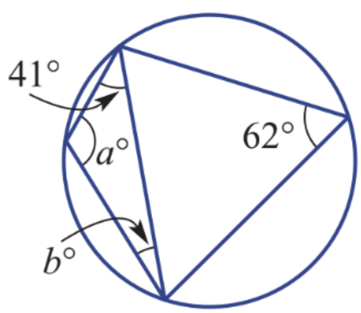
a.



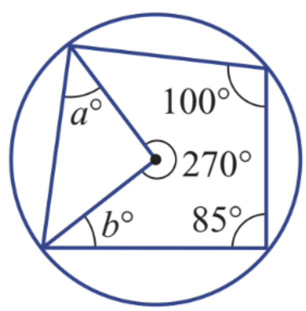
b.



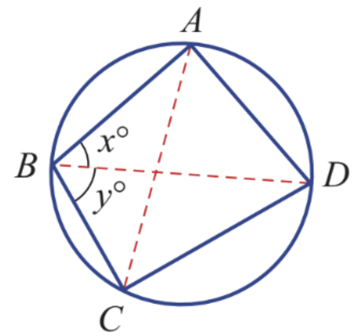
c.



d.



5. Prove that opposite angles in a cyclic quadrilateral are supplementary by following these steps.

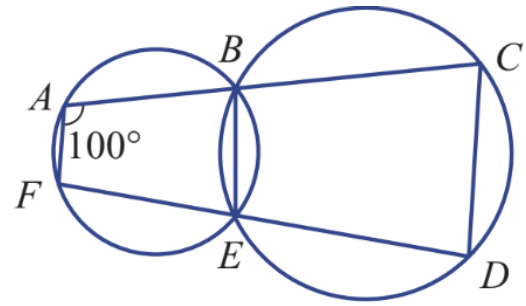


a. Explain why $\angle ACD = x°$ and $\angle DAC = y°$.

b. Prove that $\angle ADC = 180° - (x + y)°$.

c. What does this say about $\angle ABC$ and $\angle ADC$?

6. If $\angle BAF = 100°$ complete the following.



a. Find:

i. $\angle FEB$

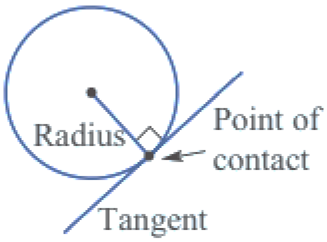
ii. $\angle BED$

iii. $\angle DCB$

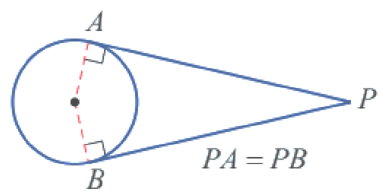
b. Explain why $AF \parallel CD$.

Chapter 2.9 Tangents

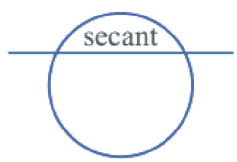
- A **tangent** is a line that touches a circle at a point called the **point of contact**.



  - A tangent intersects the circle exactly once.
  - A tangent is perpendicular to the radius at the point of contact.
  - Two different tangents drawn from an external point to the circle have equal length.



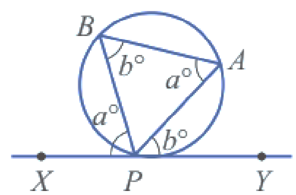
- A **secant** is a line that cuts a circle twice.



- **Alternate segment theorem:** The angle between a tangent and a chord is equal to the angle in the alternate segment.

$\angle APY = \angle ABP$ and

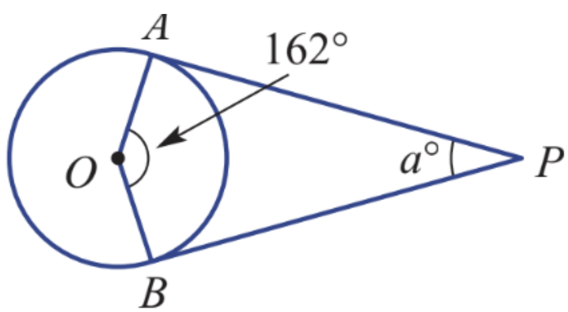
$\angle BPX = \angle BAP$



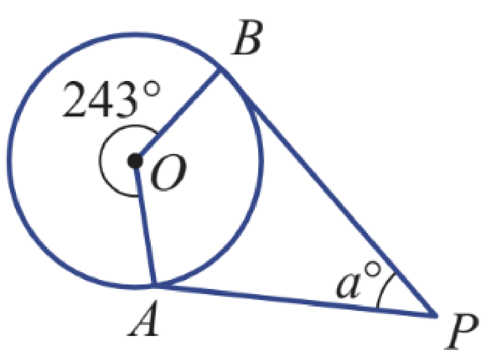
Practice Questions

1. Find the value of a in these diagrams that include tangents.

a.

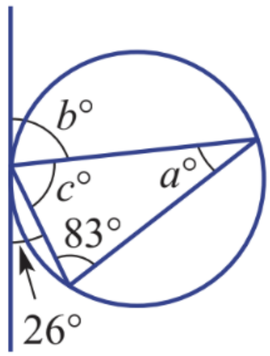


b.

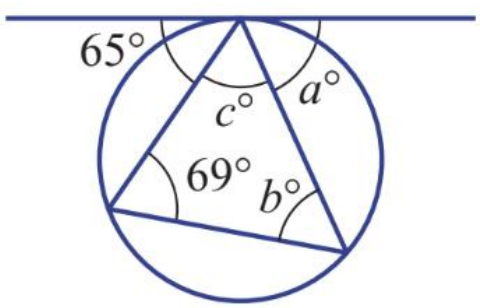


2. Find the value of a, b and c in these diagrams.

a.

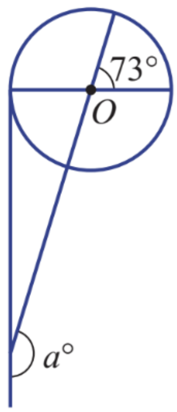


b.

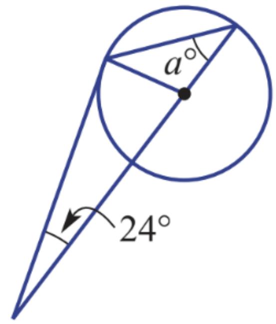


3. Find the value of a.

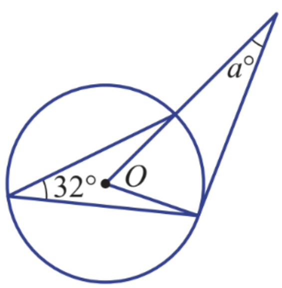
a.



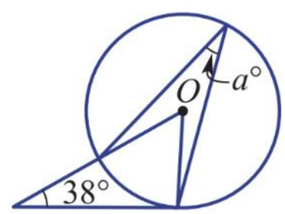
b.



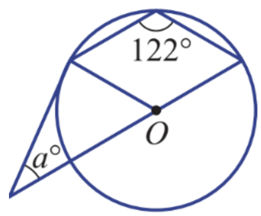
c.



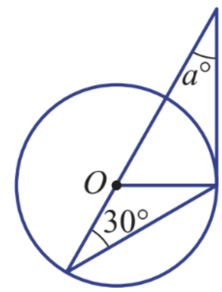
d.



e.

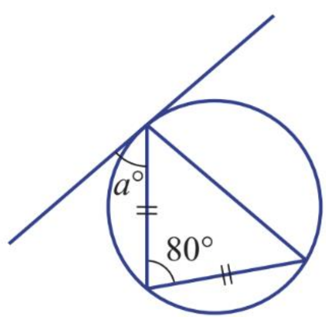


f.

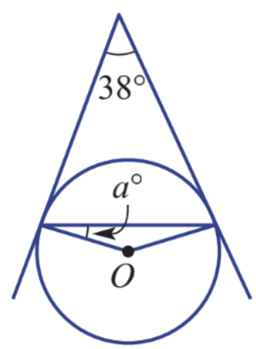


4. Find the value of a.

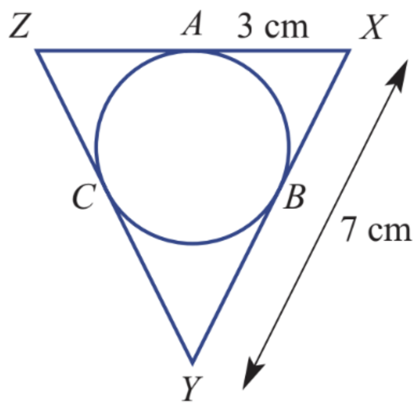
a.



b.



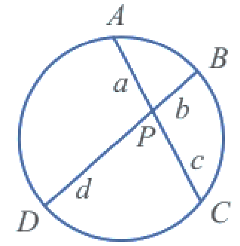
5. Find the length CY in this diagram.



Chapter 2.10 Intersecting Chords, Secants and Tangents

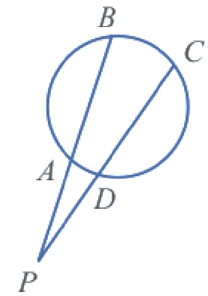
- When two chords intersect as shown then:

  $AP \times CP = BP \times DP$ or $ac = bd$



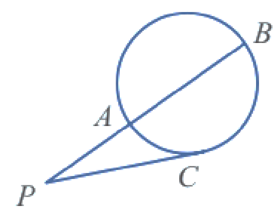
- When two secants intersect at an external point P as shown then:

  $AP \times BP = DP \times CP$



- When a secant intersects a tangent at an external point as shown then:

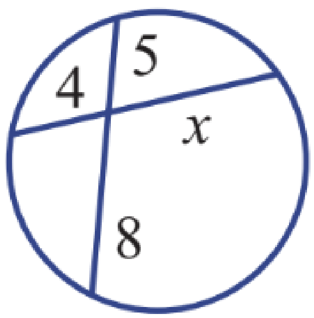
  $AP \times BP = CP^2$



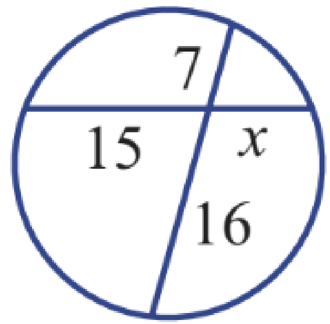
Practice Questions

1. Find the value of x in each figure.

a.

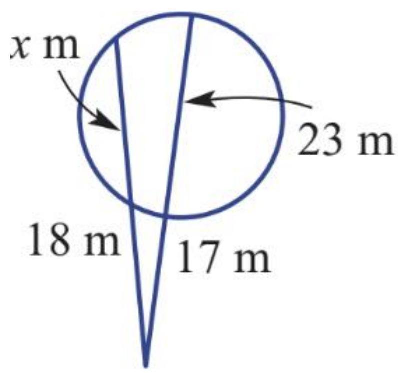


b.

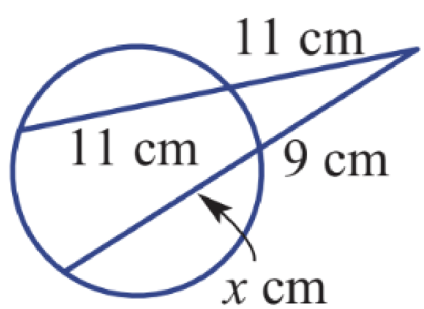


2. Find the value of x in each figure.

a.

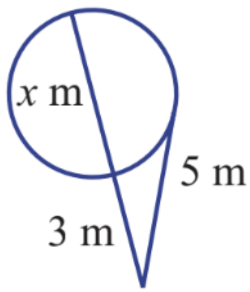


b.

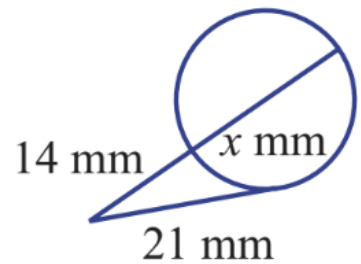


3. Find the value of x in each figure.

a.

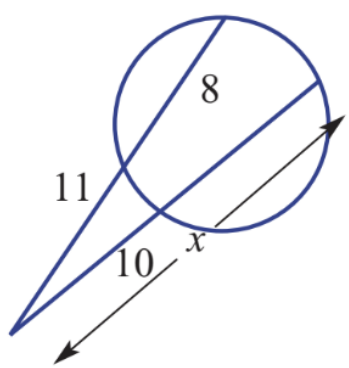


b.

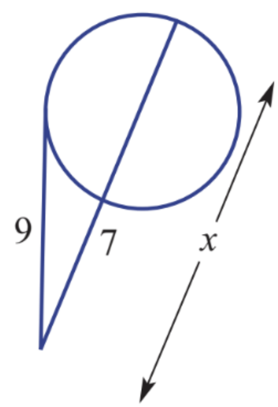


4. Find the value of x.

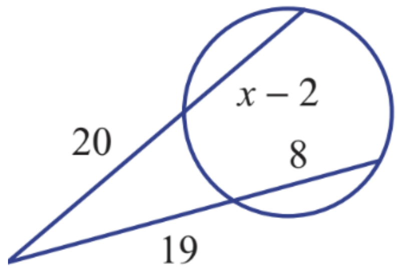
a.



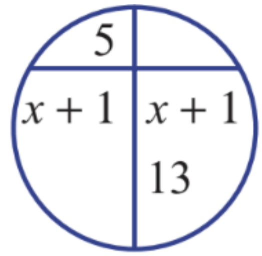
b.



c.

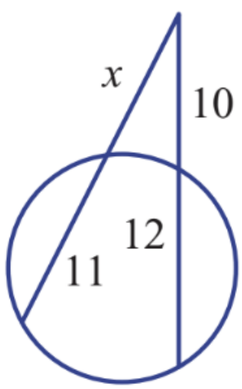


d.

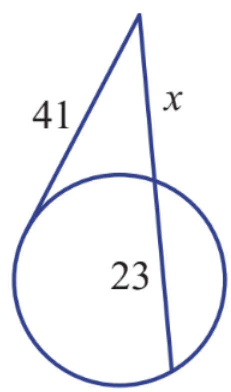


5. For each diagram derive the given equations.

a. $x^2 + 11x - 220 = 0$



b. $x^2 + 23x - 1681 = 0$



CHAPTER 3 INDICES AND SURDS

Table of Contents

Chapter 3.1 Irrational Numbers and Surds

Real Numbers & Surds

Real Numbers can be **located on a number line** and include:

Rational Numbers (expressed as fractions)

- Examples:

$$\frac{3}{7}, \frac{4}{39}, -3.6, 2.7, 0.19$$

- Their decimal forms are **terminating** (e.g., 0.5) or **recurring** (e.g., 0.3333...).

Irrational Numbers (cannot be expressed as fractions)

- Examples:

$$\sqrt{3}, -2\sqrt{7}, 12 - 1, \pi, 2\pi - 3$$

- Their decimal forms are **non-repeating and non-terminating**.

Surds (Irrational Roots)

Definition: A **surd** is an irrational number written with a root sign (√).

Examples:

$$\sqrt{2}, \sqrt{5}, \sqrt{11}, -\sqrt{200}, 1+\sqrt{5}$$

These are NOT surds (since they simplify to whole numbers):

$$\sqrt{4} = 2, \sqrt[3]{125} = 5, \sqrt{16} = 4$$

Surd Rules & Simplifications

1. **Basic Root Properties**

$$\left(\sqrt{x}\right)^2 = x, \sqrt{x} \times \sqrt{x} = x \ \ (\text{if } x \geq 0)$$

2. **Multiplication of Surds**

$$\sqrt{x} \times \sqrt{y} = \sqrt{xy} \ \ (\text{if } x \geq 0, y \geq 0)$$

Example:

$$\sqrt{3} \times \sqrt{12} = \sqrt{36} = 6$$

3. **Division of Surds**

$$\frac{\sqrt{x}}{\sqrt{y}} = \sqrt{\frac{x}{y}} \ \ (\text{if } x \geq 0, y > 0)$$

Example:

$$\frac{\sqrt{18}}{\sqrt{2}} = \sqrt{9} = 3$$

Simplifying Surds using Square Factors

Perfect Squares: 4, 9, 16, 25, 36, 49, 64, 81, 100...

Key Rule:

$$\sqrt{a \times b} = \sqrt{a} \times \sqrt{b}$$

Example:

$$\sqrt{50} = \sqrt{25 \times 2} = \sqrt{25} \times \sqrt{2} = 5\sqrt{2}$$

Key Takeaway: Understanding surds and real numbers helps in algebra and problem-solving!

Practice Questions

1. Simplify the following surds.

a. $\sqrt{24}$

b. $\sqrt{98}$

c. $\sqrt{162}$

2. Simplify the following.

a. $4\sqrt{48}$

b. $2\sqrt{63}$

c. $\frac{\sqrt{45}}{3}$

d. $\frac{\sqrt{28}}{2}$

e. $\frac{\sqrt{80}}{20}$

f. $\frac{\sqrt{99}}{18}$

g. $\frac{2\sqrt{98}}{7}$

h. $\frac{3\sqrt{68}}{21}$

i. $\frac{2\sqrt{108}}{18}$

j. $\frac{3\sqrt{147}}{14}$

3. Simplify the following.

a. $\sqrt{\frac{18}{25}}$

b. $\sqrt{\frac{11}{25}}$

c. $\sqrt{\frac{26}{32}}$

d. $\sqrt{\frac{28}{50}}$

e. $\sqrt{\frac{45}{72}}$

f. $\sqrt{\frac{56}{76}}$

4. Express these surds as a square root of a positive integer.

a. $5\sqrt{2}$

b. $3\sqrt{3}$

c. $8\sqrt{2}$

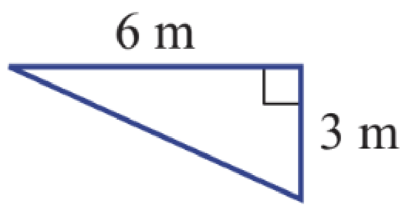
d. $10\sqrt{7}$

e. $7\sqrt{5}$

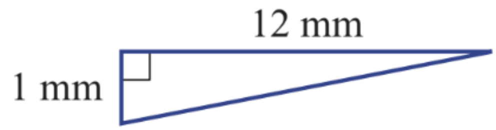
f. $11\sqrt{3}$

5. Use Pythagoras' theorem to find the unknown length in these triangles in simplest form.

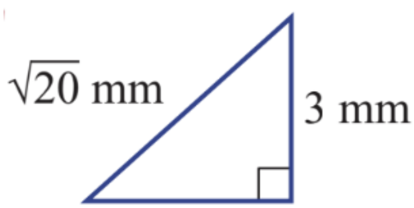
a.



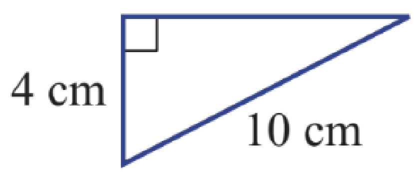
b.



c.



d.



Chapter 3.2 Adding and Subtracting Surds

Like Surds & Operations with Surds

Definition: Like surds are multiples of the **same surd** (same square root value).

Examples of Like Surds:

$$3\sqrt{3}, -5\sqrt{3}, 12\sqrt{3} \ \ (\text{since } 12\sqrt{3} = 2 \times 6\sqrt{3})$$

$$275 = 103 \ \ (\text{since } \sqrt{75} = 5\sqrt{3})$$

Adding & Subtracting Like Surds

Key Rule: **Only like surds can be added or subtracted** by factoring out the common surd.

Steps:

1. **Simplify all surds** first.

2. **Factor out common surd** and perform addition/subtraction.
   - **Example 1**:

$$3\sqrt{2} + 5\sqrt{2} = (3 + 5)\sqrt{2} = 8\sqrt{2}$$

   - **Example 2**:

$$7\sqrt{5} - 2\sqrt{5} = (7 - 2)\sqrt{5} = 5\sqrt{5}$$

   - **Example 3 (Simplifying First)**:

$$2\sqrt{50} + 3\sqrt{8}$$

1. Simplify:

$$2\sqrt{50} = 2(5\sqrt{2}) = 10\sqrt{2}$$
$$3\sqrt{8} = 3(2\sqrt{2}) = 6\sqrt{2}$$

2. Add like surds:

$$10\sqrt{2} + 6\sqrt{2} = (10 + 6)\sqrt{2} = 16\sqrt{2}$$

Key Takeaway:

- **Like surds** can be **added & subtracted**.
- **Always simplify surds first** before performing operations!

Practice Questions

1. Simplify the following.

a. $5\sqrt{3} - 2\sqrt{3}$

b. $8\sqrt{2} - 5\sqrt{2}$

c. $6\sqrt{3} - 5\sqrt{3}$

d. $6\sqrt{2} - 4\sqrt{2} + 3\sqrt{2}$

e. $3\sqrt{11} - 8\sqrt{11} - \sqrt{11}$

f. $10\sqrt{30} - 15\sqrt{30} - 2\sqrt{30}$

2. Simplify the following.

a. $5\sqrt{6} + 4\sqrt{11} - 2\sqrt{6} + 3\sqrt{11}$

b. $5\sqrt{2} + 2\sqrt{5} - 7\sqrt{2} - \sqrt{5}$

c. $5\sqrt{11} + 3\sqrt{6} - 3\sqrt{6} - 5\sqrt{11}$

d. $-4\sqrt{5} - 2\sqrt{15} + 5\sqrt{15} + 2\sqrt{5}$

Chapter 3.3 Multiplying and Dividing Surds

Key Rules:

1. **Multiplication of Surds**

$$\sqrt{x} \times \sqrt{y} = \sqrt{xy}$$

$$\mathrm{a}\sqrt{x} \times \mathrm{b}\sqrt{y} = (\mathrm{a} \times \mathrm{b}) \times \sqrt{x \times y}$$

Examples:

- **Example 1:**

$$\sqrt{3} \times \sqrt{5} = \sqrt{15}$$

- **Example 2:**

$$2\sqrt{6} \times 3\sqrt{2} = (2 \times 3) \times \sqrt{6 \times 2} = 6\sqrt{12}$$

- **Example 3 (Simplify Further):**

$$6\sqrt{12} = 6 \times 2\sqrt{3} = 12\sqrt{3}$$

2. **Division of Surds**

$$\frac{\sqrt{x}}{\sqrt{y}} = \sqrt{\frac{x}{y}}$$

$$\frac{a\sqrt{x}}{b\sqrt{y}} = \frac{a}{b} \times \sqrt{\frac{x}{y}}$$

Examples:

- **Example 1:**

$$182 = 182 = 9 = 3$$

- **Example 2:**

$$\frac{4\sqrt{50}}{2\sqrt{2}} = \frac{4}{2} \times \frac{\sqrt{50}}{\sqrt{2}}$$

$$= 2 \times \sqrt{\frac{50}{2}} = 2 \times \sqrt{25} = 2 \times 5 = 10$$

3. **Expanding Brackets Using the Distributive Law**

$$a(b + c) = ab + ac$$

Examples:

- **Example 1:**

$$\sqrt{3}(\sqrt{5} + 2) = \sqrt{3} \times \sqrt{5} + \sqrt{3} \times 2$$

$$= \sqrt{15} + 2\sqrt{3}$$

- **Example 2**:

$$(2 + \sqrt{7})(3 + \sqrt{2})$$

Use distributive law:

$$2 \times 3 + 2 \times \sqrt{2} + \sqrt{7} \times 3 + \sqrt{7} \times \sqrt{2}$$

$$= 6 + 2\sqrt{2} + 3\sqrt{7} + \sqrt{14}$$

Key Takeaways:

- Multiply and divide surds directly using the rules
- Always simplify surds where possible
- Expand brackets carefully using the distributive law

Practice Questions

1. Simplify the following.

   a. $\sqrt{2} \times \sqrt{13}$

   b. $-\sqrt{6} \times \sqrt{5}$

   c. $\sqrt{10} \times \sqrt{7}$

2. Simplify the following.

a. $\sqrt{33} \div -\sqrt{11}$

b. $\frac{\sqrt{30}}{\sqrt{3}}$

c. $-\frac{\sqrt{50}}{\sqrt{10}}$

3. Simplify the following.

a. $\sqrt{10} \times \sqrt{3}$

b. $\sqrt{9} \times \sqrt{9}$

c. $\sqrt{3} \times \sqrt{18}$

d. $\sqrt{5} \times \sqrt{20}$

4. Simplify the following.

a. $4\sqrt{6} \times \sqrt{21}$

b. $5\sqrt{3} \times \sqrt{15}$

c. $2\sqrt{10} \times -2\sqrt{5}$

d. $9\sqrt{12} \times 4\sqrt{21}$

5. Simplify the following.

a. $\frac{4\sqrt{30}}{8\sqrt{6}}$

b. $\frac{12\sqrt{70}}{18\sqrt{14}}$

6. Use the distributive law to expand the following and simplify the surds where necessary.

a. $\sqrt{2}\left(\sqrt{7} - \sqrt{5}\right)$

b. $-2\sqrt{3}\left(\sqrt{5} + \sqrt{7}\right)$

c. $4\sqrt{5}\left(\sqrt{5} - \sqrt{10}\right)$

d. $-2\sqrt{6}\left(3\sqrt{2} - 2\sqrt{3}\right)$

e. $6\sqrt{5}\left(3\sqrt{15} - 2\sqrt{8}\right)$

f. $2\sqrt{3}\left(7\sqrt{6} + 5\sqrt{3}\right)$

7. Simplify the following.

a. $-\left(5\sqrt{3}\right)^2$

b. $\sqrt{5}\left(\sqrt{2}+1\right)-\sqrt{40}$

c. $2\sqrt{3}\left(\sqrt{6}-\sqrt{3}\right)-\sqrt{50}$

8. Look at this example before simplifying the following.

$$(2\sqrt{3})^3 = 2^3(\sqrt{3})^3$$
$$= 2 \times 2 \times 2 \times \sqrt{3} \times \sqrt{3} \times \sqrt{3}$$
$$= 8 \times 3 \times \sqrt{3}$$
$$= 24\sqrt{3}$$

a. $(5\sqrt{3})^3$

b. $(\sqrt{5})^4$

c. $(2\sqrt{2})^5$

d. $2(-3\sqrt{2})^3$

e. $\frac{\left(2\sqrt{7}\right)^{3}}{4}$

f. $\frac{\left(3\sqrt{2}\right)^{4}}{4}$

g. $\frac{\left(2\sqrt{3}\right)^{2}}{9} \times \frac{\left(-3\sqrt{2}\right)^{4}}{3}$

h. $\frac{\left(3\sqrt{3}\right)^{3}}{2} \div \frac{\left(5\sqrt{2}\right)^{2}}{4}$

i. $\frac{\left(2\sqrt{2}\right)^{3}}{9} \div \frac{\left(2\sqrt{8}\right)^{2}}{\left(\sqrt{27}\right)^{3}}$

Chapter 3.4 Binomial Products

Expanding Binomial Products and Simplifying

1. **Distributive Law for Binomial Products**

   The distributive law is used to expand binomial products, where each term in one binomial is multiplied by each term in the other binomial.

$$(a+b)(c+d) = ac + ad + bc + bd$$

   Example:

   Expand $(x+2)(x+3)$:

$$\begin{aligned}(x+2)(x+3) &= x(x) + x(3) + 2(x) + 2(3)\\ &= x^2 + 3x + 2x + 6\\ &= x^2 + 5x + 6\end{aligned}$$

2. **Perfect Squares**

   A perfect square occurs when a binomial is squared. There are two important formulas to remember:

   a. Sum of Squares $(a+b)^2$:

$$(a+b)^2 = (a+b)(a+b) = a^2 + 2ab + b^2$$

   Example:

   Expand $(x+4)^2$:

$$\begin{aligned}(x+4)^2 &= x^2 + 2(x)(4) + 4^2\\ &= x^2 + 8x + 16\end{aligned}$$

b. Difference of Squares $(a-b)^2$:

$$(a-b)^2 = (a-b)(a-b) = a^2 - 2ab + b^2$$

Example:

Expand $(x-3)^2$:

$$(x-3)^2 = x^2 - 2(x)(3) + 3^2$$
$$= x^2 - 6x + 9$$

3. Difference of Perfect Squares

The difference of perfect squares occurs when you multiply a sum and a difference of the same two terms. The formula is:

$$(a+b)(a-b) = a^2 - b^2$$

Example:

Expand $(x+5)(x-5)$:

$$(x+5)(x-5) = x^2 - 5^2$$
$$= x^2 - 25$$

Key Takeaways:

- Distribute each term in the binomial product.
- Perfect squares: Expand using the formulas $(a+b)^2 = a^2 + 2ab + b^2$ and $(a-b)^2 = a^2 - 2ab + b^2$.
- Difference of squares: Use $(a+b)(a-b) = a^2 - b^2$.
- Simplify by collecting like terms when necessary.

Practice Questions

1. Simplify the following.

   a. $2\sqrt{3}\ \times\ 3\sqrt{2}$

   b. $\left(2\sqrt{3}\right)^2$

   c. $\left(9\sqrt{2}\right)^2$

2. Simplify the following.

a. $6 \times \sqrt{7} - \sqrt{7} \times 6$

b. $5\sqrt{12} - \sqrt{48}$

3. Use the distributive law $(a + b)(c + d) = ac + ad + bc + bd$ to expand and simplify these algebraic expressions.

a. $(x + 4)(x - 3)$

b. $(6x + 7)(x - 4)$

c. $(5x - 6)(5x + 6)$

d. $(3x - 7)^2$

4. Expand and simplify.

a. $(\sqrt{6} + 2)(\sqrt{6} - 1)$

b. $(\sqrt{2} - 5)(3 + \sqrt{2})$

c. $(\sqrt{7} - 4)(\sqrt{7} - 4)$

5. Expand and simplify.

a. $(6\sqrt{5} - 5)(2\sqrt{5} + 7)$

b. $(2\sqrt{7} - 3)(3 - 4\sqrt{7})$

c. $(4\sqrt{5} - 3)(3 - 4\sqrt{5})$

6. Expand and simplify these perfect squares.

a. $\left(4+\sqrt{7}\right)^2$

b. $\left(\sqrt{5}\ -\ 7\right)^2$

c. $\left(\sqrt{10}-\sqrt{3}\right)^2$

d. $\left(\sqrt{31}\ -\ \sqrt{29}\right)^2$

7. Expand and simplify these difference of perfect squares.

a. $(4 + \sqrt{3})(4 - \sqrt{3})$

b. $(\sqrt{10} - 4)(\sqrt{10} + 4)$

c. $(\sqrt{3} - \sqrt{7})(\sqrt{3} + \sqrt{7})$

8. Expand and simplify these perfect squares.

a. $\left(5\sqrt{6} + 3\sqrt{5}\right)^2$

b. $\left(3\sqrt{7} - 2\sqrt{6}\,\right)^2$

c. $\left(5\sqrt{3} - 2\sqrt{8}\right)^2$

9. Expand and simplify these difference of perfect squares.

a. $\left(4\sqrt{3} + \sqrt{7}\right)\left(4\sqrt{3} - \sqrt{7}\right)$

b. $\left(4\sqrt{10} - 5\sqrt{6}\right)\left(4\sqrt{10} + 5\sqrt{6}\right)$

c. $\left(2\sqrt{10} + 4\sqrt{5}\right)\left(2\sqrt{10} - 4\sqrt{5}\right)$

10. Use your knowledge of the simplification of surds to fully simplify the following.

a. $\left(2\sqrt{6} + 5\right)\left(\sqrt{30} - 2\sqrt{5}\right)$

b. $\left(4\sqrt{2} + \sqrt{7}\right)\left(3\sqrt{14} - 5\right)$

c. $\left(5 - 3\sqrt{2}\right)\left(2\sqrt{10} + 3\sqrt{5}\right)$

11. Fully expand and simplify these surds.

a. $\left(\sqrt{5} - \sqrt{3}\right)^2 + \left(\sqrt{5} + \sqrt{3}\right)^2$

b. $-10\sqrt{3} - \left(2\sqrt{3} - 5\right)^2$

c. $\left(2\sqrt{7} - 3\right)^2 - \left(3 - 2\sqrt{7}\right)^2$

d. $\sqrt{2}\left(2\sqrt{5} - 3\sqrt{3}\right)^2 + \left(\sqrt{6} + \sqrt{5}\right)^2$

Chapter 3.5 Rationalising the Denominator

Rationalizing the denominator involves removing square roots or other irrational numbers from the denominator of a fraction. This is typically done by multiplying both the numerator and the denominator by a number that will eliminate the radical from the denominator.

Here's how to do it:

Steps to Rationalize the Denominator:

1. When the denominator has a single square root $\left(\text{e.g.}, \frac{x}{\sqrt{y}}\right)$:

   Multiply both the numerator and the denominator by $\sqrt{y}$ to get rid of the square root in the denominator.

$$\frac{x}{\sqrt{y}} \times \frac{\sqrt{y}}{\sqrt{y}} = \frac{x\sqrt{y}}{y}$$

   Now, the denominator is a whole number, y.

2. When the denominator has a binomial with a square root $\left(\text{e.g.}, \frac{x}{\sqrt{y}+z}\right)$:

   Multiply both the numerator and the denominator by the conjugate of the denominator $(\sqrt{y} - z)$ to rationalize the denominator.

$$\frac{x}{\sqrt{y}+z} \times \frac{\sqrt{y}-z}{\sqrt{y}-z} = \frac{x(\sqrt{y}-z)}{y-z^2}$$

   Now, the denominator is a rational number.

Examples:

1. Example 1: Rationalize $\frac{5}{\sqrt{3}}$

   Multiply both the numerator and denominator by $\sqrt{3}$:

$$\frac{5}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{5\sqrt{3}}{3}$$

   Now, the denominator is rationalized.

2. Example 2: Rationalize $\frac{7}{\sqrt{2}+1}$

   Multiply both the numerator and denominator by the conjugate $\sqrt{2} - 1$:

$$\frac{7}{\sqrt{2}+1} \times \frac{\sqrt{2}-1}{\sqrt{2}-1} = \frac{7(\sqrt{2}-1)}{(\sqrt{2})^2 - 1^2}$$

   Simplifying the denominator:

$$\frac{7(\sqrt{2}-1)}{2-1} = 7(\sqrt{2}-1)$$

   The denominator is now rationalized.

Conclusion:

By multiplying by an appropriate factor (which is equivalent to 1), we can change the denominator of a fraction to a rational number while preserving the value of the expression.

Practice Questions

1. Simplify.

   a. $-\frac{7\sqrt{3}}{14\sqrt{3}}$

   b. $-\frac{3\sqrt{45}}{9\sqrt{5}}$

2. Rationalise the denominators.

a. $\frac{3}{\sqrt{11}}$

b. $\frac{\sqrt{5}}{\sqrt{3}}$

3. Rewrite in the form $\frac{\sqrt{a}}{\sqrt{b}}$ and then rationalise the denominators.

a. $\sqrt{\frac{6}{11}}$

b. $\sqrt{\frac{10}{3}}$

4. Rationalise the denominators.

a. $\frac{3\sqrt{5}}{\sqrt{2}}$

b. $\frac{2\sqrt{7}}{\sqrt{15}}$

5. Rationalise the denominators.

a. $\frac{4\sqrt{5}}{5\sqrt{10}}$

b. $\frac{7\sqrt{90}}{2\sqrt{70}}$

6. Rationalise the denominators.

a. $\dfrac{\sqrt{3}-\sqrt{5}}{\sqrt{2}}$

b. $\dfrac{\sqrt{5}+\sqrt{2}}{\sqrt{10}}$

c. $\dfrac{3\sqrt{10}+5\sqrt{3}}{\sqrt{2}}$

7. Simplify the following by first rationalising denominators and then using a common denominator.

a. $\frac{3}{\sqrt{7}} - \frac{2}{\sqrt{3}}$

b. $\frac{3}{2\sqrt{5}} + \frac{2}{5\sqrt{3}}$

c. $\frac{5\sqrt{2}}{3\sqrt{5}} - \frac{4\sqrt{7}}{3\sqrt{6}}$

8. Rationalise the denominators and simplify the following.

a. $\frac{\sqrt{2} + a}{\sqrt{6}}$

b. $\frac{\sqrt{7} - 7a}{\sqrt{7}}$

c. $\frac{2a + \sqrt{7}}{\sqrt{14}}$

9. Rationalise the denominators in the following using the difference of perfect squares.

For example: $\frac{2}{\sqrt{2}+1} = \frac{2}{\sqrt{2}+1} \times \frac{\sqrt{2}-1}{\sqrt{2}-1}$

$$= \frac{2(\sqrt{2}-1)}{(\sqrt{2}+1)(\sqrt{2}-1)}$$

$$= \frac{2(\sqrt{2}-2)}{2-1} = 2\sqrt{2} - 2$$

a. $\frac{3}{1-\sqrt{3}}$

b. $\frac{6}{\sqrt{2}+\sqrt{5}}$

c. $\frac{2\sqrt{5}}{\sqrt{5}+2}$

d. $\frac{\sqrt{ab}}{\sqrt{a}-\sqrt{b}}$

Chapter 3.6 Review of Index Laws

Index Laws (Exponential Rules)

These are the rules for working with exponents (indices), which govern how powers and roots interact.

Law 1: $a^m \times a^n = a^{m+n}$

When multiplying powers with the same base, keep the base and add the exponents (indices).

- Example:

$$a^3 \times a^2 = a^{3+2} = a^5$$

Law 2: $\frac{a^m}{a^n} = a^{m-n}$

When dividing powers with the same base, keep the base and subtract the exponents (indices).

- Example:

$$\frac{a^5}{a^2} = a^{5-2} = a^3$$

Law 3: $(a^m)^n = a^{m \times n}$

When raising a power to another power, keep the base and multiply the exponents (indices).

- Example:

$$(a^2)^3 = a^{2\times 3} = a^6$$

Law 4: $a^m \times b^m = (a \times b)^m$

When multiplying terms with the same exponent, multiply the bases together and apply the exponent.

- Example:

$$a^2 \times b^2 = (a \times b)^2$$

Law 5: $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$

When raising a fraction to an exponent, apply the exponent to both the numerator and the denominator.

- Example:

$$\left(\frac{a}{b}\right)^3 = \frac{a^3}{b^3}$$

Special Exponent Rule: The Zero Power

- $a^0 = 1$ (for any non-zero base a)
- Any number raised to the power of 0 is equal to 1.
- Example:

$$5^0 = 1 \quad \text{(as long as 5 is not 0)}$$

Examples of Applying Index Laws:

1. Multiplying with the same base:

$$2^4 \times 2^3 = 2^{4+3} = 2^7 = 128$$

2. Dividing with the same base:

$$\frac{3^5}{3^2} = 3^{5-2} = 3^3 = 27$$

3. Raising a power to a power:

$$(4^2)^3 = 4^{2\times3} = 4^6 = 4096$$

4. Multiplying terms with the same exponent:

$$2^3 \times 3^3 = (2 \times 3)^3 = 6^3 = 216$$

5. Raising a fraction to a power:

$$\left(\frac{2}{3}\right)^2 = \frac{2^2}{3^2} = \frac{4}{9}$$

Summary of Key Rules:

- **Add exponents** when multiplying powers with the same base.
- **Subtract exponents** when dividing powers with the same base.
- **Multiply exponents** when raising a power to another power.
- **Distribute exponents** when multiplying terms with the same exponent or when raising a fraction to a power.
- Any number to the power of 0 equals 1.

Practice Questions

1. Simplify using the first index law.

a. $b \times b^5$

b. $t^8 \times 2t^8$

c. $\frac{3}{5}s \times \frac{3s}{5}$

d. $3v^7w \times 6v^2w$

e. $4m^6n^7 \times mn \times 5mn^2$

2. Simplify using the second index law.

a. $q^9 \div q^6$

b. $\frac{d^8}{d^3}$

c. $2x^2y^3 \div x$

d. $16m^7x^5 \div (8m^3x^4)$

e. $\frac{2v^5}{8v^3}$

f. $\frac{-8x^2y^3}{16x^2y}$

3. Simplify using the third, fourth and fifth index laws.

a. $5(y^5)^3$

b. $(3p^4)^4$

c. $\left(\frac{u^4w^2}{v^2}\right)^4$

d. $\left(\frac{4p^2q^3}{3r}\right)^4$

4. Evaluate the following using the zero power.

a. $(10ab^2)^0$

b. $7x^0 - 4(2y)^0$

5. Use appropriate index laws to simplify the following.

a. $x^2y \div (xy) \times xy^2$

b. $\dfrac{x^2y^3 \times x^2y^4}{x^3y^5}$

c. $\dfrac{r^4s^7 \times r^4s^7}{r^4s^7}$

d. $\dfrac{4x^2y^3 \times 12x^2y^2}{24x^4y}$

e. $(3m^2n^4)^3 \times mn^2$

f. $(4f^2g)^2 \times f^2g^4 \div (3(fg^2)^3)$

g. $\frac{(7y^2z)^2 \times 3yz^2}{7(yz)^2}$

h. $\frac{(2m^3)^2}{3(mn^4)^0} \times \frac{(6n^5)^2}{(-2n)^3m^4}$

6. Evaluate without the use of a calculator.

a. $\frac{9^8}{9^6}$

b. $\frac{4^{10}}{4^7}$

c. $\frac{27^2}{3^4}$

d. $\frac{32^2}{2^7}$

Chapter 3.7 Negative Indices

Practice Questions

1. Express the following using positive indices.

   a. $3y^{-7}$

   b. $3x^{-4}y^{-2}z^{3}$

   c. $\frac{2}{5}b^{3}c^{-5}d^{-2}$

2. Express the following using positive indices.

a. $\frac{3}{b^{-5}}$

b. $\frac{5h^3}{2g^{-3}}$

3. Use index laws 1 and 2 to simplify the following. Write your answers using positive indices.

a. $3y^{-6} \times y^{3}$

b. $-3a^{-7} \times 6a^{-3}$

c. $\dfrac{3b^{-2}}{4b^{-4}}$

d. $\dfrac{15t^{-4}}{18t^{-2}}$

4. Express the following with positive indices.

a. $4(d^{-2})^3$

b. $-8(x^5)^{-3}$

c. $2(t^{-3})^{-2}$

5. Express the following in simplest form with positive indices.

a. $6a^4b^3 \times 3a^{-6}b$

b. $\dfrac{m^3n^2}{mn^3}$

c. $\dfrac{12r^4s^6}{9rs^{-1}}$

d. $\dfrac{15c^3d}{12c^{-2}d^{-3}}$

6. Simplify the following and express your answers with positive indices.

a. $(2p^2)^4 \times (3p^2q)^{-2}$

b. $2(x^2y^{-1})^2 \times (3xy^4)^3$

c. $\dfrac{(3rs^2)^4}{r^{-3}s^4} \times \dfrac{(2r^2s)^2}{s^7}$

d. $\dfrac{4(x^{-2}y^4)^2}{x^2y^{-3}} \times \dfrac{xy^4}{2x^{-2}y}$

e. $\left(\dfrac{m^4n^{-2}}{r^3}\right)^2 \div \left(\dfrac{m^{-3}n^2}{r^3}\right)^2$

f. $\dfrac{3(x^2y^{-4})^2}{2(xy^2)^2} \div \dfrac{(xy)^{-3}}{(3x^{-2}y^4)^2}$

7. Evaluate without the use of a calculator.

a. $5 \times (-3^4)$

b. $\frac{-3}{4^{-2}}$

c. $\frac{(10^{-4})^{-2}}{(10^{-2})^{-3}}$

8. Find the value of x.

a. $3^x = \frac{1}{27}$

b. $\left(\frac{3}{4}\right)^x = \frac{4}{3}$

c. $\frac{1}{2^x} = 8$

d. $\frac{1}{3^x} = 81$

e. $3^{x-3} = \frac{1}{9}$

f. $10^{x-5} = \frac{1}{1000}$

g. $\left(\frac{3}{2}\right)^{3x+2} = \frac{16}{81}$

h. $\left(\frac{7}{4}\right)^{1-x} = \frac{4}{7}$

Chapter 3.8 Scientific Notation

Scientific Notation

Scientific notation is a way of expressing very large or very small numbers in a compact form.

The general form of scientific notation is:

$$a \times 10^m$$

where:

- $1 \leq a < 10$ is the coefficient (a number between 1 and 10),
- m is the exponent (an integer), and
- 10^m indicates the power of 10.

Examples of Scientific Notation:

- Large Numbers:
  - $24{,}800{,}000 = 2.48 \times 10^7$
  - $9{,}020{,}000{,}000 = 9.02 \times 10^9$

- Small Numbers:
  - $0.00307 = 3.07 \times 10^{-3}$
  - $0.0000012 = 1.2 \times 10^{-6}$

Significant Figures

Significant figures (also known as significant digits) are the digits in a number that are meaningful in terms of precision. The rules for determining significant figures are:

1. **Non-zero digits** are always significant.
   - Example: 456 has 3 significant figures.
2. **Any zeros between non-zero digits** are significant.
   - Example: 2003 has 4 significant figures.
3. **Leading zeros** (zeros before the first non-zero digit) are not significant.
   - Example: 0.0032 has 2 significant figures.
4. **Trailing zeros** in a decimal number are significant.
   - Example: 2.300 has 4 significant figures.
5. **Trailing zeros in a whole number** without a decimal point are **not necessarily significant**.
   - Example: 3000 has 1 significant figure unless written as 3.000×10^3, which would have 4 significant figures.

Scientific Notation and Significant Figures

When a number is written in scientific notation, the digits in the coefficient are considered the significant figures. For example:

- $20{,}190{,}000 = 2.019 \times 10^7$ has **4 significant figures** (2, 0, 1, and 9).
- 3.070×10^5 has **4 significant figures** (3, 0, 7, and 0).

Using Scientific Notation on Calculators:

Many scientific calculators have the "EE" or "Exp" key, which allows you to input numbers in scientific notation. For example:

- 2.3×10^{-4} can be written as $2.3E - 4$ on a calculator.
  This means 2.3×10^{-4}, or 0.00023.

Summary of Key Points:

- **Scientific notation** expresses numbers in the form $a \times 10^m$, where $1 \leq a < 10$ and m is an integer.
- **Significant figures** count the meaningful digits in a number, starting from the leftmost non-zero digit.
- **Scientific notation** preserves significant figures, especially the digits in the coefficient a.

Practice Questions

1. Write these numbers as a basic numeral.

   a. 7.105×10^5

   b. 8.002×10^5

2. Write these numbers as a basic numeral.

   a. 3.085×10^{-4}

   b. 2.65×10^{-1}

3. Write these numbers in scientific notation using 3 significant figures.

a. $30\ 248$

b. $34\ 971\ 863$

c. $110\ 438\ 523$

d. $9\ 826\ 100\ 005$

4. Write in scientific notation using the number of significant figures given in the brackets.

   a. 703 780 030 (2)

   b. 0.00070507 (3)

   c. 0.000050034 (3)

Chapter 3.9 Rational Indices

Rational Indices (Fractional Indices)

Rational indices, or fractional exponents, can be understood as roots. Here are the key rules:

1. **Fractional Indices as Roots:**

A fractional index is another way to express a root. The general form is:

$$a^{\frac{m}{n}} = \sqrt[n]{a^m}$$

This means that the expression $a^{\frac{m}{n}}$ represents the nth root of a^m, or equivalently, a^m raised to the power of $\frac{1}{n}$.

Examples:

- $3^{\frac{1}{2}} = \sqrt{3}$ (the square root of 3),
- $5^{\frac{1}{3}} = \sqrt[3]{5}$ (the cube root of 5),
- $7^{\frac{1}{10}} = \sqrt[10]{7}$ (the 10th root of 7).

2. **Simplifying Rational Indices:**

You can manipulate rational indices as follows:

- $a^{\frac{m}{n}} = (a^m)^{\frac{1}{n}} = \sqrt[n]{a^m}$
- Alternatively: $a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m$

For example:

- $8^{\frac{2}{3}} = (8^2)^{\frac{1}{3}} = \sqrt[3]{8^2} = \sqrt[3]{64} = 4$
- $8^{\frac{1}{3}} = \sqrt[3]{8} = 2$
- $8^{\frac{2}{3}} = 2^2 = 4$

3. Applying Index Laws to Rational Indices:

The same index laws that apply to integer exponents also apply to rational exponents.

- Multiplication: $a^{\frac{m}{n}} \times a^{\frac{p}{n}} = a^{\frac{m+p}{n}}$
  - For example: $8^{\frac{1}{3}} \times 8^{\frac{2}{3}} = 8^{\frac{3}{3}} = 8^1 = 8$
- Division: $a^{\frac{m}{n}} \div a^{\frac{p}{n}} = a^{\frac{m-p}{n}}$
  - For example: $8^{\frac{2}{3}} \div 8^{\frac{1}{3}} = 8^{\frac{1}{3}} = 2$
- Power of a Power: $\left(a^{\frac{m}{n}}\right)^p = a^{\frac{m}{n} \times p} = a^{\frac{mp}{n}}$
  - For example: $\left(8^{\frac{1}{3}}\right)^2 = 8^{\frac{2}{3}} = 4$

Summary of Key Concepts:

Fractional Exponents are equivalent to roots. The fraction $\frac{m}{n}$ means the nth root of a^m.

The Index Laws apply to fractional indices just as they do to integer exponents.

Practice Questions

1. Express the following in index form.

   a. $\sqrt[4]{b^3}$

   b. $\sqrt[8]{8m^4}$

2. Express the following in index form.

a. $3\sqrt[4]{y^{12}}$

b. $2\sqrt[4]{g^{3}h^{5}}$

c. $4\sqrt[3]{4}$

3. Express the following in surd form.

a. $11^{\frac{1}{10}}$

b. $3^{\frac{4}{7}}$

4. Evaluate without using a calculator.

a. $64^{\frac{1}{3}}$

b. $125^{\frac{1}{3}}$

c. $81^{-\frac{1}{4}}$

d. $10000^{-\frac{1}{4}}$

5. Evaluate without using a calculator.

a. $16^{\frac{5}{4}}$

b. $25^{-\frac{3}{2}}$

c. $\frac{10}{100^{\frac{3}{2}}}$

6. Use index laws to simplify the following.

a. $b^{\frac{5}{4}}\ b^{\frac{3}{4}}$

b. $\left(\frac{a^{\frac{2}{3}}}{b^{\frac{4}{3}}}\right)^{\frac{3}{4}}$

7. Simplify the following.

a. $(343r^9t^6)^{\frac{1}{3}}$

b. $\left(\frac{10^2x^4}{0.01}\right)^{\frac{1}{4}}$

Chapter 3.10 Exponential Equations

Practice Questions

1. Solve for x in each of the following.

a. $6^x = 36$

b. $4^x = 64$

c. $5^x = 625$

d. $7^x = 343$

2. Solve for x in each of the following.

a. $11^x = \frac{1}{121}$

b. $5^{-x} = \frac{1}{125}$

c. $7^{-x} = \frac{1}{343}$

3. Solve for x in each of the following.

a. $25^x = 125$

b. $216^x = 6$

c. $7^{-x} = 49$

d. $25^{-x} = 125$

4. Solve for x in each of the following.

a. $7^{x + 9} = 49^{2x}$

b. $9^{x + 12} = 81^{x + 5}$

c. $32^{2x + 3} = 128^{2x}$

d. $49^{2x - 3} = 343^{2x - 1}$

5. Recall that $\sqrt{x} = x^{\frac{1}{2}}$ and $\sqrt[3]{x} = x^{\frac{1}{3}}$. Now solve the following.

a. $6^x = \sqrt[3]{36}$

b. $4^x = \sqrt[4]{64}$

c. $25^x = \sqrt[5]{125}$

d. $9^x = \frac{1}{\sqrt[3]{27}}$

6. Solve for n in the following.

a. $5^{3n} \times 25^{-2n+1} = 125$

b. $2^{-3n} \times 4^{2n-2} = 16$

c. $7^{2n+1} = \frac{1}{49}$

d. $5^{3n+2} = \frac{1}{625}$

e. $11^{3n-1} = 11$

f. $8^{5n-1} = 1$

g. $\frac{5^{3n-3}}{25^{n-3}} = 125$

h. $\frac{36^{3+2n}}{6^{n}} = 1$

Chapter 3.11 Exponential Growth and Compound Interest

Exponential Growth and Decay

Exponential growth and decay are modeled using the equation:

$$A = ka^t$$

Where:

- A is the amount at time t
- k is the initial amount
- a is the base (which determines growth or decay)
- t is the time

Growth and Decay Behavior:

- Exponential Growth: If $a > 1$, the quantity grows over time (growth).
- Exponential Decay: If $0 < a < 1$, the quantity decreases over time (decay).

Calculating the Base for Growth and Decay:

- For a growth rate of $r\%$ per annum, the base a is calculated as:

  $$a = 1 + \frac{r}{100}$$

  where r is the growth rate percentage.

- For a decay rate of $r\%$ per annum, the base a is calculated as:

  $$a = 1 - \frac{r}{100}$$

  where r is the decay rate percentage.

Compound Interest Formula:

The formula for compound interest is:

$$A = A_0\left(1 + \frac{r}{100}\right)^t$$

Where:

- A_0 is the initial amount (principal)
- r is the annual interest rate (in percentage)
- t is the time (in years or other periods)
- A is the amount after t years

Practice Questions

1. Define variables and form exponential rules for the following situations.

   a. $\$200000$ is invested at 17% per annum.

   b. A house initially valued at $\$530\ 000$ is losing value at 5% per annum.

   c. The value of a car, bought for $\$14200$, is decreasing at 3% per annum.

   d. A population, initially $172\ 500$, is increasing at 15% per year.

e. A tank with 1200 litres of water is leaking at a rate of 10% of the water in the tank every hour.

f. A cell of area $0.01\ \mathrm{cm}^2$ doubles its size every minute.

g. An oil spill, initially covering an area of 2 square metres, is increasing at 5% per minute.

h. A substance of mass $30\ \mathrm{g}$ is decaying at a rate of 8% per hour.

2. A share portfolio initially worth $\$300\ 000$ is reduced by 15% p.a. over a number of years. Let $\$A$ be the share portfolio value after t years.

a. Copy and complete the rule connecting A and t.

A = ______ $\times\ 0.85^t$

b. Use your rule to find the value of the shares after the following number of years. Round to the nearest cent.

i. 2 years

ii. 7 years

iii. 12 years

c. Use trial and error to estimate when the share portfolio will be valued at $\$180\ 000$. Round to one decimal place.

3. A water tank containing $15\ 000$ L has a small hole that reduces the amount of water by 6% per hour.

   a. Determine a rule for the volume of water (V) left after t hours.

   b. Calculate (to the nearest litre) the amount of water left in the tank after:

      i. 3 hours

      ii. 7 hours

4. Megan invests $\$50\ 000$ in a superannuation scheme that has an annual return of 11%.

a. Determine the rule for the value of her investment (V) after n years.

b. How much will Megan's investment be worth in:

i. 4 years?

ii. 20 years?

c. Find the approximate time before her investment is worth $\$100\ 000$. Round to two decimal places.

CHAPTER 4 TRIGONOMETRY

Table of Contents

Chapter 4.1 Trigonometric Ratios

Trigonometric Ratios in a Right-Angled Triangle

In a right-angled triangle, given an angle θ, the three main trigonometric ratios are:

1. **Sine (sin)** of angle θ:

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

Where:

- Opposite is the side opposite to angle θ
- Hypotenuse is the longest side, opposite the right angle

2. **Cosine (cos)** of angle θ:

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

Where:

- Adjacent is the side adjacent to angle θ
- Hypotenuse is the longest side, opposite the right angle

3. **Tangent (tan)** of angle θ:

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

Where:

- Opposite is the side opposite to angle θ
- Adjacent is the side next to angle θ

SOHCAHTOA:

- **SOH**: Sine = Opposite / Hypotenuse
- **CAH**: Cosine = Adjacent / Hypotenuse
- **TOA**: Tangent = Opposite / Adjacent

Finding Unknown Side Lengths:

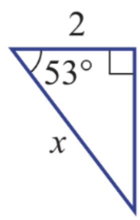
To find an unknown length on a right-angled triangle, follow these steps:

1. **Choose the appropriate trigonometric ratio** that links the known angle and a known side length with the unknown side length.
2. **Solve for the unknown side** using the chosen trigonometric formula.

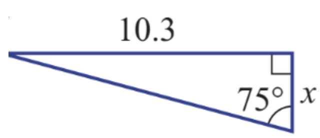
Practice Questions

1. Decide which ratio ($\sin\theta = \frac{O}{H}$, $\cos\theta = \frac{A}{H}$ or $\tan\theta = \frac{O}{A}$) would be best to help find the value of x in these triangles. Do not find the value of x.

a.



b.



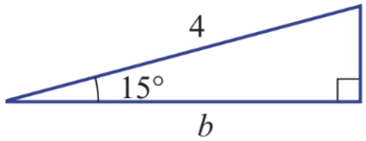
2. Solve for x in these equations correct to two decimal places.

a. $\frac{x}{12.7} = \sin 15.6°$

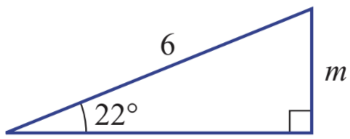
b. $\tan 71.6° = \frac{37.5}{x}$

3. Use trigonometric ratios to find the values of the pronumerals to two decimal places.

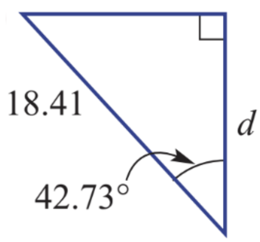
a.



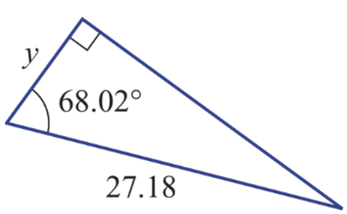
b.



c.

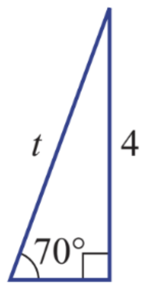


d.

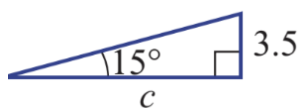


4. Use trigonometric ratios to find the values of the pronumerals to two decimal places for these right-angled triangles.

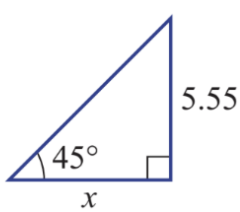
a.



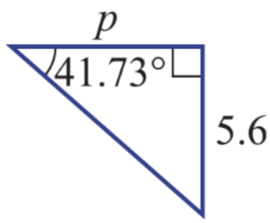
b.



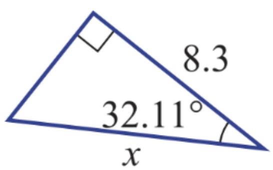
c.



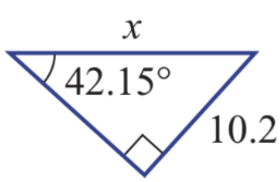
d.



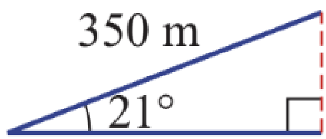
e.



f.



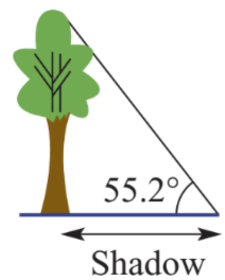
5. A 4WD climbs a $350m$ straight slope at an angle of $21°$ to the horizontal.



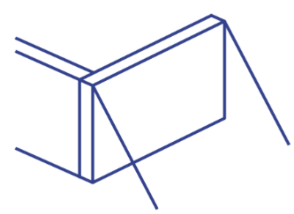
a. Find the vertical distance travelled correct to the nearest metre.

b. Find the horizontal distance travelled correct to the nearest metre.

6. The angle, from the horizontal of the line of sight, from the end of a tree's shadow to the top of the tree is $55.2°$. The length of the shadow is 15.5m. Find the height of the tree correct to one decimal place.

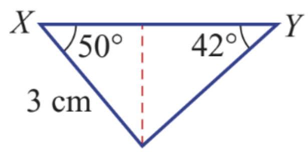


7. On a construction site, large concrete slabs of height 5.6 metres are supported at the top by steel beams positioned at an angle of $42°$ from the vertical. Find the length of the steel beams to two decimal places.

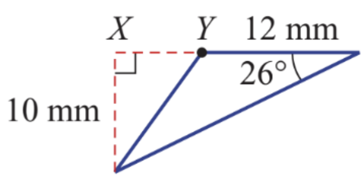


8. Find the length XY in these diagrams correct to one decimal place.

a.

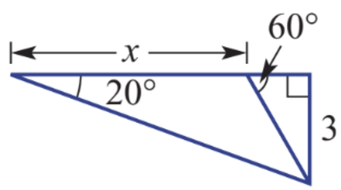


b.

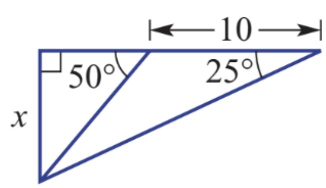


9. Find the value of x, correct to one decimal place, in these triangles.

a.



b.



Chapter 4.2 Finding Angles

Inverse Trigonometric Functions

Inverse trigonometric functions are used to find the angle θ when the value of a trigonometric ratio is given. They are the reverse of the standard trigonometric functions (sine, cosine, and tangent).

The general form for the inverse trigonometric functions is:

- **Inverse sine ($\sin^{-1}$)**:
  If $\sin\theta = k$, then:
  $$\theta = \sin^{-1}(k)$$
- **Inverse cosine ($\cos^{-1}$)**:
  If $\cos\theta = k$, then:
  $$\theta = \cos^{-1}(k)$$
- **Inverse tangent ($\tan^{-1}$)**:
  If $\tan\theta = k$, then:
  $$\theta = \tan^{-1}(k)$$

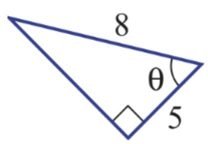
Practice Questions

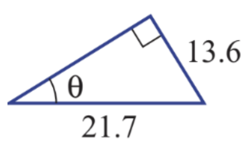
1. Find θ in the following, rounding to two decimal places where necessary.

   a. $\tan\theta = 0.2$

   b. $\sin\theta = 0.25$

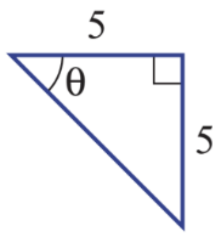
2. Decide which trigonometric ratio (sine, cosine or tangent) would be used to find θ in these triangles.

a. 

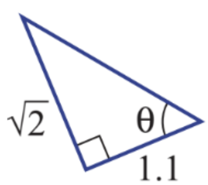
b. 

3. Find the value of θ in the following right-angled triangles, rounding to two decimal places where necessary.

a.

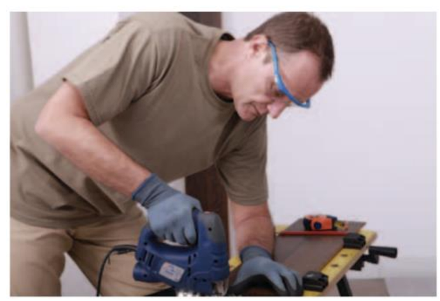


b.

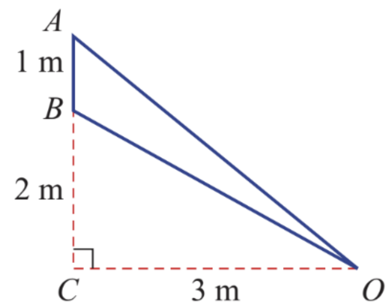


4. A ladder reaches 5.5m up a wall and sits 2m from the base of the wall. Find the angle the ladder makes with the horizontal, correct to two decimal places.

5. A diagonal cut of length 2.85 metres is to be made on a rectangular wooden slab from one corner to the other. The front of the slab measures 1.94 metres. Calculate the angle with the front edge at which the carpenter needs to begin the cut. Round to one decimal place.



6. Consider ΔOAC and ΔOBC.



a. Find, correct to one decimal place where necessary:

i. $\angle AOC$

ii. $\angle BOC$

b. Hence find the angle $\angle AOB$.

Chapter 4.3 Applications in Two Dimensions

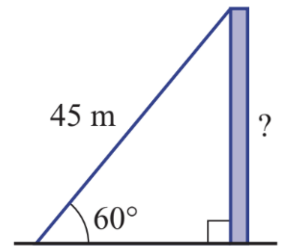
Angle of Elevation and Angle of Depression

1. Angle of Elevation:
    - The angle of elevation is the angle measured upwards from the horizontal line to an object or point. It is the angle formed between the horizontal ground (or line of sight) and the line of sight to the object above.

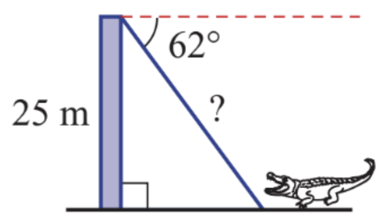
2. Angle of Depression:
    - The angle of depression is the angle measured downwards from the horizontal line to an object or point. It is the angle formed between the horizontal line of sight and the line of sight to the object below.

Practice Questions

1. A cable of length 45 metres is anchored from the ground to the top of a communications mast. The angle of elevation of the cable to the top of the mast is $60°$. Find the height of the communications mast, to the nearest metre.

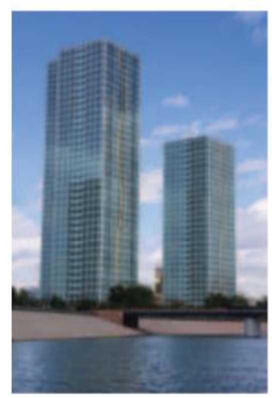


2. The angle of depression from the top of a 25 metre high viewing tower to a crocodile on the ground is $62°$. Find the direct distance from the top of the tower to the crocodile, to the nearest cm.

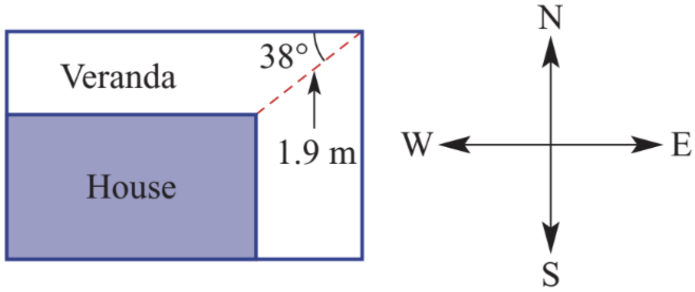


3. The distance between two buildings is 24.5 metres. Find the height of the taller building, to the nearest metre, if the angle of elevation from the base of the shorter building to the top of the taller building is $85°$ and the height of the shorter building is $40\ \mathrm{m}$.

4. Two vertical buildings 91 metres apart are 136 metres and 192 metres high respectively. Find the angle of elevation from the top of the shorter building to the top of the taller building to the nearest degree.



5. An L-shaped veranda has dimensions as shown. Find the width, to the nearest centimetre, of the veranda for the following sides of the house.



a. north side

b. east side

6. Initially a ship and a submarine are stationary at sea level 1.78km apart. The submarine then manoeuvres to position A, 45 metres directly below its starting point. In a second manoeuvre the submarine dives a further 62 metres to position B.

Give all answers to two decimal places.

a. Find the angle of elevation of the ship from the submarine when the submarine is at position A.

b. Find the angle of elevation of the ship from the submarine when the submarine is at position B.

c. Find the difference in the angles of elevation from the submarine to the ship when the submarine is at positions A and B.

Chapter 4.4 Bearings

True Bearings and Directions

1. **True Bearings**:
   - **True bearings** are used to specify a direction from a reference point, typically from **due north**. Bearings are measured **clockwise** from due north ($0°$) and are usually written in **three digits** (e.g., $045°$ for northeast).
   - The full range of true bearings is from $\mathbf{0°}$ **to** $\mathbf{360°}$, where:
     - $0°$ represents **due north**,
     - $90°$ represents **due east**,
     - $180°$ represents **due south**,
     - $270°$ represents **due west**.

2. **Example Directions**:
   - **Northeast (NE)**: A bearing of $\mathbf{045°}$. It's $45°$ clockwise from due north.
   - **Southeast (SE)**: A bearing of $\mathbf{135°}$. It's $45°$ clockwise from due east (or 135° from due north).
   - **Southwest (SW)**: A bearing of $\mathbf{225°}$. It's $45°$ clockwise from due south.
   - **Northwest (NW)**: A bearing of $\mathbf{315°}$. It's $45°$ clockwise from due west.

3. **Opposite Directions**:
   - Opposite directions (such as north and south, east and west) are separated by $\mathbf{180°}$. For example:
     - The opposite of a bearing of $\mathbf{045°}$ (northeast) is $\mathbf{225°}$ (southwest).
     - The opposite of a bearing of $\mathbf{090°}$ (due east) is $\mathbf{270°}$ (due west).

Practice Questions

1. A bushwalker hikes due north from a resting place for 1.5km to a waterhole and then on a true bearing of $315°$ for 2km to base camp.

   a. Find how far west the base camp is from the waterhole, to the nearest metre.

   b. Find how far north the base camp is from the waterhole, to the nearest metre.

   c. Find how far north, the base camp is from the initial resting place to the nearest metre.

2. A military desert tank manoeuvres 13.5km from point A on a true bearing of $042°$ to point B. From point B, how far due south will the tank need to travel to be at a point due east of point A. Give the answer correct to the nearest metre.

3. An overall direction and distance of a journey can be calculated by considering two (or more) smaller parts (or legs). Find the bearing of C from A and the length AC in this journey by answering these parts.

   a. Find correct to two decimal places where necessary how far north:

      i. point B is from A

      ii. point C is from B

iii. point C is from A

b. Find correct to two decimal places how far east:

i. point B is from A

ii. point C is from B

iii. point C is from A

4. Tour groups A and B view a rock feature from different positions on a road heading east–west. Group A views the rock at a distance of $235\mathrm{m}$ on a bearing of $155°$ while group B views the rock feature on a bearing of $162°$ at a different point on the road. Find the following, rounding all answers to two decimal places.

   a. Find how far south the rock feature is from the road.

   b. Find how far east the rock feature is from:

      i. group A

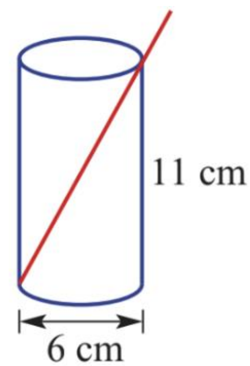
ii. group B

c. Find the distance between group A and group B.

Chapter 4.5 Applications in Three Dimensions

Practice Questions

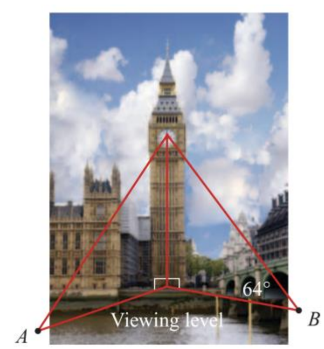
1. Find the angle of elevation this red drinking straw makes with the base of the can with diameter $6\ \mathrm{cm}$ and height $11\ \mathrm{cm}$. Round to one decimal place.



2. A vertical mast is supported at the top by two cables reaching from two points, A and B. The cable reaching from point A is 43 metres long and is at an angle of $61°$ to the horizontal. Point B is 37 metres from the base of the mast.

a. Find the height of the mast correct to three decimal places.

b. Find the angle to the horizontal of the cable reaching from point B to two decimal places.



3. Viewing points A and B are at a horizontal distance from a clock tower of 36 metres and 28 metres respectively. The viewing angle to the clock face at point B is $64°$.

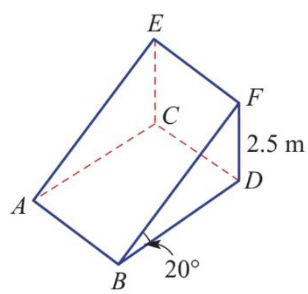
   a. Find the height of the clock face above the viewing level, to three decimal places.

   b. Find the viewing angle to the clock face at point A, to two decimal places.

4. A ramp $ABCDEF$ rests at an angle of $20°$ to the horizontal and the highest point on the ramp is 2.5 metres above the ground, as shown. Use two decimal places in the following questions.

   a. Find the length of the ramp BF.

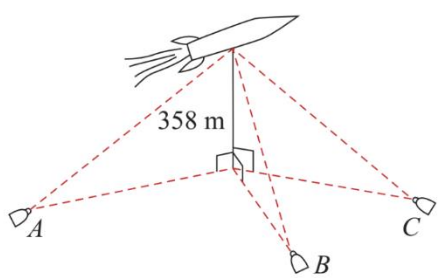
   b. Find the length of the horizontal BD.



5. Three cameras operated at ground level view a rocket being launched into space. At 5 seconds immediately after launch, the rocket is 358m above ground level and the three cameras A, B and C are positioned at an angle of $28°, 32°$ and $36°$ respectively to the horizontal. At the 5 second mark find:

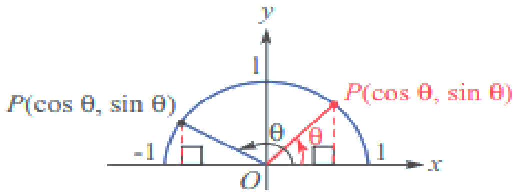
   a. which camera is closest to the rocket

   b. the distance between the rocket and the closest camera, to the nearest centimetre.



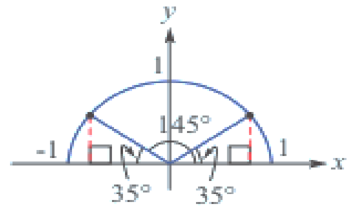
Chapter 4.6 Obtuse Angles and Exact Values

- The **unit circle** is a circle of radius 1 unit.
- The unit circle is used to define $\cos\theta$, $\sin\theta$ and $\tan\theta$ for all angles θ.
  - A point $P(x, y)$ is a point on the unit circle defined by an angle θ measured anticlockwise from the positive x-axis.
  - **$\cos\theta$ is the x-coordinate of P**
  - **$\sin\theta$ is the y-coordinate of P**
  - **$\tan\theta = \dfrac{x}{y}$**

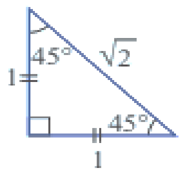


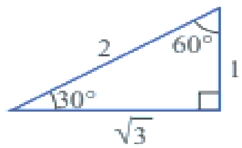
- For supplementary angles θ and $180° - \theta$:
  - $\cos(180° - \theta) = -\cos\theta$
  - $\sin(180° - \theta) = \sin\theta$
  - $\tan(180° - \theta) = -\tan\theta$

  For example:
  - $\cos 145° = -\cos 35°$
  - $\sin 145° = \sin 35°$
  - $\tan 145° = -\tan 35°$



- For acute angles ($0° < \theta < 90°$) all three trigonometric ratios are positive.
- For obtuse angles ($90° < \theta < 180°$) $\sin\theta$ is positive, $\cos\theta$ is negative and $\tan\theta$ is negative.
- Exact values for $\sin\theta$, $\cos\theta$ and $\tan\theta$ can be obtained using two special triangles. Pythagoras' theorem can be used to confirm the length of each side.





$\sin 45° = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$ $\quad$ $\sin 30° = \dfrac{1}{2}$ $\quad$ $\sin 60° = \dfrac{\sqrt{3}}{2}$

$\cos 45° = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$ $\quad$ $\cos 30° = \dfrac{\sqrt{3}}{2}$ $\quad$ $\cos 60° = \dfrac{1}{2}$

$\tan 45° = \dfrac{1}{1} = 1$ $\quad$ $\tan 30° = \dfrac{1}{\sqrt{3}} = \dfrac{\sqrt{3}}{3}$ $\quad$ $\tan 60° = \dfrac{\sqrt{3}}{1} = \sqrt{3}$

- Exact values for $\sin\theta$, $\cos\theta$ and $\tan\theta$ for angles of 30°, 45°, 60° and 90° are given in this table.

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
0°	0	1	0
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90°	1	0	Undefined

Practice Questions

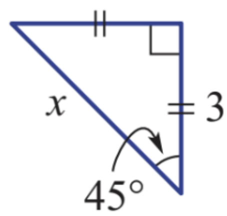
1. State the two values of θ if $0° < \theta < 180°$.

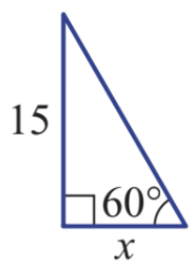
   a. $\sin\theta = \frac{1}{2}$

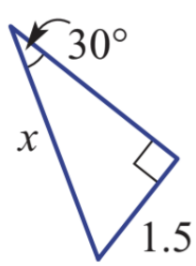
   b. $\sin\theta = \frac{\sqrt{2}}{2}$

   c. $\sin\theta = \frac{\sqrt{3}}{2}$

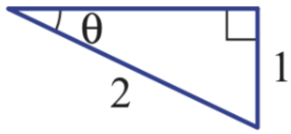
2. Use trigonometric ratios to find the exact value of x. Calculators are not required.

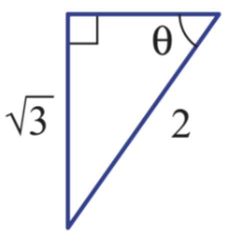
a. 

b. 

c. 

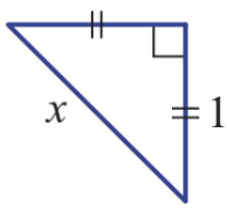
3. Find the exact value of θ without the use of a calculator.

a. 

b. 

4. Use Pythagoras' theorem to find the exact value of x in these special triangles.

a.



5. If $0° < \theta < 90°$ find $\tan\theta$ when:

a. $\sin\theta = \frac{1}{\sqrt{10}}$ and $\cos\theta = \frac{3}{\sqrt{10}}$

b. $\sin\theta = \frac{2}{\sqrt{11}}$ and $\cos\theta = \frac{\sqrt{7}}{\sqrt{11}}$

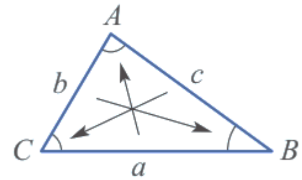
6. If $90° < \theta < 180°$ find $\tan\theta$ when:

a. $\sin\theta = \frac{5}{\sqrt{34}}$ and $\cos\theta = \frac{-3}{\sqrt{34}}$

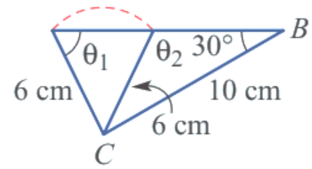
b. $\sin\theta = \frac{\sqrt{20}}{6}$ and $\cos\theta = -\frac{2}{3}$

Chapter 4.7 The Sine Rule

- When using the sine rule, label triangles with capital letters for vertices and the corresponding lower case letter for the side opposite the angle.



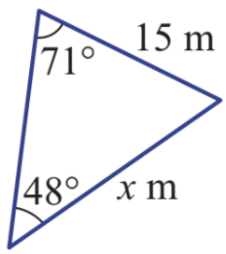
- The **sine rule** states that the ratios of each side of a triangle to the sine of the opposite angle are equal. $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$
  - The sine rule holds true for both acute- and obtuse-angled triangles.
- The **ambiguous case** arises when we are given two sides and an angle that is not the included angle.
  - This example shows a diagram with two given side lengths and one angle. Two triangles are possible.
  - Using $\dfrac{6}{\sin 30} = \dfrac{10}{\sin \theta}$ could give two results for θ (θ_1 or θ_2). You will need to choose the correct angle (acute or obtuse) to suit your triangle.



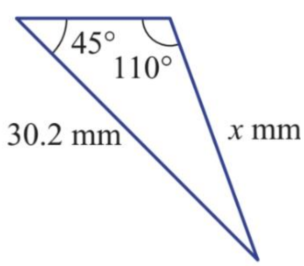
Practice Questions

1. Find the value of x in these triangles correct to one decimal place.

a.

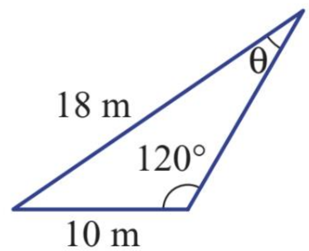


b.

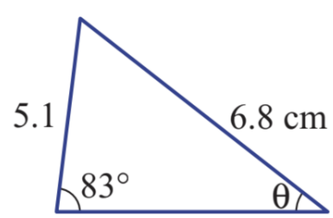


2. Find the value of θ correct to one decimal place if θ is acute.

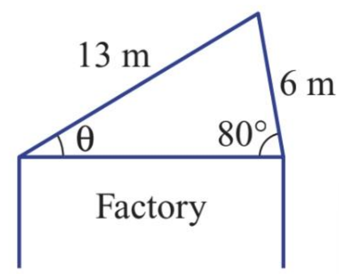
a.



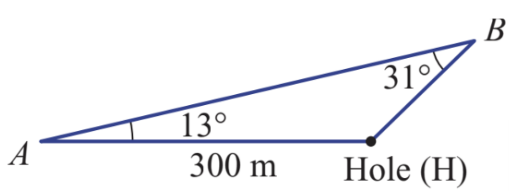
b.



3. A factory roof has a steep 6m section at $80°$ to the horizontal and another 13m section. What is the angle of elevation of the 13m section of roof? Answer to one decimal place.



4. A golf ball is hit off-course by $13°$ to point B. The shortest distance to the hole is $300m$ and the angle formed by the new ball position is $31°$ as shown.
Find the new distance to the hole (BH) correct to one decimal place.



5. A ship heads due north from point A for $40\ \mathrm{km}$ to point B, and then heads on a true bearing of $100°$ to point C. The bearing from C to A is $240°$.

   a. Find $\angle ABC$.

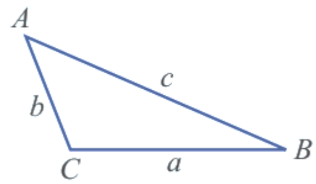
   b. Find the distance from A to C correct to one decimal place.

   c. Find the distance from B to C correct to one decimal place.

Chapter 4.8 The Cosine Rule

- The **cosine rule** relates one angle and three sides of any triangle.
- The cosine rule is used to find:
  - the third side of a triangle given two sides and the included angle
  - an angle given three sides.

$$c^2 = a^2 + b^2 - 2ab\cos C$$

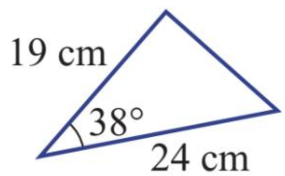


- If θ is obtuse, then note that $\cos\theta$ is negative.

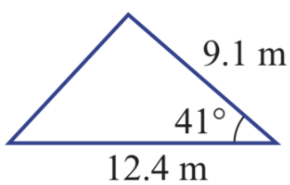
Practice Questions

1. Find the length of the third side correct to two decimal places.

   a.

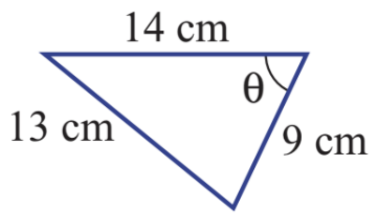


   b.

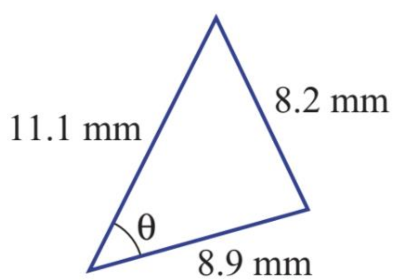


2. Find the angle θ correct to two decimal places.

a.

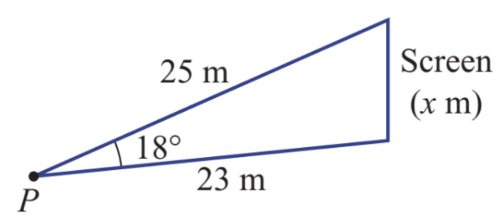


b.



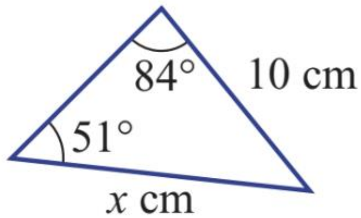
3. Find the size of all three angles in a triangle that has side lengths 10m, 7m and 13m. Round each angle to one decimal place.

4. The viewing angle to a vertical screen is $18°$ and the distances between the viewing point P and the top and bottom of the screen are 25m and 23m respectively. Find the height of the screen (x m) correct to the nearest centimetre.

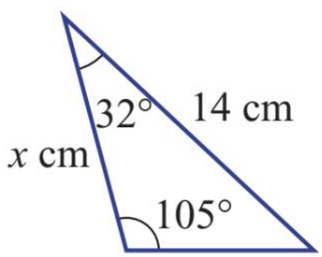


5. Decide whether the cosine rule or sine rule would be used to calculate the value of x in these triangles.

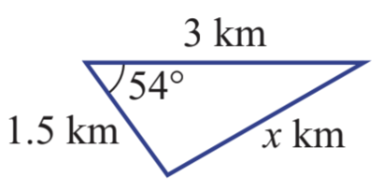
a.



b.



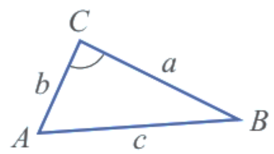
c.



Chapter 4.9 Area of A Triangle

- The area of a triangle is equal to half the product of two sides and the sine of the included angle.

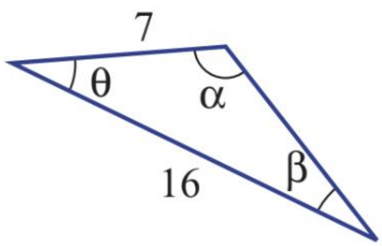
  Area $= \frac{1}{2}ab\sin C$



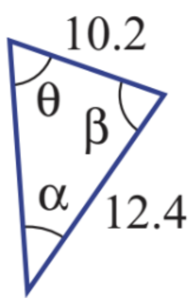
Practice Questions

1. Name the included angle between the two given sides in these triangles.

   a.

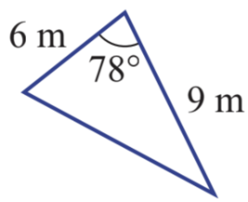


   b.

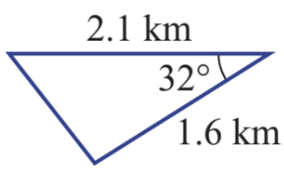


2. Find the area of these triangles correct to one decimal place.

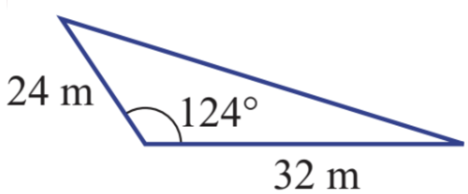
a.



b.

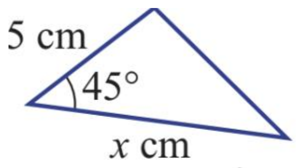


c.



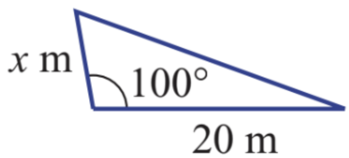
3. Find the value of x correct to one decimal place for these triangles with given areas.

a.



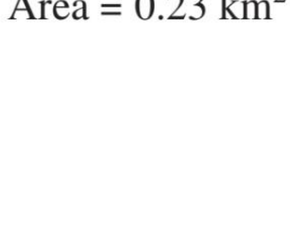
Area = 14 cm^2

b.



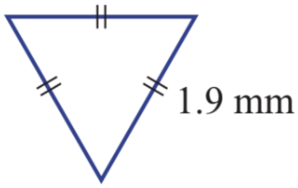
Area = 90 m^2

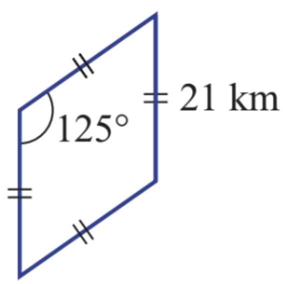
c.



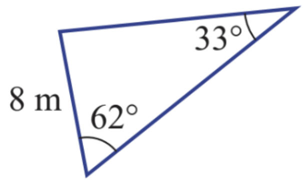
Area = 0.23 km^2

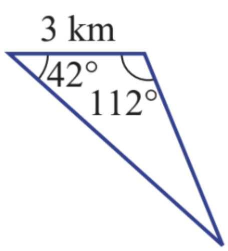
4. Find the area of these shapes correct to two decimal places.

a. 

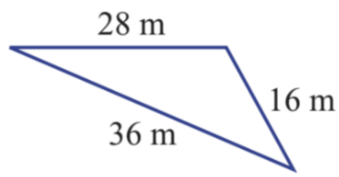
b. 

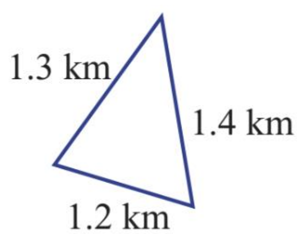
5. First use the sine rule to find another side length, and then find the area of these triangles correct to two decimal places.

a. 

b. 

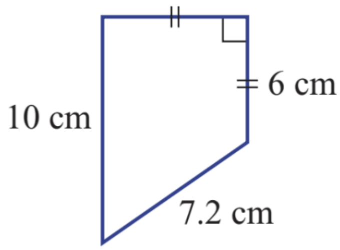
6. First use the cosine rule to find an angle, and then calculate the area of these triangles correct to two decimal places.

a. 

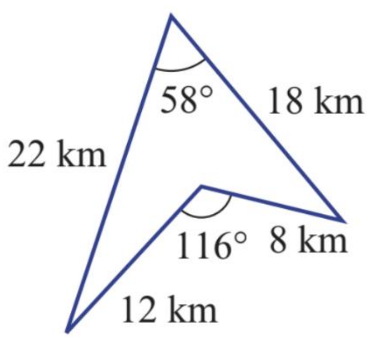
b. 

7. Find the area of these quadrilaterals correct to one decimal place.

a.

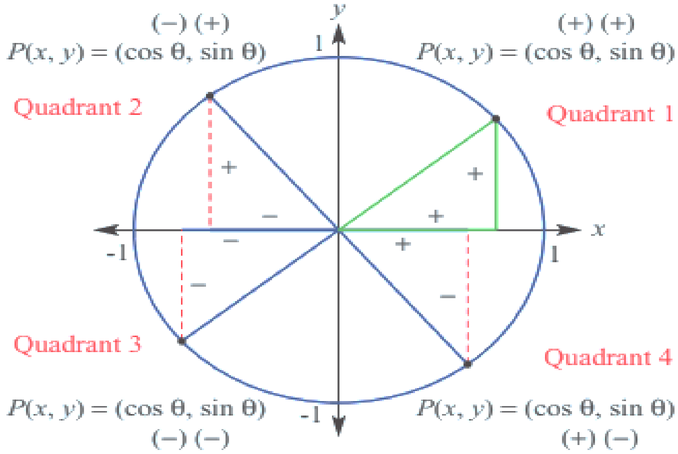


b.



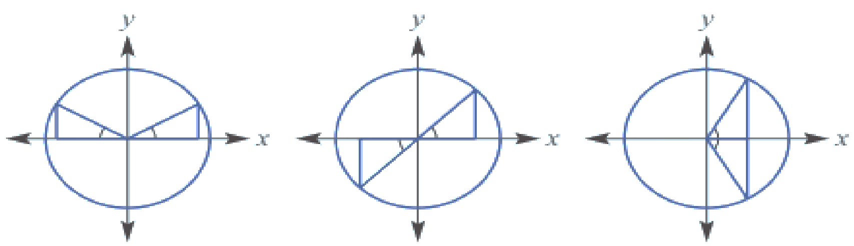
Chapter 4.10 The Four Quadrants

- Every point $P(x, y)$ on the unit circle can be described in terms of the angle θ such that: $x = \cos\theta$ and $y = \sin\theta$ where $-1 \leq \sin\theta \leq 1$ and $-1 \leq \cos\theta \leq 1$.



  - For different quadrants, $\cos\theta$ and $\sin\theta$ can be positive or negative.
  - Recall that $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$.
  - ASTC means:
    - a Quadrant 1: All $\sin\theta$, $\cos\theta$ and $\tan\theta$ are positive.
    - b Quadrant 2: only Sin θ is positive
    - c Quadrant 3: only Tan θ is positive
    - d Quadrant 4: only Cos θ is positive
- A **reference angle** (sometimes called a related angle) is an acute angle that helps to relate $\cos\theta$ and $\sin\theta$ to the first quadrant.

Angle θ	90° – 180°	180° – 270°	270° – 360°
Reference angle	180° – θ	θ – 180°	360° – θ



- **Exact values** can be used when the reference angles are 30°, 45° or 60°.
- Multiples of 90°.

θ	0°	90°	180°	270°	360°
sin θ	0	1	0	-1	0
cos θ	1	0	-1	0	1
tan θ	0	undefined	0	undefined	0

Practice Questions

1. Complete this table.

θ	$0°$	$90°$	$180°$	$270°$	$360°$
$\sin\theta$					0
$\cos\theta$			-1		
$\tan\theta$		undefined			

2. Decide in which quadrant θ lies and state whether $\sin\theta$, $\cos\theta$, $\tan\theta$ are positive or negative.

a. $\theta = 252°$

b. $\theta = 73°$

c. $\theta = 197°$

d. $\theta = 221°$

e. $\theta = 346°$

f. $\theta = 147°$

3. If θ is acute, find the value of θ.

a. $\sin 240° = -\sin \theta$

b. $\sin 336° = -\sin \theta$

c. $\cos 109° = -\cos \theta$

d. $\cos 284° = \cos \theta$

e. $\tan 155° = -\tan \theta$

f. $\tan 278° = -\tan \theta$

4. Recall the exact sine, cosine and tangent values for $30°$, $45°$ and $60°$.

θ	$30°$	$45°$	$60°$
$\sin\theta$	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
$\cos\theta$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$
$\tan\theta$	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$

a. State the reference angle for $225°$.

b. Hence give the exact value of the following.

i. $\sin 225°$

ii. $\cos\ 225°$

iii. $\tan 225°$

c. State the reference angle for $330°$.

d. Hence give the exact value of the following.

i. $\sin 330°$

ii. $\cos 330°$

iii. $\tan 330°$

e. State the reference angle for $120°$.

f. Hence give the exact value of the following.

i. $\sin 120°$

ii. $\cos 120°$

iii. $\tan 120°$

5. Give the exact value of the following.

a. $\cos 150°$

b. $\sin 240°$

c. $\sin 330°$

d. $\tan 120°$

e. $\tan 330°$

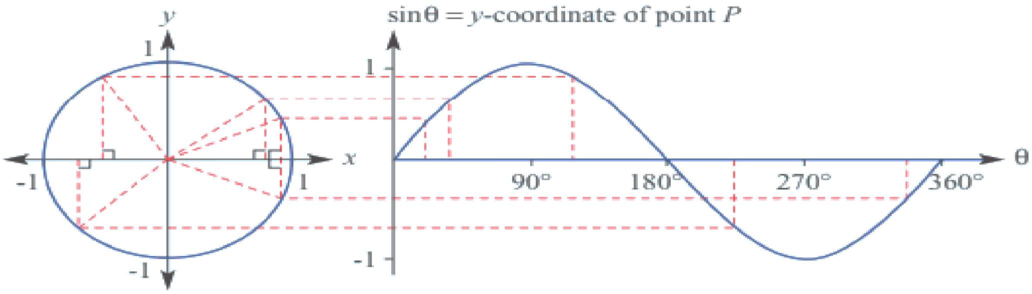
f. $\cos 300°$

g. $\tan 225°$

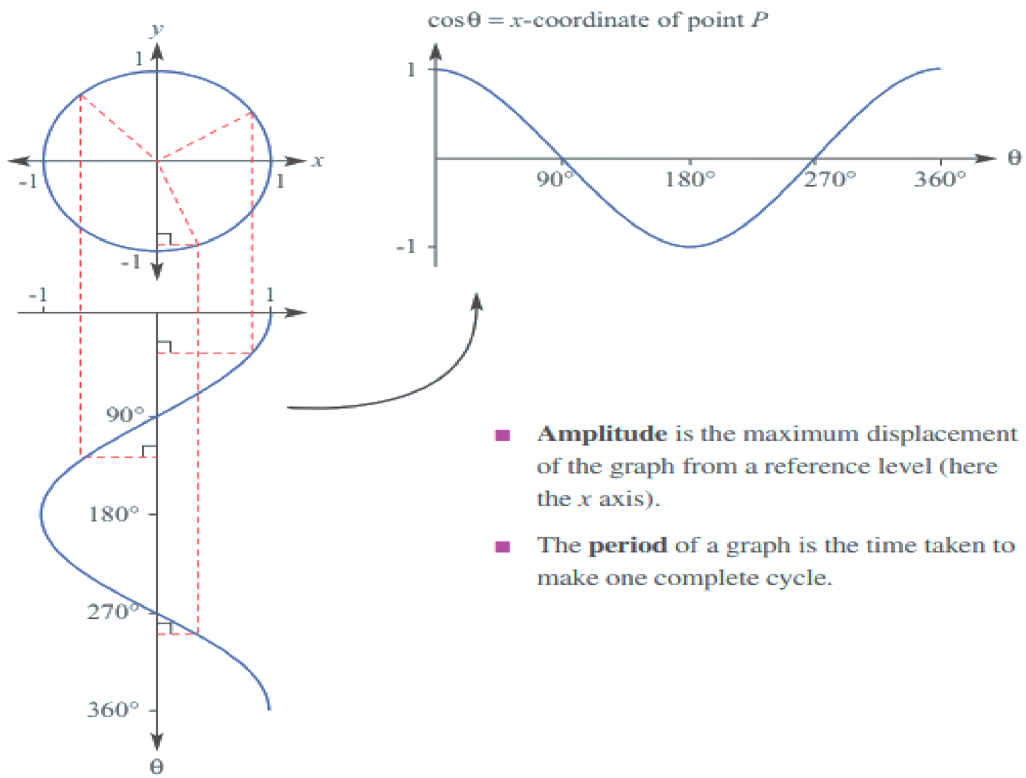
h. $\cos 180°$

Chapter 4.11 Graphs of Trigonometric Functions

- By plotting θ on the x-axis and $\sin\theta$ on the y-axis we form the graph of $\sin\theta$.
  $\sin\theta = y$-coordinate of point P on unit circle.
  - $y = \sin\theta$ Amplitude = 1 Period = 360°

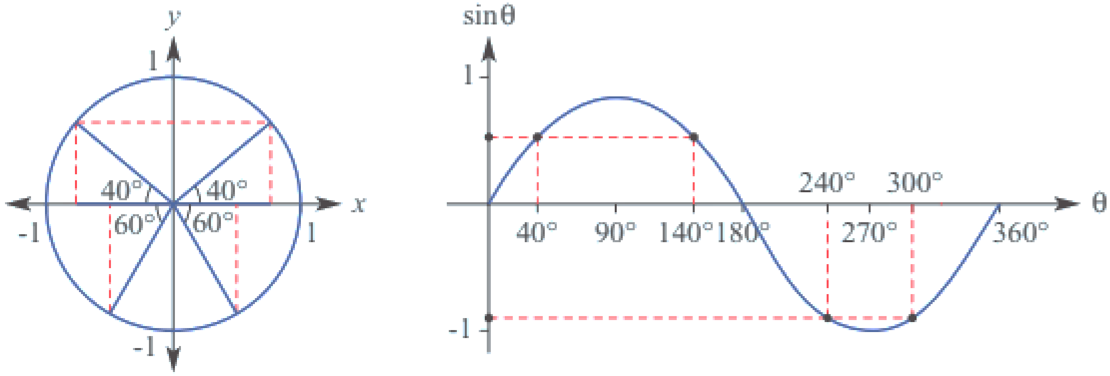


- By plotting θ on the x-axis and $\cos\theta$ on the y-axis we form the graph of $\cos\theta$.
  $\cos\theta = x$-coordinate of point P on unit circle
  - When we write $y = \cos\theta$, the y variable is not to be confused with the y-coordinate of the point P on the unit circle.
  - $y = \cos\theta$ Amplitude = 1 Period = 360°

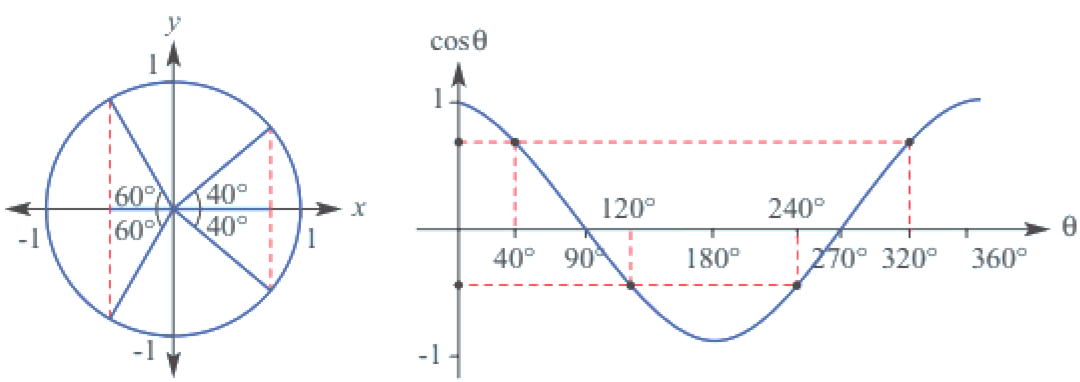


- **Amplitude** is the maximum displacement of the graph from a reference level (here the x axis).
- The **period** of a graph is the time taken to make one complete cycle.

- **Symmetry** within the unit circle using reference angles can be illustrated using graphs of trigonometric functions.
  - This shows $\sin 40° = \sin 140°$ (reference angle 40°) and $\sin 240° = \sin 300°$ (reference angle 60°).



  - This shows $\cos 40° = \cos 320°$ (reference angle 40°) and $\cos 120° = \cos 240°$ (reference angle 60°).

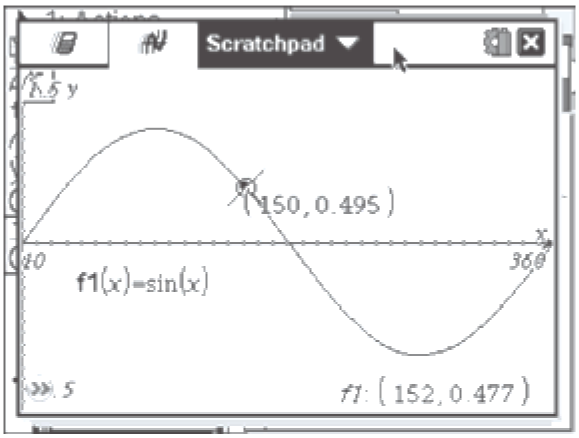


1 Sketch the graph of $y = \sin(x)$ for $0° \le x \le 360°$ and trace to explore the behaviour of y.

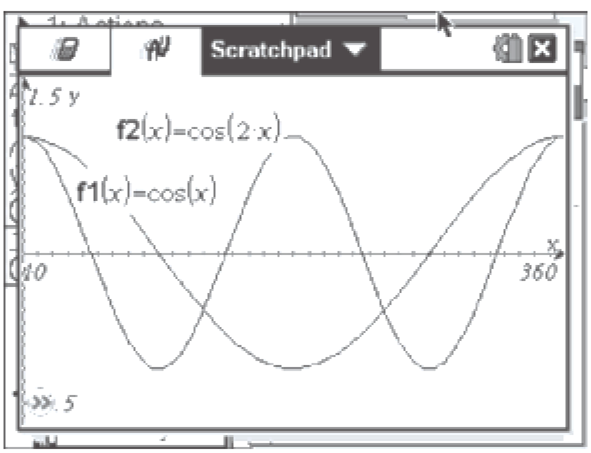
2 Sketch the graph of $y = \cos(x)$ and $y = \cos(2x)$ for $0° \le x \le 360°$ on the same set of axes.

Using the TI-Nspire:

1 In a graphs and geometry page define $f1(x) = \sin(x)$ and press **enter**. Select **menu**, **Window/Zoom**, **Window Settings**, and set x from 0 to 360 and y from about -1.5 to 1.5. Select **menu**, **Trace**, **Graph Trace** then scroll along the graph. Ensure you are in degree mode.

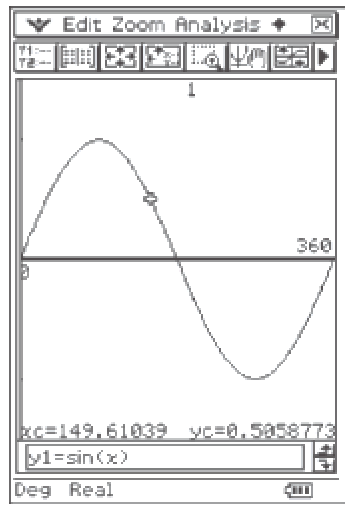


2 In a graphs and geometry page define $f1(x) = \cos(x)$ and $f2(x) = \cos(2x)$ and press **enter**. Use settings as before.

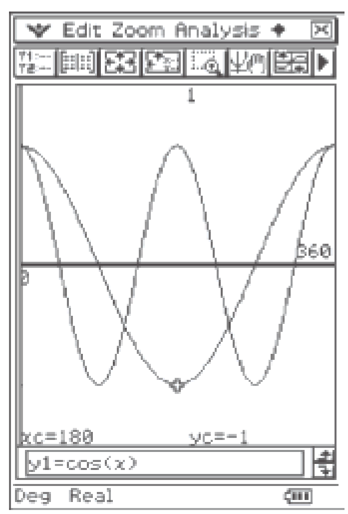


Using the ClassPad:

1 With the calculator in **Degree** mode, go to the **Graph&Table** application. Enter the rule y1 = sin(x) followed by **EXE**. Tap to see the graph. Tap and set x from 0 to 360 and y from about -1.5 to 1.5. Tap **Analysis**, **Trace** then scroll along the graph.



2 In the **Graph&Table** application define $y1 = \cos(x)$ and $y2 = \cos(2x)$. Tap . Use settings as before.

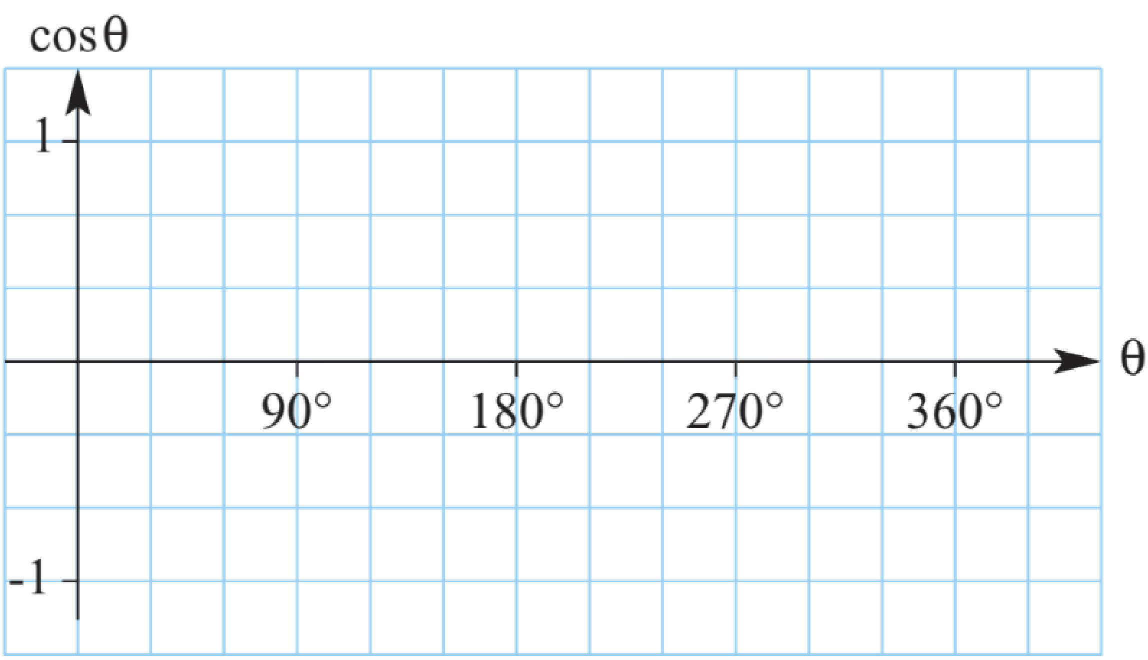


Practice Questions

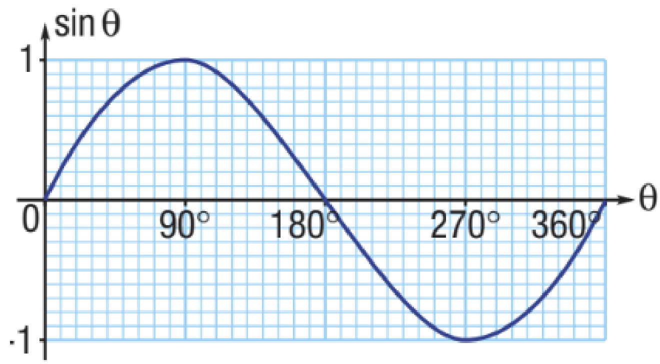
1. a. Using the diagram in question 1, complete the table below for $\cos\theta$, writing the x-coordinate of each point at which the angle intersects the unit circle.

θ	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
cos θ	0	0.87			-0.5			-0.87					

b. Graph the above points and join to make a smooth curve for $\cos\theta$.



2. This graph shows $\sin\theta$ for $0° \leq \theta \leq 360°$.
Use this graph to estimate the value of $\sin\theta$ for the following.



a. $\theta = 25°$

b. $\theta = 115°$

c. $\theta = 220°$

d. $\theta = 310°$

e. $\theta = 160°$

f. $\theta = 235°$

g. $\theta = 320°$

h. $\theta = 70°$

3. For each of the following angles, state the second angle between 0 and $360°$ that gives the same value for $\sin\theta$.

a. $120°$

b. $280°$

c. $214°$

d. $183°$

4. For each of the following angles, state the second angle between 0 and $360°$ that gives the same value for $\cos\theta$.

a. $10°$

b. $285°$

c. $147°$

d. $199°$

5. For θ between $0°$ and $360°$, fi nd the two values of θ that satisfy the following.

a. $\sin\theta = \frac{\sqrt{3}}{2}$

b. $\sin\theta = \frac{1}{2}$

c. $\cos\theta = -\frac{1}{2}$

d. $\cos\theta = -\frac{\sqrt{3}}{2}$

CHAPTER 5 QUADRATIC EQUATIONS

Table of Contents

Chapter 5.1 Expanding Expressions

Like Terms and Operations

1. **Like Terms:**
   - Like terms are terms that have the same pronumeral part (i.e., the same variables raised to the same powers).
   - Example:
     - $7x - 11x = -4x$ (Both terms have the same variable "x", so they can be combined).
     - $4a^2b - 7ab^2 = -3a^2b$ (Even though the terms have different coefficients and powers, they are "like" because they contain the same variables).

2. **Distributive Law:**
   - The distributive law is used to expand expressions by multiplying each term inside the brackets by the term outside the brackets.
   - Example:
     - $a(b + c) = ab + ac$ (Distribute a to both b and c).
     - $a(b - c) = ab - ac$ (Distribute a to both b and c, applying subtraction).
     - For two binomials, such as $(a + b)(c + d)$, you expand each term:
       $$(a + b)(c + d) = ac + ad + bc + bd$$
     - Binomial Product: The expansion involves multiplying each term in one bracket by every term in the other bracket.

3. Perfect Squares:

- A perfect square is the product of a binomial by itself.
- Example:
  - $(a+b)^2 = (a+b)(a+b) = a^2 + 2ab + b^2$
  - $(a-b)^2 = (a-b)(a-b) = a^2 - 2ab + b^2$

4. Difference of Perfect Squares (DOPS):

- The difference of squares is a special product where the terms are the difference of two squares, and it factors into two binomials.
- Formula:
  - $(a+b)(a-b) = a^2 - b^2$
- This works because the cross terms (ab and $-ab$) cancel each other out, leaving only a^2 and $-b^2$.

Practice Questions

1. Expand and simplify.

   a. $x(4x - 3) - 2x(x - 5)$

   b. $2x(2 - 3x) - 3x(2x - 7)$

2. Expand the following.

a. $(x + 7)(x + 5)$

b. $(x - 2)(x + 3)$

c. $(x - 8)(x - 5)$

d. $(x + 6)^2$

e. $(x - 10)^2$

f. $(2x - 3)(2x + 3)$

g. $(8x - 7)(8x + 7)$

3. Expand the following using the distributive law.

a. $(5x + 3)(2x + 7)$

b. $(2x + 5)(3x - 5)$

c. $(5x - 7)(5x + 7)$

d. $(7x - 2)(8x - 2)$

e. $(7x - 1)^2$

4. Expand the following.

a. $-2(x + 8)(x + 2)$

b. $3(x + 5)(x - 3)$

c. $4(a - 3)(a - 6)$

d. $-6(y - 4)(y - 3)$

e. $-2(x + 4)(3x - 7)$

f. $2(a - 7)^2$

g. $-3(2y - 6)^2$

5. Expand and simplify the following.

a. $(x + 8)(x + 3) + (x + 4)(x + 5)$

b. $(y - 7)(y + 4) + (y + 5)(y - 3)$

c. $(4b + 8)(b + 5) - (3b - 5)(b - 7)$

d. $(x - 7)^2 - 9$

e. $14 - (5x + 3)^2$

Chapter 5.2 Factorising Expressions

Factorisation Techniques

1. **Factorising by Common Factors**:
   - When an expression has a common factor, we can factor out (or "take out") that common factor.

   - **Examples**:
     - $-5x - 20 = -5(x + 4)$
       - Here, -5 is the common factor, so we factor it out.
     - $4x^2 - 8x = 4x(x - 2)$
       - The common factor is $4x$, so we factor it out from both terms.

2. **Factorising the Difference of Perfect Squares (DOPS)**:
   - The difference of perfect squares can be factored using the formula:

$$a^2 - b^2 = (a + b)(a - b)$$

   - **Example with Surds**:
     - $x^2 - 5 = (x + \sqrt{5})(x - \sqrt{5})$
     - In this case, 5 is not a perfect square, so we write $\sqrt{5}$ and use the same difference of squares formula.

3. **Factorising Four-Term Expressions by Grouping**:
   - Sometimes you can factor four-term expressions by grouping the terms in pairs and then factoring each pair.

   - **Example**:
     - $x^2 + 5x - 2x - 10$
       - First, group the terms: $(x^2 + 5x) - (2x + 10)$.
       - Now, factor each group:
         - $x(x+5) - 2(x+5)$.
       - Finally, factor out the common binomial factor:

$$(x+5)(x-2)$$

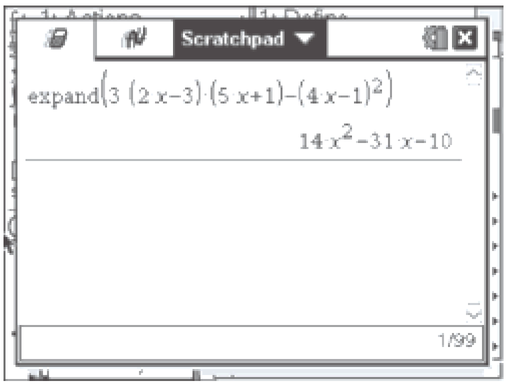
1 Expand and simplify $3(2x-3)(5x+1)-(4x-1)^2$

2 Factorise:

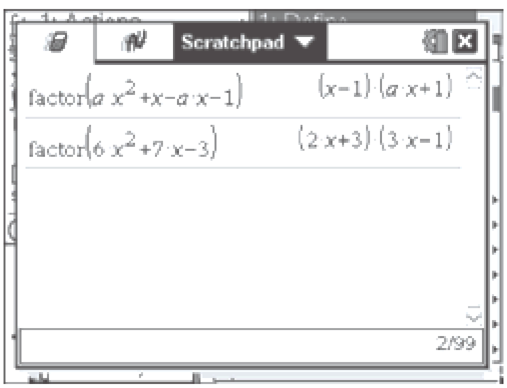
a $ax^2+x-ax-1$ b $6x^2+7x-3$

Using the TI-Nspire:

1 In a calculator page select **menu**, **Algebra**, **Expand** and type in as shown.

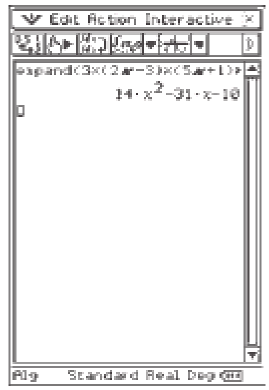


2 Select **menu**, **Algebra**, **Factor** and type in as shown.



Using the ClassPad:

1 In the **Main** application tap **Action**, **Transformation**, **expand** and type in as shown.



2 Tap **Action**, **Transformation**, **factor** and type in as shown.

Practice Questions

1. Factorise.

a. $a(x + 5) - 4(x + 5)$

b. $3(x + 1) - x(x + 1)$

c. $(x - 6) - x(x - 6)$

2. Factorise the following difference of perfect squares.

a. $y^2 - 1$

b. $100 - 9x^2$

c. $144a^2 - 49b^2$

3. Factorise the following.

a. $3y^2 - 48$

b. $63a^2 - 112b^2$

c. $(a - 7)^2 - 1$

d. $(3x - 5y)^2 - 25y^2$

4. Factorise using surds.

a. $x^2 - 21$

b. $x^2 - 11$

c. $x^2 - 20$

d. $x^2 - 200$

e. $(x - 1)^2 - 7$

f. $(x - 7)^2 - 26$

5. Factorise by grouping.

a. $x^2 - 3x + ax - 3a$

b. $x^2 + 3x - 4ax - 12a$

c. $3x^2 - 6ax - 7x + 14a$

6. Factorise fully and simplify surds.

a. $x^2 - \frac{5}{36}$

b. $(x - 7)^2 - 40$

c. $6x^2 - 11$

d. $-7 + 13x^2$

7. Factorise by first rearranging.

a. $ax - 10 + 5x - 2a$

b. $2ax - 20 + 8a - 5x$

8. Factorise fully.

a. $2x^2 - 96$

b. $5(x + 6)^2 - 90$

Chapter 5.3 Factorising Trinomials of the Form $x^2 + bx + c$

Monic Quadratics:

- A **monic quadratic** is a quadratic equation where the coefficient of x^2 is 1. In general, a monic quadratic equation has the form:

$$x^2 + bx + c$$

where b and c are constants.

Factorising Monic Quadratics:

To factorise a monic quadratic of the form $x^2 + bx + c$, follow these steps:

1. **Identify the two numbers**:
   - Find two numbers, m and n, that multiply to give the constant term c and add up to give the coefficient of x, which is b.
     - **Product**: $m \times n = c$
     - **Sum**: $m + n = b$

2. **Write the factorised form**:
   - Once you have the two numbers m and n, the quadratic can be factorised as:

$$x^2 + bx + c = (x + m)(x + n)$$

Example:

Factorise the quadratic equation $x^2 + 7x + 12$.

1. Find two numbers that multiply to 12 and add to 7:
    - The two numbers are 3 and 4 because:
        - $3 \times 4 = 12$
        - $3 + 4 = 7$

2. The factorised form is:

$$x^2 + 7x + 12 = (x + 3)(x + 4)$$

Summary:

- A monic quadratic is one where the coefficient of x^2 is 1.
- To factorise a monic quadratic, find two numbers that multiply to the constant term c and add to the coefficient of x, b.
- The factorised form of $x^2 + bx + c$ is $(x + m)(x + n)$, where m and n are the numbers that satisfy the conditions above.

Practice Questions

1. Factorise these quadratic trinomials.

a. $x^2 + 6x + 9$

b. $x^2 + 11x + 18$

c. $x^2 + 2x - 8$

d. $x^2 + 9x - 22$

e. $x^2 - 7x + 12$

f. $x^2 - 11x + 18$

g. $x^2 - 5x - 14$

h. $x^2 - 3x - 10$

2. Factorise by first taking out the common factor.

a. $2x^2 + 22x + 36$

b. $3x^2 - 9x - 30$

c. $-2x^2 + 10x + 28$

d. $-7x^2 + 49x - 42$

3. Factorise these perfect squares.

a. $x^2 + 12x + 36$

b. $x^2 - 20x + 100$

c. $5x^2 - 50x + 125$

d. $-4x^2 - 72x - 324$

4. Use factorisation to simplify these algebraic fractions.

a. $\frac{x^2-6x+9}{x-3}$

b. $\frac{x+1}{x^2-5x-6}$

c. $\frac{x^2-15x+56}{5(x-8)}$

5. Simplify by factorising.

a. $\frac{x^2+3x+2}{x^2+4x+3} \times \frac{x^2-9}{3x+6}$

b. $\frac{x^2-9}{x^2-5x+6} \times \frac{4x-8}{x^2+8x+15}$

c. $\frac{x^2+6x+8}{x^2-4} \times \frac{6x-24}{x^2-16}$

d. $\frac{x^2-4x-12}{x^2-4} \times \frac{x^2-6x+8}{x^2-36}$

6. Simplify these expressions that involve surds.

a. $\frac{x^2-12}{x+2\sqrt{3}}$

b. $\frac{7x^2-5}{\sqrt{7}x+\sqrt{5}}$

c. $\frac{(x-6)^2-6}{x-6+\sqrt{6}}$

7. Simplify by factorising.

a. $\frac{x^2+3x+2}{x^2+4x+3} \div \frac{4x+8}{x^2-9}$

b. $\frac{x^2-49}{x^2-3x-28} \div \frac{4x+28}{6x+24}$

c. $\frac{x^2+8x+15}{x^2+5x-6} \div \frac{x^2+6x+5}{x^2+7x+6}$

Chapter 5.4 Factorising Quadratic Trinomials of the Form $ax^2 + bx + c$

Factorising a Non-Monic Trinomial:

To factorise a non-monic trinomial of the form $ax^2 + bx + c$, where $a \neq 1$, follow these steps:

1. **Find two numbers that multiply to give $\boldsymbol{a \times c}$**:
   - You need to find two numbers that **multiply** to give $a \times c$ and **add** to give b.

2. **Split the middle term**:
   - Use the two numbers to split the middle term bx into two terms. These two numbers will be used to break up the middle term so that the equation can be factorised by grouping.

3. **Factor by grouping**:
   - Once the middle term is split, factor each group separately and then factor out the common binomial factor.

Example: Factorise $\boldsymbol{15x^2 - x - 6}$

1. **Find $\boldsymbol{a \times c}$**:
   - For the quadratic $15x^2 - x - 6$, $a = 15, b = -1$, and $c = -6$.
   - Multiply $a \times c$:

$$a \times c = 15 \times (-6) = -90$$

2. **Find two numbers that multiply to give $\boldsymbol{-90}$ and add to $\boldsymbol{-1}$ (the coefficient of $\boldsymbol{x}$)**:
   - The two numbers are -10 and 9 because:
     - $-10 \times 9 = -90$
     - $-10 + 9 = -1$

3. **Split the middle term using $-\mathbf{10}$ and $\mathbf{9}$**:
   - Rewrite the quadratic as:

$$15x^2 - x - 6 = 15x^2 - 10x + 9x - 6$$

4. **Factor by grouping**:
   - Group the terms in pairs:

$$= (15x2 - 10x) + (9x - 6)$$

   - Factor out the common factors in each group:

$$= 5x(3x - 2) + 3(3x - 2)$$

5. **Factor out the common binomial factor**:
   - Both terms have the common factor $(3x - 2)$, so factor this out:

$$= (3x - 2)(5x + 3)$$

Final Factorised Form:

$$15x^2 - x - 6 = (3x - 2)(5x + 3)$$

Practice Questions

1. Factorise by grouping pairs.

a. $x^2 + 3x + 7x + 21$

b. $x^2 - 5x + 3x - 15$

c. $8x^2 - 4x + 6x - 3$

d. $12x^2 - 6x - 10x + 5$

2. Factorise the following.

a. $3x^2 + 8x + 4$

b. $5x^2 + 2x - 3$

c. $7x^2 + 2x - 5$

d. $2x^2 + 5x - 12$

e. $5x^2 - 22x + 8$

f. $10x^2 + 11x - 6$

g. $8x^2 - 14x + 5$

h. $9x^2 + 9x - 10$

3. Factorise the following.

a. $21x^2 + 22x - 8$

b. $28x^2 - 13x - 6$

c. $25x^2 - 50x + 16$

4. Factorise by first taking out the common factor.

a. $48x^2 - 18x - 3$

b. $90x^2 + 90x - 100$

c. $20x^2 - 25x + 5$

5. Simplify by first factorising.

a. $\frac{9x^2-21x+10}{3x-5}$

b. $\frac{20x-12}{10x^2-21x+9}$

c. $\frac{10x^2+3x-4}{14x^2-11x+2}$

d. $\frac{8x^2-2x-15}{16x^2-25}$

6. Combine all your knowledge of factorising to simplify the following.

a. $\frac{4x^2-1}{6x^2-x-2} \times \frac{9x^2-4}{8x-4}$

b. $\frac{20x^2+21x-5}{16x^2+8x-15} \times \frac{16x^2-24x+9}{25x^2-1}$

c. $\frac{3x^2-12}{30x+15} \div \frac{2x^2-3x-2}{4x^2+4x+1}$

d. $\frac{16x^2-25}{4x^2-7x-15} \div \frac{4x^2-17x+15}{16x^2-40x+25}$

7. Factorise the quadratics in the expressions then simplify using a common denominator.

a. $\frac{3}{3x-1} - \frac{x}{6x^2+13x-5}$

b. $\frac{4x}{12x^2-11x+2} - \frac{3x}{3x-2}$

c. $\frac{2}{9x^2-25} - \frac{3}{9x^2+9x-10}$

d. $\frac{1}{10x^2-19x+6} + \frac{2}{4x^2+8x-21}$

Chapter 5.5 Factorising by Completing the Square

Completing the Square:

Completing the square is a method used to factorise quadratic expressions and solve quadratic equations. Here's how it works:

1. **Standard Form:** Begin with a quadratic expression in the form:

$$x^2 + bx$$

2. **Add and Subtract the Term $\left(\frac{b}{2}\right)^2$**: To complete the square, you add and subtract the same value $\left(\frac{b}{2}\right)^2$, where b is the coefficient of x. This makes the expression a perfect square trinomial.

   The expression $x^2 + bx$ becomes:

$$x^2 + bx + \left(\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$$

3. **Factor the Perfect Square:** The part of the expression $x^2 + bx + \left(\frac{b}{2}\right)^2$ is a perfect square trinomial, which can be written as:

$$\left(x + \frac{b}{2}\right)^2$$

   So, the complete expression becomes:

$$\left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$$

   This is the completed square form.

Example:

Let's complete the square for the quadratic expression $x^2 + 6x$.

1. **Original Expression:**

$$x^2 + 6x$$

2. **Add and Subtract $\left(\frac{6}{2}\right)^2$:**

$$x^2 + 6x + \left(\frac{6}{2}\right)^2 - \left(\frac{6}{2}\right)^2 = x^2 + 6x + 9 - 9$$

3. **Factor the Perfect Square:**

$$(x + 3)^2 - 9$$

Factorising by Completing the Square:

To factorise a quadratic by completing the square:

1. **Add** $\left(\frac{b}{2}\right)^2$ and **subtract** $\left(\frac{b}{2}\right)^2$ to balance the equation.

2. **Factor the perfect square**.

3. If necessary, use the **difference of perfect squares (DOPS)** for factorisation.

Example: Factorising $x^2 + 6x - 7$

1. Start with:

$$x^2 + 6x - 7$$

2. Add and subtract $\left(\frac{6}{2}\right)^2 = 9$:

$$x^2 + 6x + 9 - 9 - 7$$
$$= (x + 3)^2 - 16$$

3. Use DOPS:

$$(x+3)^2 - 4^2 = (x+3+4)(x+3-4)$$
$$= (x+7)(x-1)$$

Key Points:

- Completing the square turns the quadratic expression into a perfect square trinomial and a constant.
- The difference of perfect squares (DOPS) can be used if the expression ends in a negative value, like $a^2 - b^2$.
- Not all quadratic expressions can be factorised. If you end up with something like $(x+3)^2 + 6$, it cannot be factorised further.

This technique helps to solve quadratic equations and understand the properties of parabolas.

Practice Questions

1. Factorise these perfect squares.

   a. $x^2 + 10x + 25$

   b. $x^2 - 18x + 81$

2. Factorise using surds. Recall $a^2 - b^2 = (a + b)(a - b)$.

a. $(x + 4)^2 - 10$

b. $(x - 5)^2 - 3$

3. Decide what number needs to be added to these expressions to complete the square. Then factorise the resulting perfect square.

a. $x^2 + 8x$

b. $x^2 - 12x$

c. $x^2 + 11x$

d. $x^2 - 9x$

4. Factorise by completing the square.

   a. $x^2 + 10x - 4$

   b. $x^2 - 8x - 5$

5. Factorise if possible.

a. $x^2 + 4x + 2$

b. $x^2 - 6x + 6$

c. $x^2 - 4x + 6$

6. Factorise the following.

a. $x^2 + 9x - 3$

b. $x^2 - 9x - \frac{5}{2}$

7. Factorise by first taking out the common factor.

a. $4x^2 - 8x - 16$

b. $-3x^2 - 30x - 3$

c. $-3x^2 + 24x - 15$

8. Factorise by first taking out the coefficient of x^2.

a. $2x^2 - 10x + 4$

b. $-2x^2 - 14x + 8$

c. $-2x^2 + 10x + 8$

9. A non-monic quadratic such as $2x^2 - 5x + 1$ can be factorised in the following way.

$$
\begin{aligned}
2x^2 - 5x + 1 &= 2\left(x^2 - \frac{5}{2}x + \frac{1}{2}\right) \\
&= 2\left(x^2 - \frac{5}{2}x + \frac{1}{2}\right) \\
&= 2\left(\left(x - \frac{5}{4}\right)^2 - \frac{25}{16} + \frac{8}{16}\right) \\
&= 2\left(\left(x - \frac{5}{4}\right)^2 - \frac{17}{16}\right) \\
&= 2\left(x - \frac{5}{4} + \frac{\sqrt{17}}{4}\right)\left(x - \frac{5}{4} - \frac{\sqrt{17}}{4}\right)
\end{aligned}
$$

Factorise these using a similar technique.

a. $4x^2 - 7x - 16$

b. $-3x^2 - 7x - 3$

c. $2x^2 + 5x - 7$

d. $-3x^2 - 7x - 4$

Chapter 5.6 Solving Quadratic Equations

Null Factor Law and Solving Quadratic Equations

The **Null Factor Law** is a useful principle in algebra, particularly when solving quadratic equations. It states that if the product of two numbers is zero, then at least one of the numbers must be zero. This is formalized as:

$$a \times b = 0 \;\Rightarrow\; a = 0 \text{ or } b = 0$$

Steps to Solve a Quadratic Equation Using the Null Factor Law:

1. **Write the equation in standard form**: The general form of a quadratic equation is:

   $$ax^2 + bx + c = 0$$

   Where a, b, and c are constants.

2. **Factorise the quadratic equation**: To apply the Null Factor Law, you first need to factorise the quadratic equation into two binomial factors. The goal is to express the equation in the form:

   $$(x + m)(x + n) = 0$$

   Where m and n are numbers that make the equation true.

3. **Apply the Null Factor Law**: Once the quadratic is factorised, use the Null Factor Law:

   $$(x + m)(x + n) = 0 \;\Rightarrow\; x + m = 0 \text{ or } x + n = 0$$

   Solving these gives the solutions for x:

   $$x = -m \text{ or } x = -n$$

4. **If necessary, divide by any common factors**: If the coefficients of all the terms have a common factor, you should divide through by that common factor first to simplify the equation before factorising. This will make the process easier and the factorisation more straightforward.

Example 1: Solving a Simple Quadratic Equation

Solve $x^2 - 5x + 6 = 0$.

1. **Write the equation in standard form**: Already in the form $x^2 - 5x + 6 = 0$.

2. **Factorise the equation**: Look for two numbers that multiply to 6 and add to -5. These numbers are -2 and -3.

$$(x-2)(x-3) = 0$$

3. **Apply the Null Factor Law**:

$$x - 2 = 0 \text{ or } x - 3 = 0$$

   Solve for x:

$$x = 2 \text{ or } x = 3$$

So, the solutions are $x = 2$ and $x = 3$.

Example 2: Solving a Quadratic Equation with a Common Factor

Solve $2x^2 + 6x = 0$.

1. **Write the equation in standard form**: It's already in the form $2x^2 + 6x = 0$.

2. **Factor out the common factor (2x)**:

$$2x(x + 3) = 0$$

3. **Apply the Null Factor Law**:

$$2x = 0 \text{ or } x + 3 = 0$$

So, the solutions are $x = 0$ and $x = -3$.

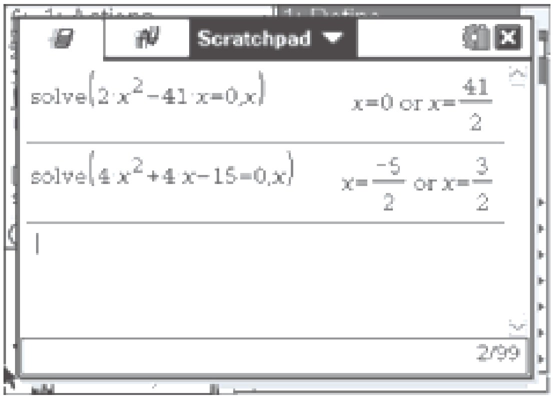
Key Points:

- **Null Factor Law**: If the product of two factors is zero, then at least one factor must be zero.
- **Factorisation**: Factorising a quadratic expression is often the most straightforward way to solve a quadratic equation.
- **Common Factor**: If all terms have a common factor, factor it out first to simplify the equation before solving.

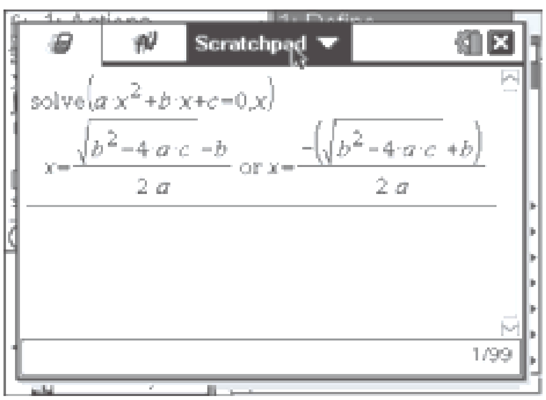
This method is essential for solving quadratic equations and plays a key role in algebraic problem-solving.

Using the TI-Nspire:

1 Select **menu**, **Algebra**, **Solve** and type the equation. End with *,x*) to finish.

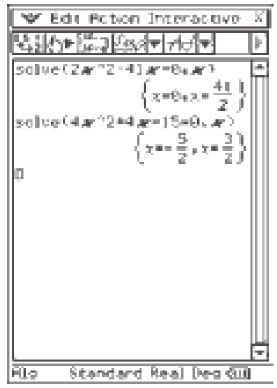


2 Select **menu**, **Algebra**, **Solve** and type the equation. End with *,x*) to finish. Enter a multiplication sign between *a* and *x* in *ax* and *b* and *x* in *bx*. This gives the general quadratic formula studied in 5.9.

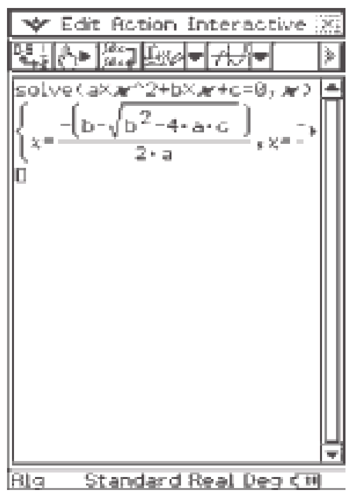


Using the ClassPad:

1 Tap **Action**, **Advanced**, **solve** and type the equation. End with *,x*) to finish.



2 Tap **Action**, **Advanced**, **solve** and type the equation. End with *,x*) to finish. Enter a multiplication sign between *a* and *x* in *ax* and *b* and *x* in *bx*. This gives the general quadratic formula studied in 5.9.



Practice Questions

1. Write the solutions to these equations, which are already in factorised form.

a. $2x(x - 4) = 0$

b. $(x + 1)(x - 1) = 0$

c. $(x + 2\sqrt{2})(x - 2\sqrt{2}) = 0$

d. $(8x + 3)(4x + 3) = 0$

2. Solve the following quadratic equations.

a. $x^2 + 2x = 0$

b. $4x^2 + 8x = 0$

c. $3x^2 - 15 = 0$

d. $7x^2 = -x$

e. $2x^2 = 72$

3. Solve the following quadratic equations.

a. $x^2 - 6x + 8 = 0$

b. $x^2 + 2x - 15 = 0$

c. $x^2 - 12x + 32 = 0$

d. $x^2 - 8x + 16 = 0$

e. $x^2 + 18x + 81 = 0$

4. Solve the following quadratic equations.

a. $4x^2 + 16x + 7 = 0$

b. $2x^2 - 23x + 11 = 0$

c. $5x^2 - 7x - 6 = 0$

d. $7x^2 + 25x - 12 = 0$

5. Solve by first taking out a common factor.

a. $2x^2 - 20x - 22 = 0$

b. $5x^2 - 20x + 20 = 0$

c. $18x^2 - 57x + 30 = 0$

6. Solve the following by first writing in the form $ax^2 + bx + c = 0$.

a. $x^2 = 3(2x - 3)$

b. $x(x + 4) = 4x + 9$

c. $x^2 + x - 9 = 5x - 4$

d. $x(x + 4) = 4(x + 16)$

e. $4x(x + 5) = 6x - 4x^2 - 3$

7. Solve the following by first writing in the form $ax^2 + bx + c = 0$.

a. $\frac{18-7x}{x} = x$

b. $\frac{7x+10}{2x} = 3x$

c. $\frac{4}{x-2} = x + 1$

8. Solve these equations by first multiplying by an appropriate expression.

a. $-\frac{5}{x} = 2x - 11$

b. $\frac{2x^2-12}{x} = -5$

c. $\frac{x-4}{2} = -\frac{2}{x}$

d. $\frac{1}{x-1} - \frac{1}{x+3} = \frac{1}{3}$

Chapter 5.7 Applications of Quadratics

Practice Questions

1. A square hut of side length 5 m is to be surrounded with a verandah of width x metres. Find the width of the verandah if its area is to be 24 m^2.

2. A rectangular painting is to have a total area (including the frame) of 1200 cm^2. The painting is 30 cm long and 20 cm wide, find the width of the frame.

3. The height h (in metres) of a golf ball is given by $h = -x^2 + 100x$ where x metres is the horizontal distance from where the ball was hit.

a. Find the values of x if $h = 0$.

b. Interpret your answer from part **a**.

c. Find how far the ball has travelled horizontally if the height is $196\ \mathrm{m}$.

Chapter 5.8 Solving Quadratic equations by Completing the Square

Practice Questions

1. Solve by first completing the square.

   a. $x^2 + 10x + 15 = 0$

   b. $x^2 + 6x - 5 = 0$

c. $x^2 - 2x - 16 = 0$

d. $x^2 - 8x + 9 = 0$

e. $x^2 - 14x - 6 = 0$

2. Decide how many solutions there are to these equations. Try factorising the equations if you are unsure.

a. $x^2 + 3 = 0$

b. $(x + 2)^2 - 7 = 0$

c. $x^2 - 3x + 10 = 0$

d. $x^2 - 2x + 17 = 0$

3. Solve by first completing the square.

a. $x^2 + 7x + 5 = 0$

b. $x^2 + 5x - 2 = 0$

c. $x^2 + x - 4 = 0$

d. $x^2 + 5x + \frac{5}{4} = 0$

4. A slightly different way to solve by completing the square is shown here. Solve the following using this method.

   a. $x^2 - 6x + 2 = 0$

   b. $x^2 + 8x + 6 = 0$

   c. $x^2 + 4x - 7 = 0$

Chapter 5.9 The Quadratic Formula

Quadratic Formula and Discriminant

The **quadratic formula** is a powerful method for solving quadratic equations of the form:

$$ax^2 + bx + c = 0 \quad \text{where} \quad a \neq 0$$

The general solution to this quadratic equation can be found using the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Discriminant:

The **discriminant** (denoted as Δ or D) is the part of the quadratic formula inside the square root:

$$D = b^2 - 4ac$$

The discriminant helps determine the nature of the solutions to the quadratic equation:

- **If $\boldsymbol{D < 0}$**: There are no real solutions because the square root of a negative number is undefined in the set of real numbers. The solutions will be complex (imaginary).
- **If $\boldsymbol{D = 0}$**: There is exactly **one real solution** (also called a repeated or double root). The solution is:

$$x = -\frac{b}{2a}$$

- **If $\boldsymbol{D > 0}$**: There are **two distinct real solutions**. These solutions are:

$$x = \frac{-b \pm \sqrt{D}}{2a}$$

Example 1: $2x^2 + 3x - 2 = 0$

For the equation $2x^2 + 3x - 2 = 0$, we have $a = 2, b = 3$, and $c = -2$.

1. **Find the discriminant**:

$$D = b2 - 4ac = 32 - 4(2)(-2) = 9 + 16 = 25$$

2. **Since $D > 0$**, there are two real solutions:

$$x = \frac{-3 \pm \sqrt{25}}{2(2)} = \frac{-3 \pm 5}{4}$$

So the two solutions are:

$$x = \frac{-3 + 5}{4} = \frac{2}{4} = \frac{1}{2}$$

And

$$x = \frac{-3 - 5}{4} = -\frac{8}{4} = -2$$

Therefore, the solutions are $x = \frac{1}{2}$ and $x = -2$.

Example 2: $x^2 - 4x + 4 = 0$

For the equation $x^2 - 4x + 4 = 0$, we have $a = 1, b = -4$, and $c = 4$.

1. **Find the discriminant**:

$$D = b^2 - 4ac = (-4)^2 - 4(1)(4) = 16 - 16 = 0$$

2. **Since $D = 0$**, there is one real solution:

$$x = \frac{-(-4)}{2(1)} = \frac{4}{2} = 2$$

Therefore, the solution is $x = 2$ (a repeated root).

Example 3: $x^2 + 2x + 5 = 0$

For the equation $x^2 + 2x + 5 = 0$, we have $a = 1, b = 2$, and $c = 5$.

1. **Find the discriminant**:

$$D = b^2 - 4ac = 2^2 - 4(1)(5) = 4 - 20 = -16$$

2. **Since $D < 0$**, there are no real solutions. The solutions are complex (imaginary).

   The solutions will be:

$$x = \frac{-2 \pm \sqrt{-16}}{2(1)} = \frac{-2 \pm 4i}{2} = -1 \pm 2i$$

Therefore, the solutions are $x = -1 + 2i$ and $x = -1 - 2i$.

Summary:

- **Discriminant** $\boldsymbol{D = b^2 - 4ac}$ helps determine the nature of the solutions.
  - $D < 0$: No real solutions (complex solutions).
  - $D = 0$: One real solution (repeated root).
  - $D > 0$: Two distinct real solutions.

- **Quadratic Formula**:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Practice Questions

1. Using the discriminant, determine the number solutions of these quadratic equations.

   a. $x^2 + 6x + 9 = 0$

   b. $x^2 + 4x - 4 = 0$

c. $2x^2 + 12x + 9 = 0$

d. $-4x^2 - 6x + 3 = 0$

2. Find the exact solutions to the following quadratic equations using the quadratic formula.

a. $x^2 + 6x - 2 = 0$

b. $-3x^2 + 8x - 2 = 0$

c. $-5x^2 + 8x + 3 = 0$

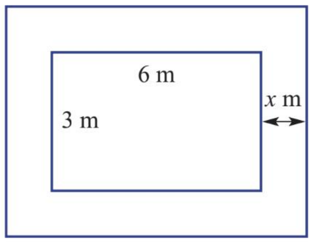
3. Solve the following using the quadratic formula.

a. $5x = 2 - 4x^2$

b. $-\frac{5}{x} = 2 - x$

c. $3x = \frac{10x-1}{2x}$

4. A pool $6\ \mathrm{m}$ by $3\ \mathrm{m}$ is to have a path surrounding it. If the total area of the pool and path is to be $31\ \mathrm{m}^2$, find the width of the path correct to the nearest cm.



CHAPTER 6 MEASUREMENT

Table of Contents

Chapter 6.1 Review of Length

Converting Between Metric Units of Length

The metric system is based on powers of ten, making conversions straightforward:

- **Kilometers (km) to meters (m):** Multiply by **1000**
- **Meters (m) to centimeters (cm):** Multiply by **100**
- **Centimeters (cm) to millimeters (mm):** Multiply by **10**
- **Millimeters (mm) to centimeters (cm):** Divide by **10**
- **Centimeters (cm) to meters (m):** Divide by **100**
- **Meters (m) to kilometers (km):** Divide by **1000**

Perimeter

The **perimeter** is the total distance around the outside of a closed shape.

For common shapes:

- **Rectangle:** $P = 2(l + w)$
- **Square:** $P = 4s$ (where s is the side length)
- **Triangle:** $P = a + b + c$ (sum of all sides)
- **Regular polygon:** $P = n \times s$ (where n is the number of sides and s is the side length)

Circumference of a Circle

The **circumference** (C) is the distance around the circle:

$$C = 2\pi r \quad \text{or} \quad C = \pi d$$

where:

- r is the radius
- $d = 2r$ is the diameter
- $\pi \approx 3.1416$

Perimeter of a Sector

A **sector** is a portion of a circle, like a slice of pizza. The perimeter of a sector includes the curved arc length plus the two radii:

$$P = 2r + \frac{\theta}{360} \times 2\pi r$$

where:

- θ is the central angle in degrees
- r is the radius
- $\frac{\theta}{360} \times 2\pi r$ gives the arc length

This formula helps find the total boundary length of a sector, including both straight and curved parts.

Practice Questions

1. Convert the following length measurements into the units given in brackets.

   a. $0.001 \text{ km } (\text{m})$

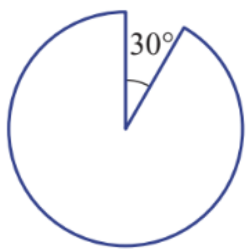
   b. $297 \text{ m } (\text{km})$

   c. $0.0032 \text{ km } (\text{m})$

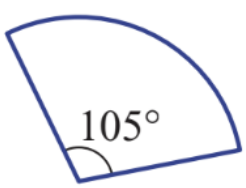
2. What fraction of a circle (in simplest form) is shown in these sectors?

a.

b.

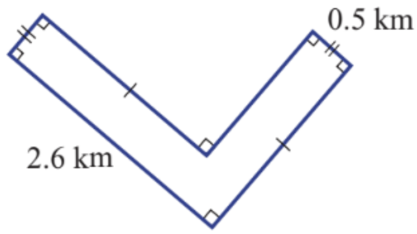


c.

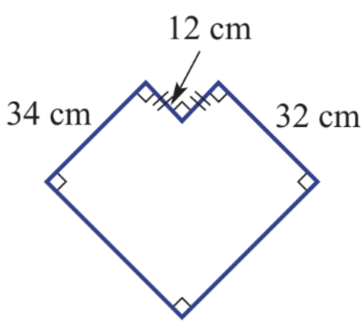


3. Find the perimeter of these shapes.

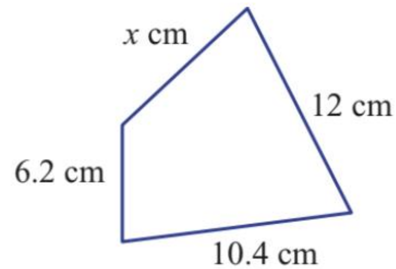
a.



b.



4. Consider the given two-dimensional shape.



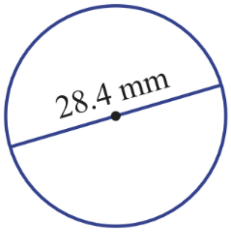
a. Find the perimeter of the shape if $x = 8$.

b. Find x if the perimeter is 33.7 cm.

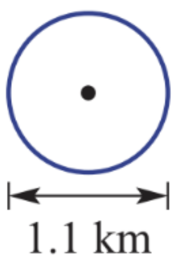
c. Write an expression for x in terms of the perimeter P.

5. Find the circumference of these circles, correct to two decimal places.

a.

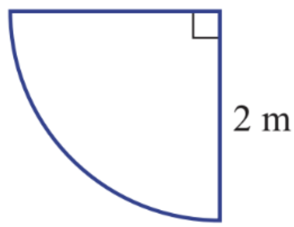


b.

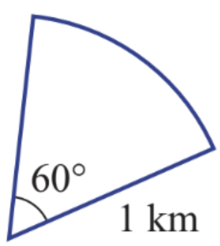


6. Find the perimeter of these sectors, by:
   i. using exact values
   ii. rounding the answer to one decimal place.

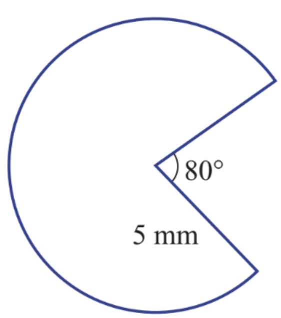
a.



b.

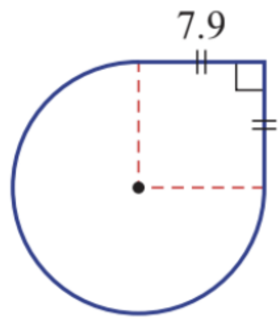


c.

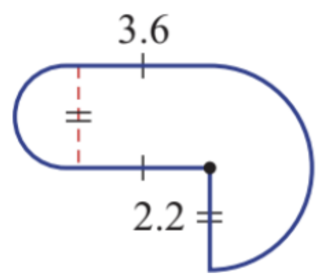


7. Find the perimeter of these composite shapes correct to two decimal places.

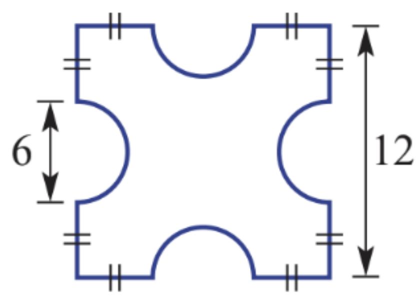
a.



b.

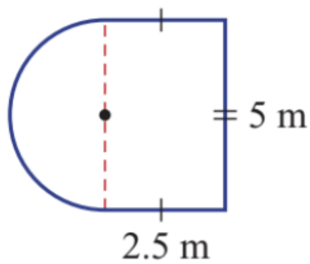


c.

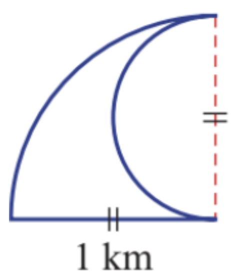


8. Find the perimeter of these shapes, giving answers as exact values.

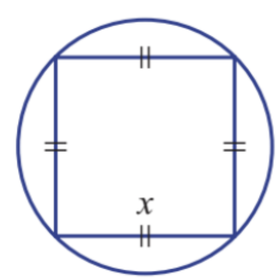
a.



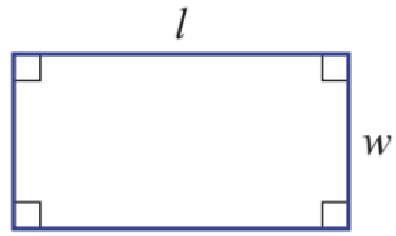
b.



9. A square of side length x just fits inside a circle. Find the exact circumference of the circle in terms of x.



10. Consider a rectangle with perimeter P, length l and width w.



a. Express l in terms of w and P.

b. Express l in terms of w if $P = 10$.

c. If $P = 10$ state the range of all possible values of w.

d. If $P = 10$ state the range of all possible values of l.

Chapter 6.2 Pythagoras’ Theorem

Pythagoras’ Theorem

In a right-angled triangle:

$$a^2 + b^2 = c^2$$

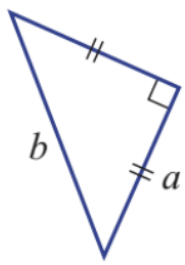
where:

- a and b are the shorter sides (legs) of the triangle
- c is the hypotenuse (the longest side, opposite the right angle)

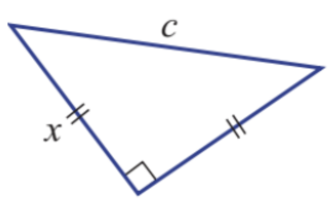
Practice Questions

1. Write an equation connecting the pronumerals in these right-angled triangles.

a.

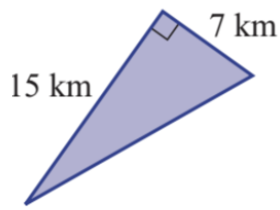


b.

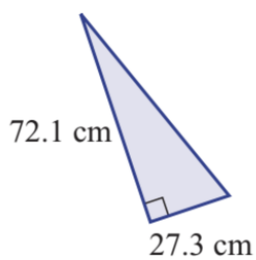


2. Use Pythagoras' theorem to find the length of the hypotenuse for these right-angled triangles. Round your answers to two decimal places where necessary.

a.

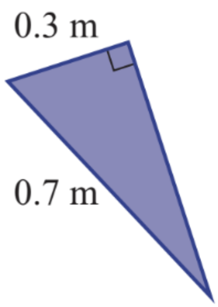


b.

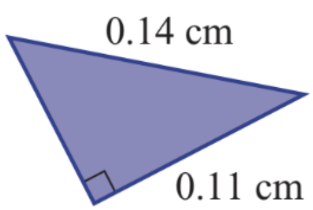


3. Find the length of the unknown side in these right-angled triangles, correct to two decimal places.

a.

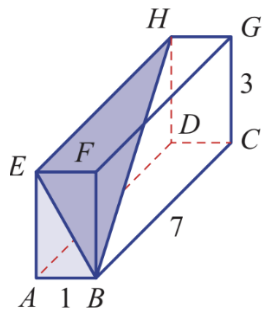


b.

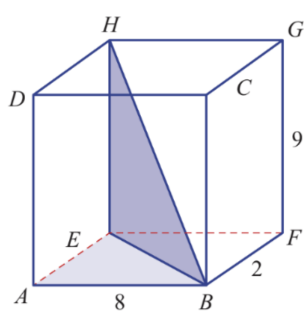


4. For each of the cuboids $ABCDEFGH$, find:
   i. BE, leaving your answer in exact form
   ii. BH, correct to two decimal places.

a.

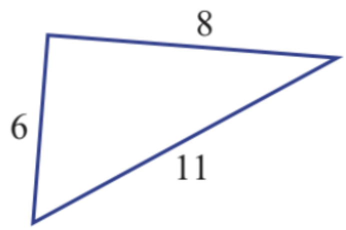


b.

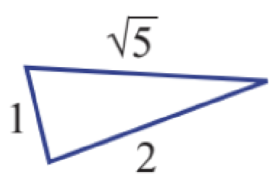


5. Use Pythagoras' theorem to help decide whether these triangles are right-angled. They may not be drawn to scale.

a.

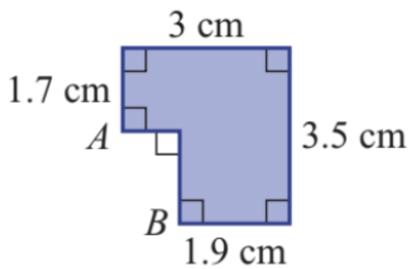


b.

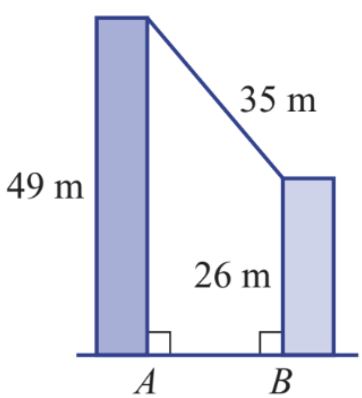


6. Use Pythagoras' theorem to find the distance between points A and B in these diagrams, correct to two decimal places.

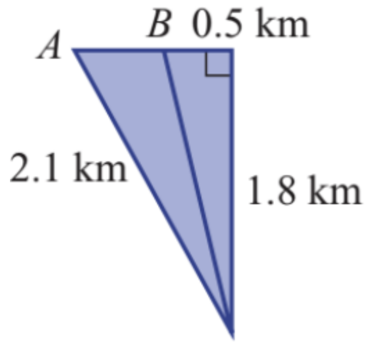
a.



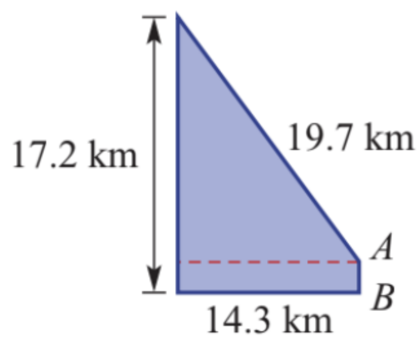
b.



c.

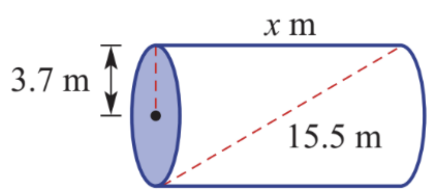


d.

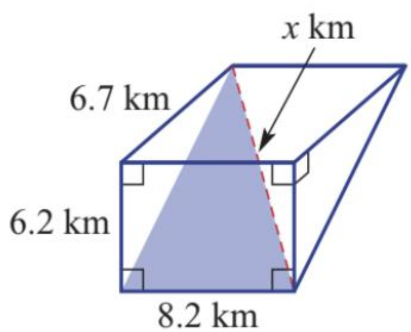


7. Find the value of x, correct to two decimal places, in these three-dimensional diagrams.

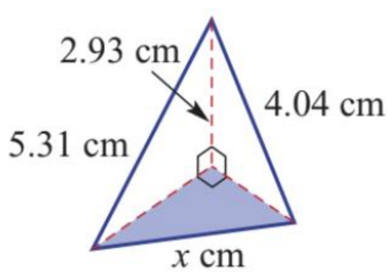
a.



b.

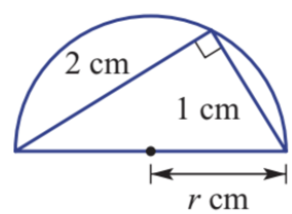


c.



8. Find the length of the longest rod that will fit inside these objects. Give your answer correct to one decimal place.
A cylinder with diameter 10 cm and height 20 cm.

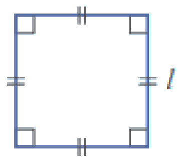
9. Two joining chords in a semicircle have lengths 1 cm and 2 cm as shown. Find the exact radius, r cm, of the semicircle. Give reasons.



Chapter 6.3 Area

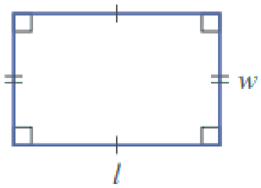
- The area of a two-dimensional shape can be defined as the number of square units contained within its boundaries. Some common area formulas are given.

Square



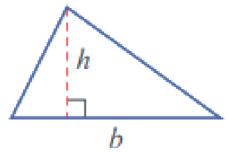
Area = l^2

Rectangle



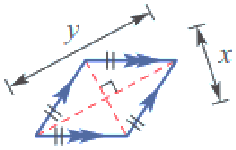
Area = lw

Triangle



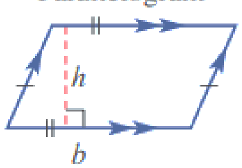
Area = $\frac{1}{2}bh$

Rhombus



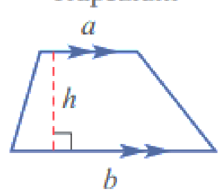
Area = $\frac{1}{2}xy$

Parallelogram



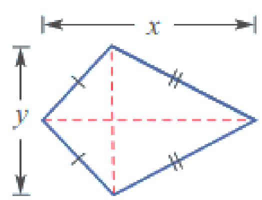
Area = bh

Trapezium



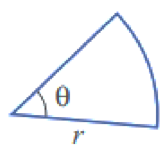
Area = $\frac{1}{2}(a+b)h$

Kite



Area = $\frac{1}{2}xy$

- The rule for the area of a circle is:
  Area = πr^2, where r is the radius
- The rule for the area of a sector is $A = \frac{\theta}{360}\pi r^2$



Practice Questions

1. Write the formula for the area of these shapes.

a. circle

b. sector

c. square

d. rectangle

e. kite

f. trapezium

g. triangle

h. rhombus

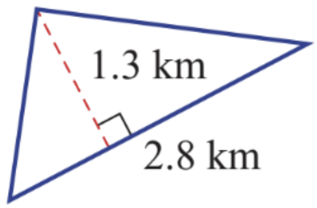
i. parallelogram

j. semicircle

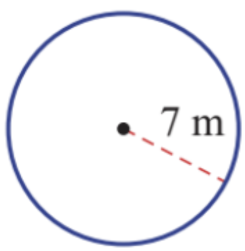
k. quadrant (quarter circle)

2. Find the area of these basic shapes, rounding to two decimal places where necessary.

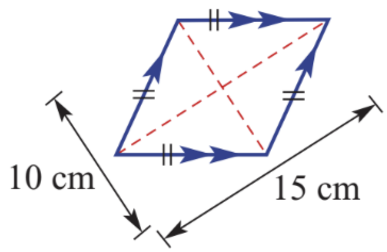
a.



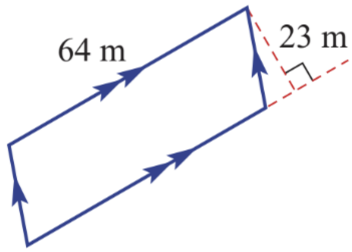
b.



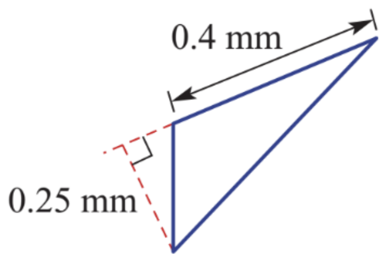
c.



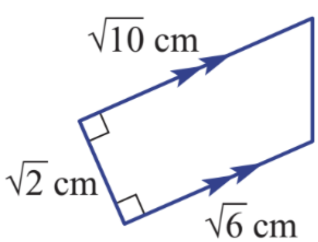
d.



e.

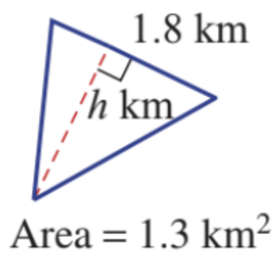


f.

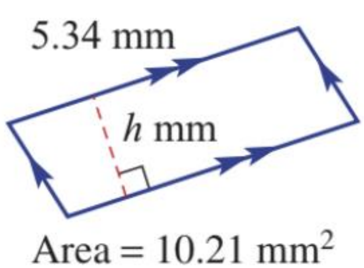


3. Find the value of the pronumeral for these basic shapes, rounding to two decimal places where necessary.

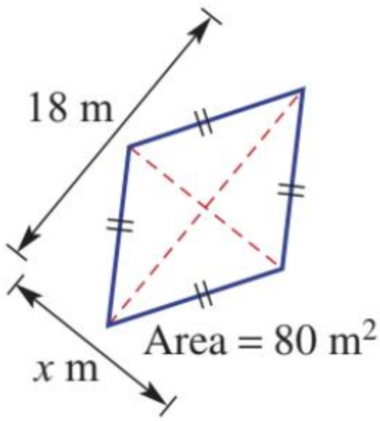
a.



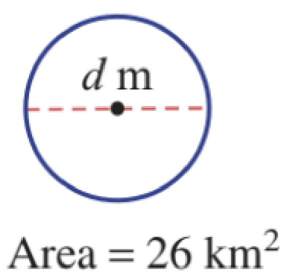
b.



c.

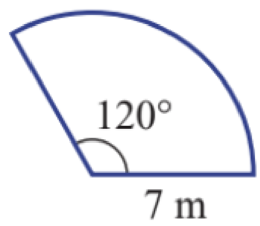


d.

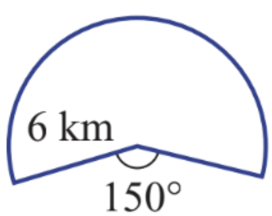


4. Find the area of each sector. Write your answer as an exact value and as a decimal rounded to two places.

a.

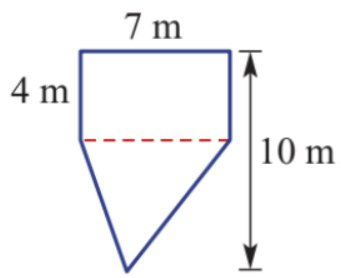


b.

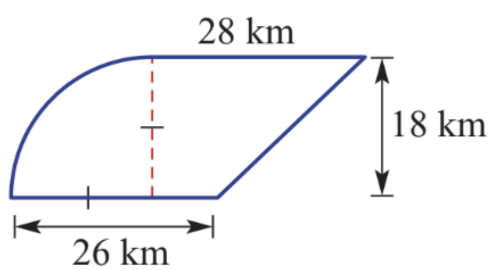


5. Find the area of these composite shapes. Write your answers as exact values and as decimals correct to two places.

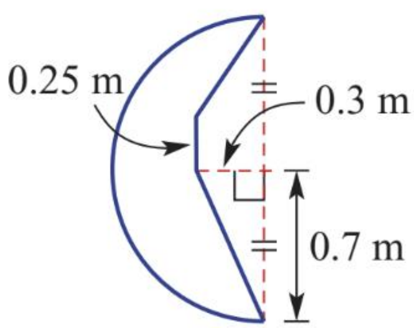
a.



b.



c.



Chapter 6.4 Surface Area — Prisms and Cylinders

Total Surface Area (TSA) of 3D Objects

To calculate the **total surface area (TSA)** of a three-dimensional shape, sum the areas of all its **faces**.

Common 3D Shapes and Their Surface Area Formulas

1. **Cube**

$$TSA = 6s^2$$

(since a cube has 6 square faces of side length s)

2. **Rectangular Prism (Cuboid)**

$$TSA = 2(lw + lh + wh)$$

(where l = length, w = width, h = height)

3. **Cylinder**

$$TSA = 2\pi r^2 + 2\pi rh$$

(sum of two circular bases and the curved surface)

4. **Sphere**

$$TSA = 4\pi r^2$$

(since a sphere has a single curved surface)

5. **Cone**

$$TSA = \pi r^2 + \pi rl$$

(where r = radius, l = slant height)

6. **Triangular Prism**

$$TSA = \text{area of two triangular bases} + \text{sum of three rectangular faces}$$

Nets of 3D Shapes

A **net** is a **2D representation** of a **3D object** when unfolded. It shows all the faces laid flat.

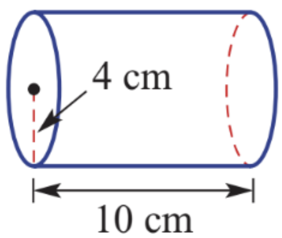
- **Example**: A **cube's net** consists of **six squares** arranged in a cross-like shape.
- **Example**: A **cylinder's net** consists of **two circles** (top & bottom) and a **rectangle** (curved surface when unrolled).

Understanding **nets** helps in **visualizing and calculating surface area efficiently**.

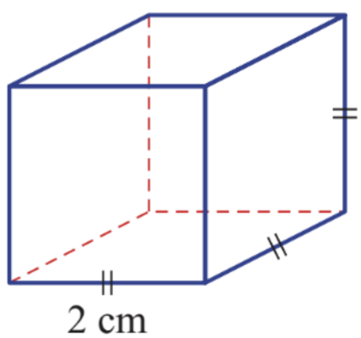
Practice Questions

1. Draw a net for each of these solids.

a.

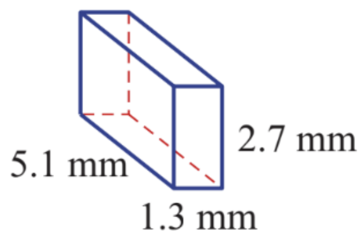


b.

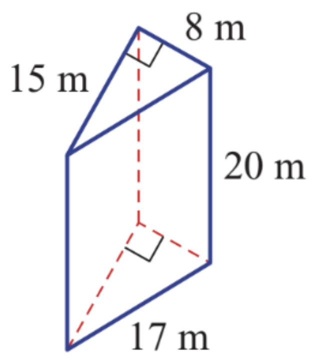


2. Find the total surface area of these solids. Round your answers to two decimal places where necessary.

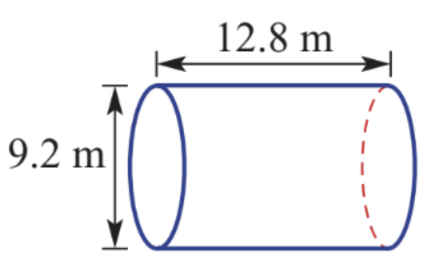
a.



b.

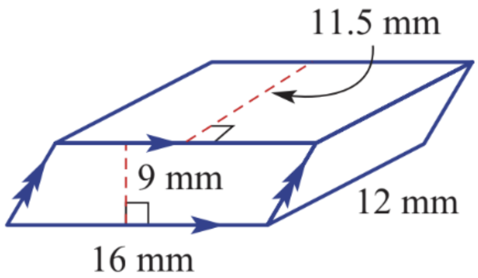


c.

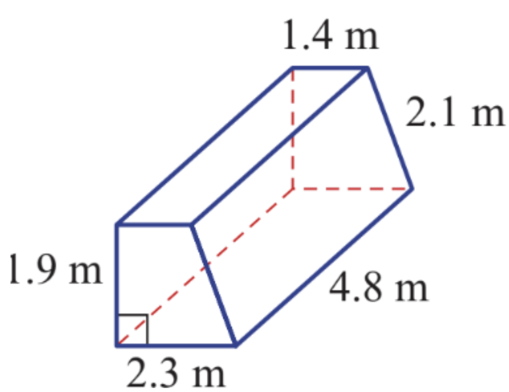


3. Find the total surface area of these solids.

a.

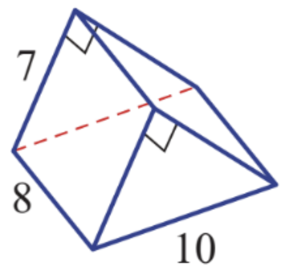


b.

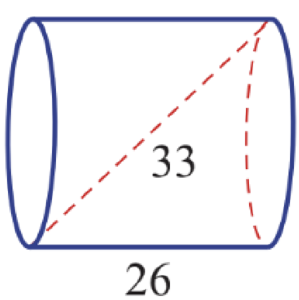


4. Use Pythagoras' theorem to determine any unknown side lengths and find the total surface area of these solids, correct to one decimal place.

a.

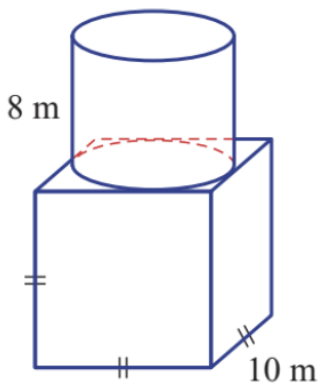


b.

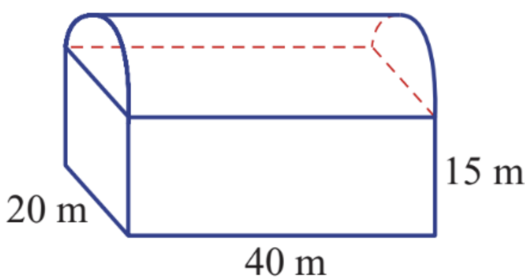


5. Find the total surface area of these composite solids. Answer correct to one decimal place.

a.

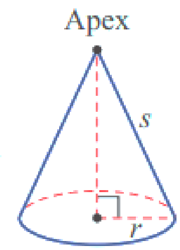


b.

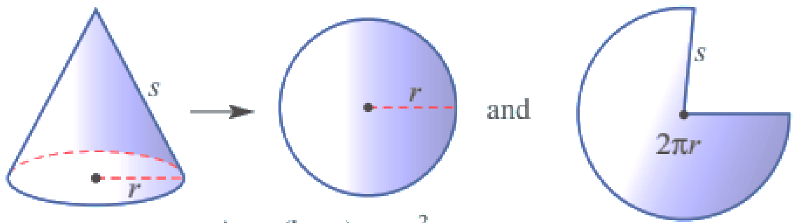


Chapter 6.5 Surface Area — Pyramids and Cones

- A **cone** is a solid with a circular base and a curved surface that reaches from the base to a point called the **apex**.
  - A right cone has its apex directly above the centre of the base.
  - The pronumeral s is used for the slant height and r is the radius of the base.
  - Cone total surface area.



Right cone

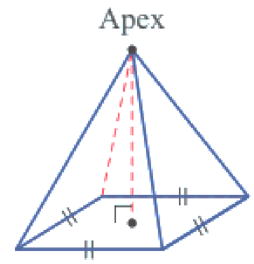


Area (base) $= \pi r^2$

$$\text{Area} = \frac{2\pi r}{2\pi s} \times \pi s^2 = \pi rs$$

$\therefore$ TSA (cone) $= \pi r^2 + \pi rs = \pi r(r + s)$

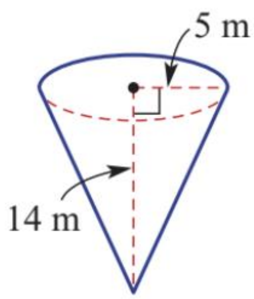
- A **pyramid** has a base that is a polygon and remaining sides that are triangles that meet at the apex.
  - A pyramid is named by the shape of its base.
  - A right pyramid has its apex directly above the centre of the base.



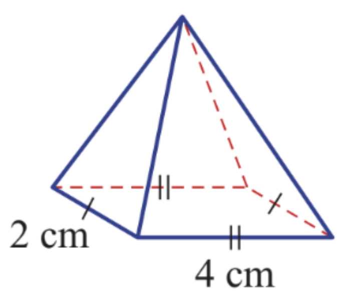
Right square-based pyramid

Practice Questions

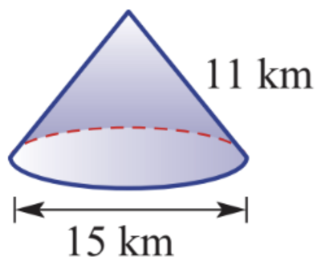
1. Find the exact slant height for these cones, using Pythagoras' theorem. Answer exactly, using a square root sign.



2. Draw a net for each of these solids.

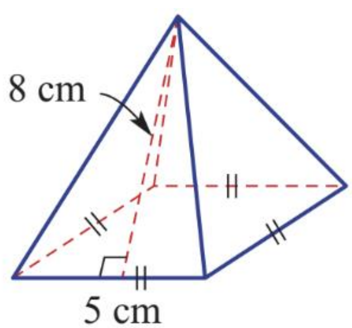


3. Find the total surface area of these cones, correct to two decimal places.

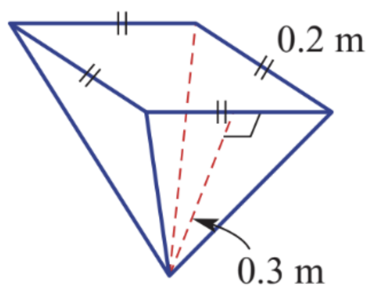


4. Find the total surface area of these pyramids.

a.



b.



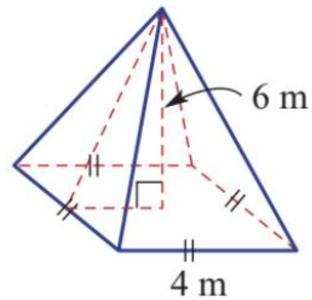
5. A cone with radius $5\ \mathrm{cm}$ has a curved surface area of $400\mathrm{cm}^2$.

a. Find the slant height of the cone correct to one decimal place.

b. Find the height of the cone correct to one decimal place.

6. This right square-based pyramid has base side length 4m and vertical height 6m.

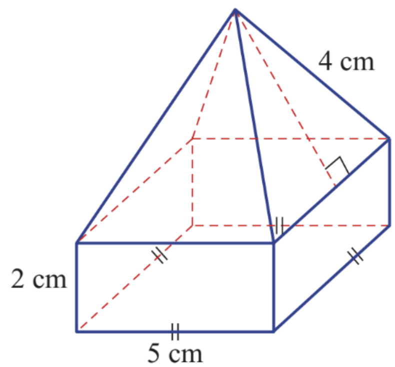
a. Find the height of the triangular faces correct to one decimal place.



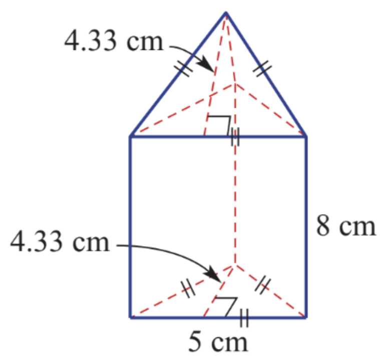
b. Find the total surface area correct to one decimal place.

7. Find the total surface area of these composite solids correct to one decimal place as necessary.

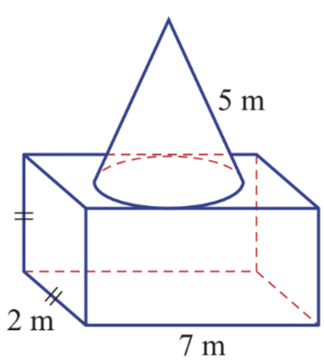
a.



b.



c.

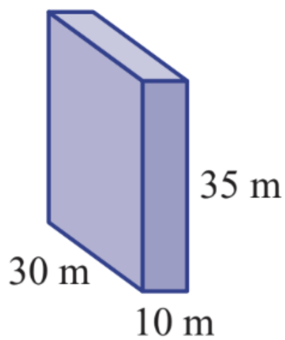


Chapter 6.6 Volume — Prisms and Cylinders

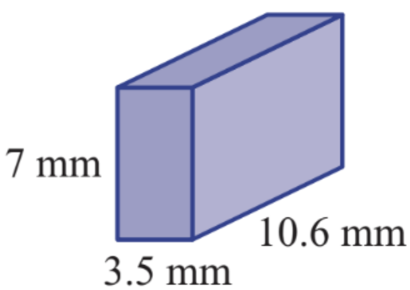
Practice Questions

1. Find the volume of each rectangular prism.

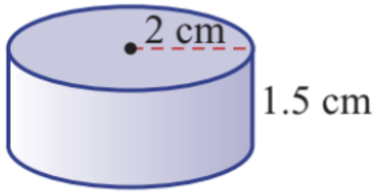
a.

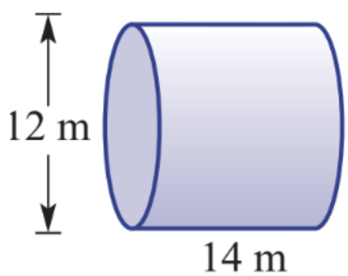


b.



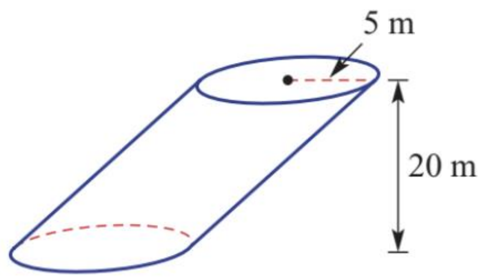
2. Find the volume of each cylinder, correct to two decimal places.

a. 

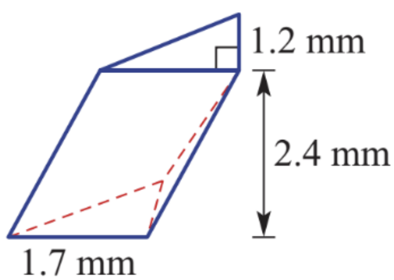
b. 

3. Find the volume of these oblique solids. Round to one decimal place for part **b**.

a.

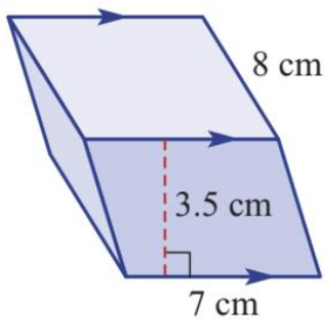


b.

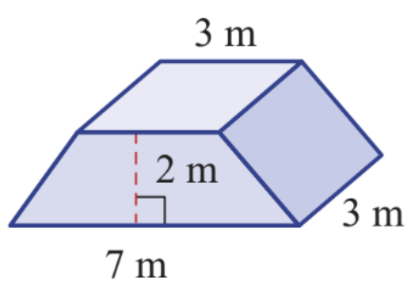


4. Find the volume of these solids, rounding your answers to three decimal places where necessary.

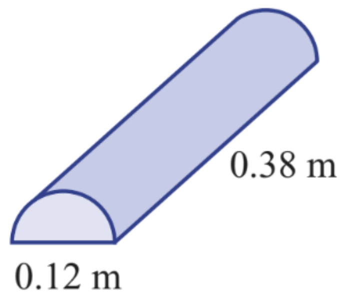
a.



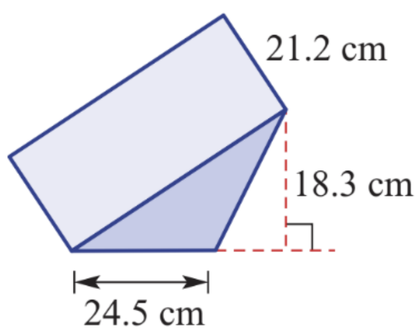
b.



c.

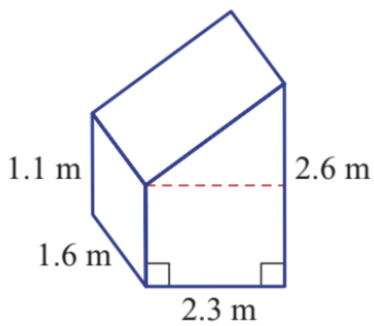


d.

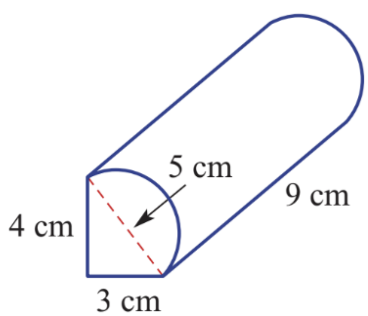


5. Find the volume of these composite objects, rounding to two decimal places where necessary.

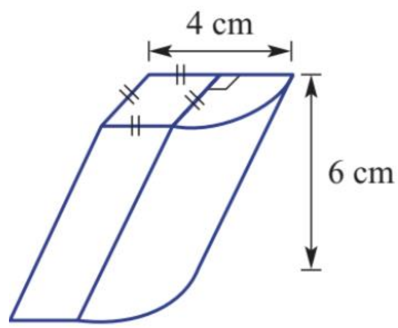
a.



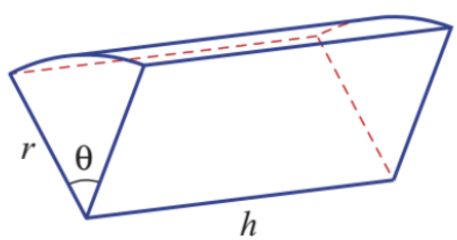
b.



c.



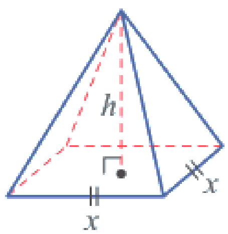
6. Find a rule for the volume of a cylindrical portion with angle θ, radius r and height h as shown.



Chapter 6.7 Volume — Pyramids and Cones

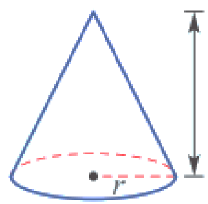
- For pyramids and cones the volume is given by: $V = \frac{1}{3}Ah$
  where A is the area of the base and h is the perpendicular height.

Right square pyramid



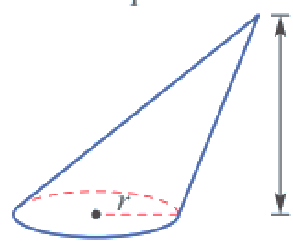
$$V = \frac{1}{3}Ah$$
$$= \frac{1}{3}x^2h$$

Right cone



$$V = \frac{1}{3}Ah$$
$$= \frac{1}{3}\pi r^2h$$

Oblique cone

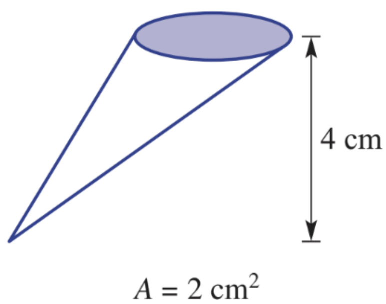


$$V = \frac{1}{3}Ah$$
$$= \frac{1}{3}\pi r^2h$$

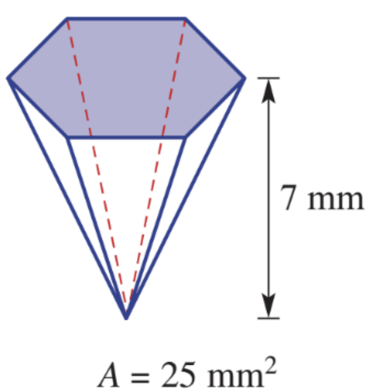
Practice Questions

1. Find the volume of these solids with the given base areas.

a.

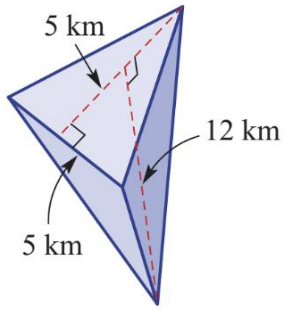


b.

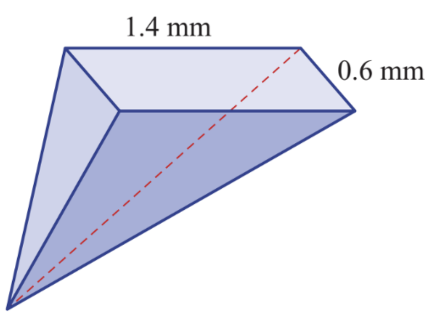


2. Find the volume of the following pyramids. For the oblique pyramids (parts **d, e, f**) use the given perpendicular height.

a.



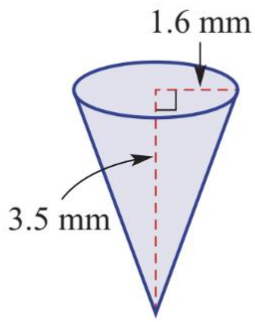
b.



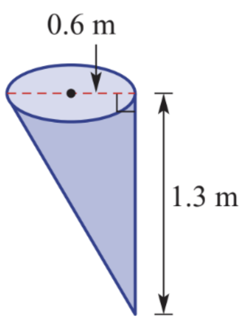
Perpendicular
height = 1.2 mm

3. Find the volume of the following cones, correct to two decimal places.

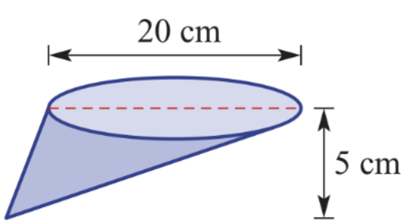
a.



b.

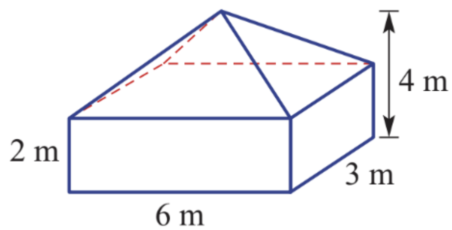


c.

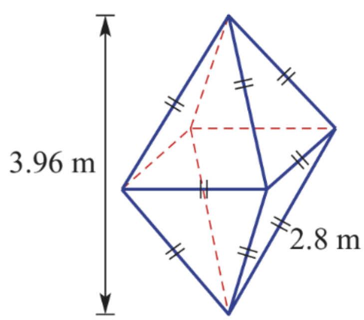


4. Find the volume of these composite objects, rounding to two decimal places where necessary.

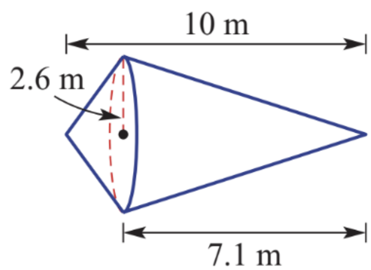
a.



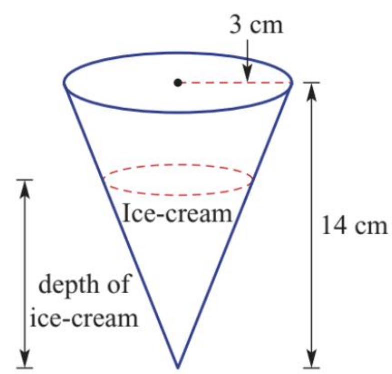
b.



c.



5. The volume of ice-cream in the cone is half the volume of the cone. The cone has a 3 cm radius and 14 cm height. What is the depth of the ice-cream correct to two decimal places?



Chapter 6.8 Spheres

Surface Area and Volume of a Sphere

1. **Surface Area of a Sphere**

$$SA = 4\pi r^2$$

- Where r is the **radius** of the sphere.
- The surface area represents the total area covering the outer surface of the sphere.

2. **Volume of a Sphere**

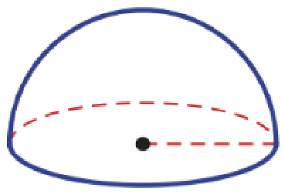
$$V = 43\pi r^3$$

- This formula calculates the total space enclosed within the sphere.
- The volume increases **cubically** with the radius.

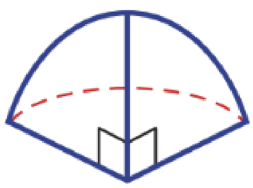
Practice Questions

1. What fraction of a sphere is shown in these diagrams?

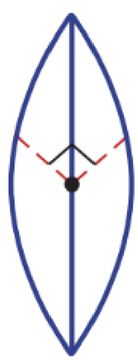
a.



b.

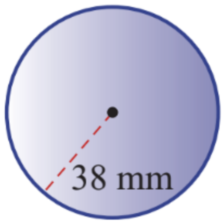


c.

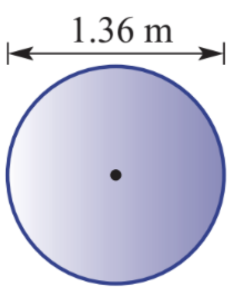


2. Find the surface area and volume of the following spheres, correct to two decimal places.

a.

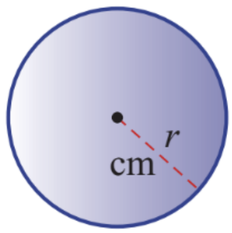


b.



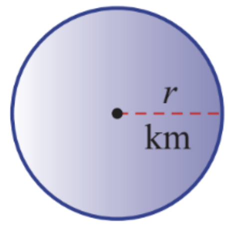
3. a. Find the radius of these spheres with the given volumes, correct to two decimal places.

i.



$V = 180\,\text{cm}^3$

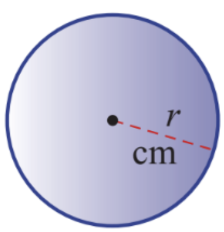
ii.



$V = 0.52\,\text{km}^3$

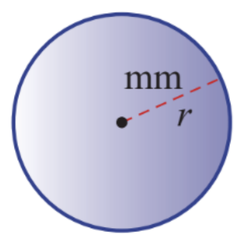
b. Find the radius of these spheres with the given surface area, correct to two decimal places.

i.



$S = 120\text{ cm}^2$

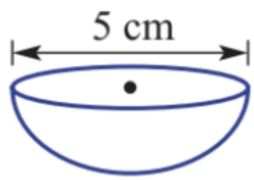
ii.



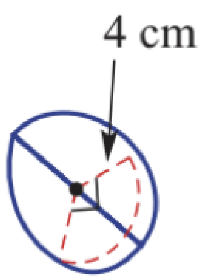
$S = 0.43\,\text{mm}^2$

4. Find the volume of these portions of a sphere correct to two decimal places.

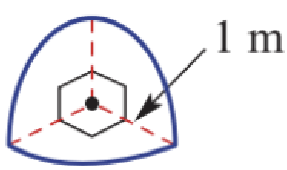
a.



b.

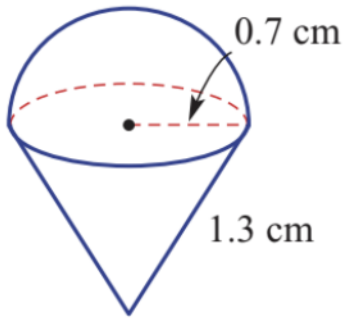


c.

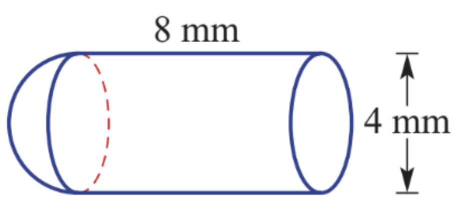


5. Find the total surface area for these solids, correct to two decimal places.

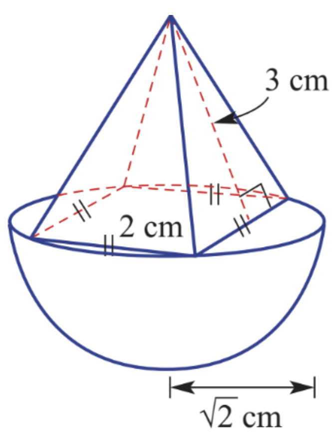
a.



b.

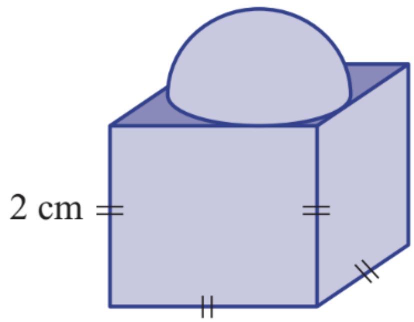


c.

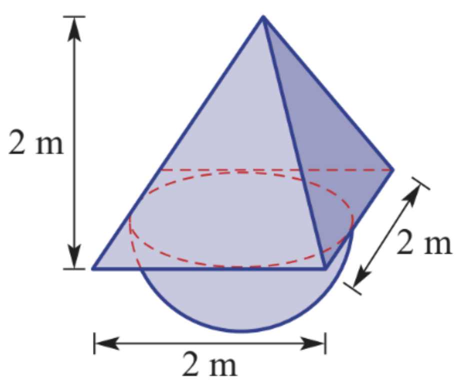


6. Find the volume of the following composite objects correct to two decimal places.

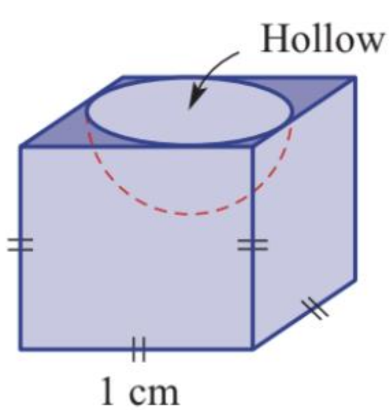
a.



b.

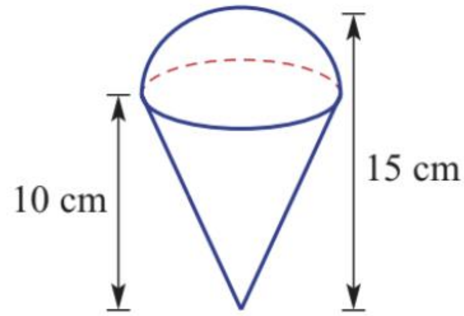


c.



7. A hemisphere sits on a cone and two height measurements are given as shown. Find:

a. the radius of the hemisphere



b. the exact slant height of the cone in surd form

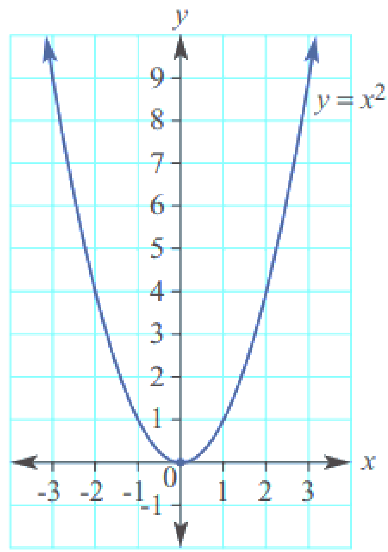
c. the total surface area of the solid, correct to one decimal place.

Chapter 7
Parabolas and Other Graphs

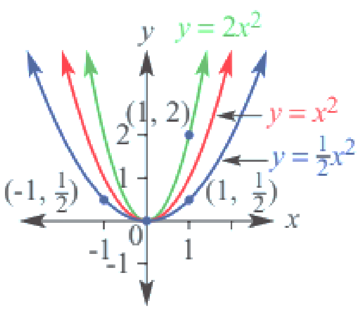
Table of Contents

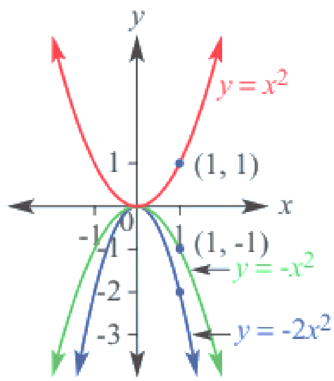
Chapter 7.1 Exploring Parabolas

- A **parabola** is the graph of a quadratic relation. The basic parabola has the rule $y = x^2$.
  - The vertex (or turning point) is (0, 0).
  - It is a minimum turning point.
  - Axis of symmetry is $x = 0$.
  - y-intercept is (0, 0).
  - x-intercept is (0, 0).

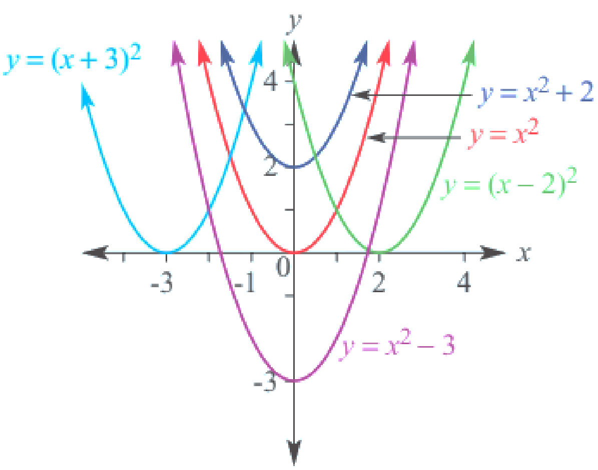


- Simple transformations of the graph of $y = x^2$ include:
  - dilation
  - reflection





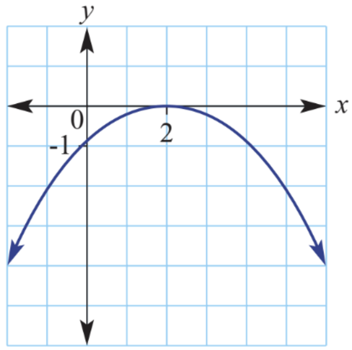
  - translation



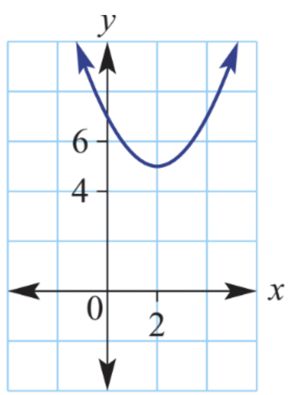
Practice Questions

1. Determine these key features of the following graphs:
   i. turning point
   ii. axis of symmetry
   iii. x-intercepts
   iv. y-intercept

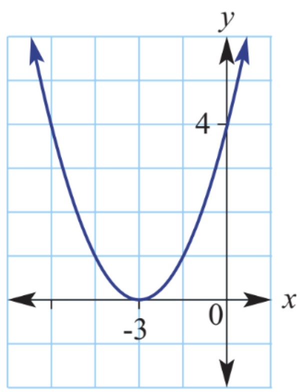
a.



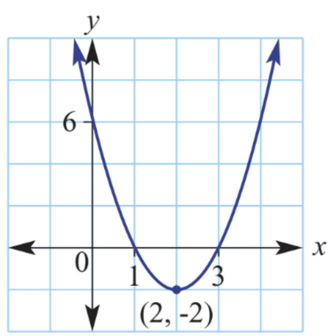
b.



c.



d.



2. Write down the equation of the axis of symmetry for the graphs of these rules.

a. $y = x^2 + 7$

b. $y = -2x^2$

c. $y = x^2 - 4$

d. $y = (x - 2)^2$

e. $y = -(x + 3)^2$

f. $y = -x^2 - 3$

g. $y = x^2 - 16$

h. $y = -(x + 4)^2$

3. Using technology, plot the following pairs of graphs on the same set of axes for $-5 \leq x \leq 5$ and compare their tables of values.

a. $y = x^2$ and $y = \frac{1}{3}x^2$

b. $y = x^2$ and $y = \frac{1}{4}x^2$

Chapter 7.2 Sketching Parabolas with Transformations

Sketching Parabolas and Identifying Key Features

1. **Basic Parabola: $y = ax^2$**

- **Turning point**: $(0,0)$
- **Axis of symmetry**: $x = 0$
- **Intercepts**: $(0,0)$ is both the y**-intercept** and x**-intercept**
- **Effect of a**:
  - $a > 0$ → Parabola **opens upwards** (U-shaped)
  - $a < 0$ → Parabola **opens downwards** (inverted U-shape)

2. **Horizontal Translation: $y = (x - h)^2$**

- The **graph shifts horizontally** based on h:
  - $h > 0$ → Translates **right** by h units
  - $h < 0$ → Translates **left** by | h | units
- **Turning point**: $(h, 0)$
- **Axis of symmetry**: $x = h$

3. **Vertical Translation: $\boldsymbol{y = x^2 + k}$**

- The **graph shifts vertically** based on k:
  - $k > 0$ → Moves **up** by k units
  - $k < 0$ → Moves **down** by $|k|$ units

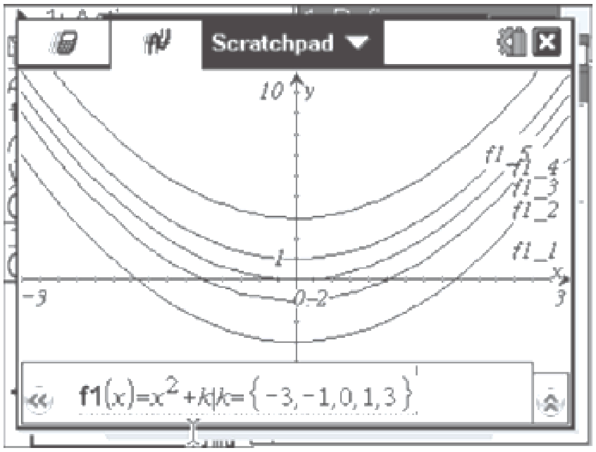
- **Turning point**: $(0, k)$
- **Axis of symmetry**: $x = 0$

4. **General Form (Turning Point Form): $\boldsymbol{y = a(x - h)^2 + k}$**

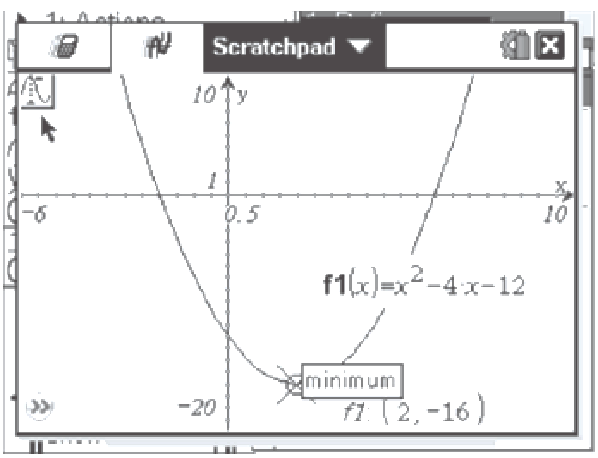
- **Turning point**: (h, k)
- **Axis of symmetry**: $x = h$
- **Effect of $\boldsymbol{a}$**:
  - If $|a| > 1$, the parabola is **narrower** (steeper)
  - If $0 < |a| < 1$, the parabola is **wider**

Using the TI-Nspire:

1 In a graphs and geometry page type the rule in $f1(x)$ using the **given** symbol (|) found in the **symbol palette**.

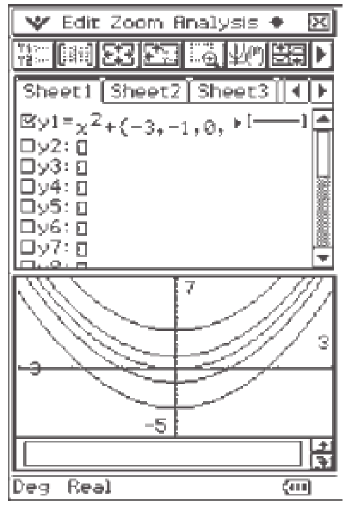


2 Enter the rule $f1(x) = x^2 - 4x - 12$. Change the scale using the window settings. Select **menu**, **Trace**, **Graph Trace** and shift left or right to show significant points.

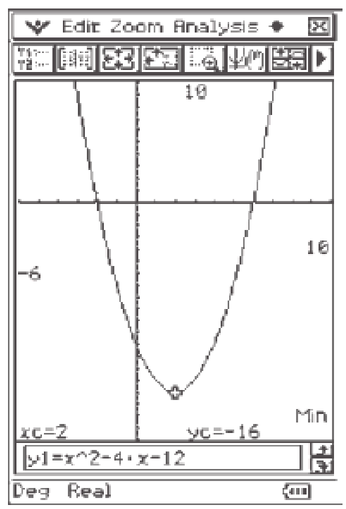


Using the ClassPad:

1 In the **Graph&Table** application enter the rule $y1 = x^2 + \{-3,-1,0,1,3\}$ followed by **EXE**. Tap [graph icon] to see the graph.



2 Enter the rule $y1 = x^2 - 4x - 12$. Tap 6 and set an appropriate scale. Tap **Analysis**, **G-Solve**, **x-Cal** to locate the x-intercepts. Tap **Analysis**, **G-Solve**, **Min** to locate the turning point.



Practice Questions

1. Sketch graphs of the following quadratics, labelling the turning point and the y-intercept.

   a. $y = \frac{1}{2}x^2$

   b. $y = x^2 - 4$

c. $y = (x + 3)^2$

d. $y = -(x - 3)^2$

2. State the coordinates of the turning point for the graphs of these rules.

a. $y = (x - 1)^2 + 3$

b. $y = (x - 2)^2 + 2$

c. $y = -(x + 1)^2 + 4$

d. $y = -(x - 4)^2 - 10$

3. Sketch graphs of the following quadratics, labelling the turning point and the y-intercept.

   a. $y = (x + 3)^2 + 2$

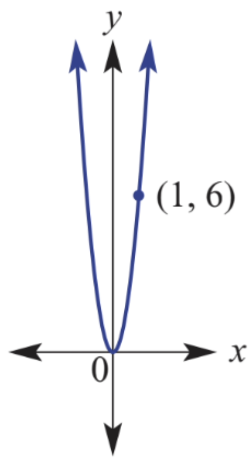
   b. $y = (x - 1)^2 - 4$

c. $y = -(x + 3)^2 - 2$

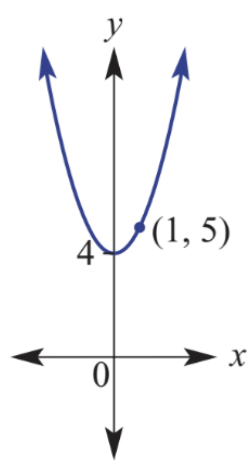
d. $y = -(x + 2)^2 + 2$

4. Determine the rule for the following parabolas.

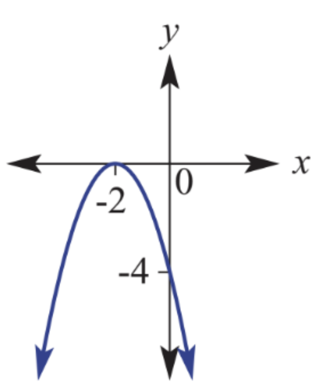
a.



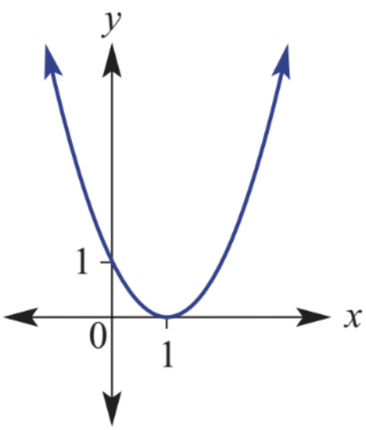
b.



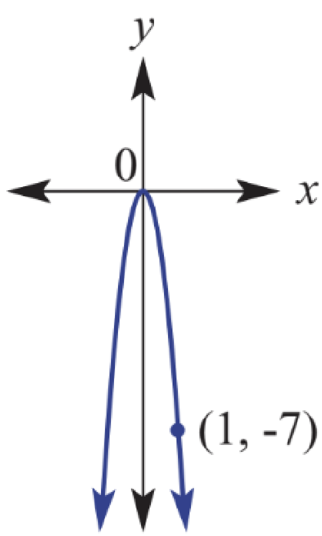
c.



d.



e.



5. The path of a basketball is given by $y = -(x - 5)^2 + 25$ where y metres is the height and x metres is the horizontal distance.

a. Is the turning point a maximum or a minimum?

b. What are the coordinates of the turning point?

c. What is the y-intercept?

d. What is the maximum height of the ball?

e. What is the height of the ball at these horizontal distances?

i. $x = 7$

Chapter 7.3 Sketching $y = x^2 + bx + c$ using Factorisation

Steps to Sketch the Graph of $y = x^2 + bx + c$

1. Find the y-intercept

- Substitute $x = 0$ into the equation.
- The y-intercept is c, so the point is $(0, c)$.

2. Find the x-intercepts

- Set $y = 0$ and solve for x:

$$x^2 + bx + c = 0$$

- If factorable, use the **Null Factor Law**:

$$(x - p)(x - q) = 0 \Rightarrow x = p \text{ or } x = q$$

- If not factorable, use the **quadratic formula**:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- The x-intercepts are $(p, 0)$ and $(q, 0)$.

3. Find the Axis of Symmetry

- The axis of symmetry is halfway between the x-intercepts:

$$x = \frac{p + q}{2}$$

- This is also the x-coordinate of the **turning point**.

4. Find the Turning Point

- Substitute $x = \frac{p+q}{2}$ into the equation to find the y-coordinate.
- The turning point is $\left(\frac{p+q}{2}, y\right)$.

Practice Questions

1. Use the Null Factor Law to find the x-intercepts ($y = 0$) for these factorised quadratics.

   a. $y = (x + 1)(x + 5)$

   b. $y = -3x(x + 2)$

   c. $y = (x + 2\sqrt{2})(x - 2\sqrt{2})$

2. Factorise and use the Null Factor Law to find the x-intercepts for these quadratics.

a. $y = x^2 - 8x + 16$

b. $y = 2x^2 + 10x$

c. $y = x^2 - 10$

3. A parabola has the rule $y = x^2 - 2x - 48$.

a. Factorise $x^2 - 2x - 48$.

b. Find the x-intercepts of $y = x^2 - 2x - 48$.

c. Hence state the equation of the axis of symmetry.

d. Find the coordinates of the turning point.

4. Sketch the graphs of the following quadratics.

a. $y = x^2 - 8x + 12$

b. $y = x^2 - 6x - 16$

c. $y = x^2 - 4x - 21$

d. $y = x^2 - 10x + 24$

4. Sketch graphs of the following quadratics.

a. $y = x^2 - 5x + 6$

b. $y = x^2 + 11x + 30$

c. $y = x^2 + 13x + 12$

d. $y = x^2 - x - 2$

e. $y = x^2 + 3x - 4$

f. $y = x^2 + 9x - 22$

5. Sketch by first finding x-intercepts.

a. $y = x^2 + 6x$

b. $y = x^2 - 5x$

c. $y = x^2 + 7x$

6. Sketch graphs of the following perfect squares.

a. $y = x^2 + 8x + 16$

b. $y = x^2 + 20x + 100$

7. Determine the turning points of the following quadratics.

a. $y = 3(x^2 - 7x + 10)$

b. $y = 4x^2 + 24x + 32$

c. $y = -4(x^2 - 49)$

d. $y = 5x^2 - 10x + 5$

8. Sketch a graph of these quadratics.

a. $y = 1 - x^2$

b. $y = 3x - x^2$

c. $y = -x^2 + 8x + 9$

9. If the graph of $y = a(x + 2)(x - 4)$ passes through the point $(2, 16)$ determine the value of a and the coordinates of the turning point for this parabola.

10. Consider the quadratics $y = x^2 - 2x - 8$ and $y = -x^2 + 2x + 8$.

a. Show that both quadratics have the same x-intercepts.

b. Find the coordinates of the turning points for both quadratics.

c. Compare the positions of the turning points.

Chapter 7.4 Sketching by Completing the Square

Sketching a Quadratic Using Completing the Square

Any quadratic of the form:

$$y = ax^2 + bx + c$$

can be rewritten in **turning point form**:

$$y = a(x-h)^2 + k$$

where (h, k) is the **turning point**.

Steps to Sketch the Graph

1. Convert to Turning Point Form by Completing the Square

For $y = ax^2 + bx + c$:

1. Factor out a from the quadratic terms if $a \neq 1$:

$$y = a(x^2 + \frac{b}{a}x) + c$$

2. Complete the square inside the bracket:
   - Take half of $\frac{b}{a}$, square it, and add/subtract it.
   - Rewrite as a perfect square:
   $$y = a((x-h)^2 - \text{extra term}) + c$$
   - Simplify to get $y = a(x-h)^2 + k$.

2. Identify Key Features

- **Turning point** (h, k).
  - If $a > 0$, the parabola has a **minimum**.
  - If $a < 0$, the parabola has a **maximum**.
- $\boldsymbol{Y}$**-intercept**: Substitute $x = 0$ into the equation.
- $\boldsymbol{X}$**-intercepts**: Solve $y = 0$ using the Null Factor Law or the quadratic formula.

Example: Sketch $\boldsymbol{y = x^2 + 6x + 5}$

Step 1: Complete the Square

$$y = (x^2 + 6x) + 5$$

Take half of 6, square it:

$$\left(\frac{6}{2}\right)^2 = 9$$

Add and subtract 9 inside the bracket:

$$y = (x^2 + 6x + 9 - 9) + 5$$

Rewrite as a perfect square:

$$y = (x + 3)^2 - 9 + 5$$

Simplify:

$$y = (x + 3)^2 - 4$$

So the equation in turning point form is:

$$y = (x + 3)^2 - 4$$

Step 2: Identify Key Features

- **Turning point**: $(-3, -4)$.
- $\boldsymbol{Y}$**-intercept**:
  - Set $x = 0$:

$$y = (0 + 3)^2 - 4 = 9 - 4 = 5$$

  - Y-intercept is $(0,5)$.
- $\boldsymbol{X}$**-intercepts**: Solve $(x + 3)^2 - 4 = 0$.
  - Rearrange:

$$(x + 3)^2 = 4$$

  - Take the square root:

$$x + 3 = \pm 2$$

  - Solve for x:

$$x = -3 \pm 2$$

  - $x = -1$ and $x = -5$.
  - X-intercepts: $(-1,0)$ and $(-5,0)$.

Practice Questions

1. Determine the y-intercept of each of the following.

a. $y = (x - 3)^2 - 2$

b. $y = -(x - 7)^2 - 6$

c. $y = x^2 + 7x - 5$

d. $y = x^2 - 12x - 5$

2. Determine the coordinates of the x-intercepts (if any) of the following.

a. $y = (x - 3)^2 - 36$

b. $y = (x - 5)^2 - 3$

c. $y = 2(x - 7)^2 + 18$

d. $y = -(x - 3)^2 + 10$

3. Determine the x-intercepts (if any) by first completing the square and rewriting the equation in turning-point form. Give exact answers.

a. $y = x^2 + 6x + 2$

b. $y = x^2 + 8x - 5$

c. $y = x^2 - 4x + 14$

d. $y = x^2 - 12x - 5$

4. Sketch the graphs of the following. Label the turning point and intercepts.

a. $y = (x + 4)^2 - 1$

b. $y = (x + 7)^2 + 2$

c. $y = -(x - 5)^2 - 4$

d. $y = -(x - 2)^2 + 4$

5. Sketch these graphs by completing the square. Label the turning point and intercepts.

a. $y = x^2 + 6x + 9$

b. $y = x^2 - 2x - 15$

6. Sketch these graphs by completing the square. Label the turning point and intercepts with exact values.

   a. $y = x^2 - 2x + 6$

   b. $y = x^2 - 3x + 1$

7. Complete the square and decide if the graphs of the following quadratics will have 0, 1 or 2 x-intercepts.

   a. $y = x^2 + 6x + 9$

   b. $y = x^2 - 5x + 5$

8. Take out a common factor and complete the square to find the x-intercepts for these quadratics.

a. $y = 2x^2 - 12x - 14$

b. $y = 2x^2 - 6x + 2$

Chapter 7.5 Sketching using the Quadratic Formula

Steps to Sketch the Graph of $y = ax^2 + bx + c$

To accurately sketch the quadratic function $y = ax^2 + bx + c$, follow these steps:

1. Find the y-Intercept

- Set $x = 0$:

$$y = a(0)^2 + b(0) + c = c$$

- The **y-intercept** is $(0, c)$.

2. Find the x-Intercepts (Roots)

- Solve $y = 0$:

$$ax^2 + bx + c = 0$$

- Use the **quadratic formula** to find the x-intercepts:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

3. Use the Discriminant $\Delta = b^2 - 4ac$ to Determine the Number of X-Intercepts

- **If $\Delta > 0$**: Two real solutions → **Two x-intercepts**.
- **If $\Delta = 0$**: One real solution → **One x-intercept (tangent to x-axis)**.
- **If $\Delta < 0$**: No real solutions → **No x-intercepts**.

4. Find the Axis of Symmetry and Turning Point

- The **axis of symmetry** is the vertical line:

$$x = -\frac{b}{2a}$$

- The **turning point** (vertex) is:

$$\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$$

Example: Sketch $y = x^2 - 4x + 3$

Step 1: Find the y-Intercept

$$y = (0)^2 - 4(0) + 3 = 3$$

- Y-intercept: (0,3).

Step 2: Find the x-Intercepts

Solve $x^2 - 4x + 3 = 0$ using the quadratic formula:

- Compute the discriminant:

$$\Delta = (-4)^2 - 4(1)(3) = 16 - 12 = 4$$

- Since $\Delta > 0$, there are **two x-intercepts**:

$$x = \frac{4 \pm \sqrt{4}}{2(1)} = \frac{4 \pm 2}{2}$$

$$x = \frac{4+2}{2} = \frac{6}{2} = 3, \qquad x = \frac{4-2}{2} = \frac{2}{2} = 1$$

- X-intercepts: $(1,0)$ and $(3,0)$.

Step 3: Find the Axis of Symmetry and Turning Point

- Axis of symmetry:

$$x = \frac{-(-4)}{2(1)} = \frac{4}{2} = 2$$

- Find the turning point:

$$y = (2)^2 - 4(2) + 3 = 4 - 8 + 3 = -1$$

- Turning point: $(2, -1)$.

Final Graph Features

- **Y-intercept**: $(0,3)$.
- **X-intercepts**: $(1,0)$ and $(3,0)$.
- **Axis of symmetry**: $x = 2$.
- **Turning point**: $(2, -1)$ (minimum point).
- Since $a = 1 > 0$, the parabola **opens upwards**.

Practice Questions

1. Use the discriminant to determine the number of x-intercepts for the parabolas given by the following quadratics.

   a. $y = -x^2 + 4x + 2$

   b. $y = 2x^2 - 12x + 18$

   c. $y = -3x^2 - 2x$

   d. $y = -5x^2 + x$

2. Determine the y-intercept for the parabolas given by the following quadratics.

a. $y = 4x^2 + 3x - 2$

b. $y = -2x^2 + 7x - 10$

c. $y = 5x^2 - 7$

3. Using $x = -\frac{b}{2a}$ determine the coordinates of the turning points for the parabolas defined by the following quadratics.

a. $y = x^2 - 4x + 3$

b. $y = -x^2 + 7x - 7$

c. $y = -4x^2 - 9$

4. Sketch the graphs of these quadratics.

a. $y = 9x^2 - 6x + 1$

b. $y = -9x^2 + 30x - 25$

c. $y = -3x^2 + 12x - 16$

d. $y = 3x^2 + 4x + 2$

5. Give the exact coordinates of the x-intercepts of the graphs of these parabolas. Simplify surds.

a. $y = -4x^2 + 8x + 6$

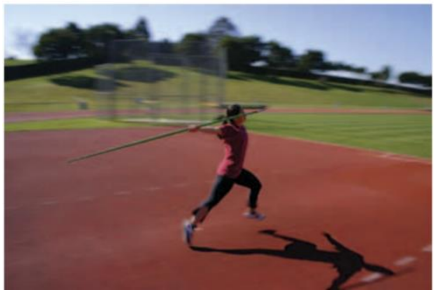
b. $y = 2x^2 - 8x + 5$

c. $y = 5x^2 - 10x - 1$

Chapter 7.6 Applications of Parabolas

Practice Questions

1. The path of a javelin thrown by Jo is given by the formula $h = -\frac{1}{16}(d-10)^2 + 9$ where h metres is the height of the javelin above the ground and d metres is the horizontal distance travelled.



a. Sketch the graph of the rule for $0 \leq d \leq 22$ by finding the intercepts and the coordinates of the turning point.

b. What is the maximum height the javelin reaches?

c. What horizontal distance does the javelin travel?

2. The equation for the arch of a particular bridge is given by $h = -\frac{1}{500}(x - 100)^2 + 20$ where h m is the height above the base of the bridge and x m is the distance from the left side.


Richmond Bridge, Tasmania, is the oldest bridge in Australia still in use.

a. Determine the coordinates of the turning point of the graph.

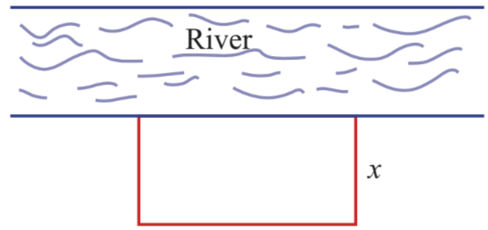
b. Determine the x-intercepts of the graph.

c. Sketch the graph of the arch for appropriate values of x.

d. What is the span of the arch?

e. What is the maximum height of the arch?

3. A farmer has 100 m of fencing to form a rectangular paddock with a river on one side (that does not require fencing) as shown.



a. Use the perimeter to write an expression for the length of the paddock in terms of x.

b. Write an equation for the area of the paddock (A m^2) in terms of x.

c. Decide on suitable values of x.

d. Sketch the graph of A versus x for suitable values of x.

e. Use the graph to determine the maximum paddock area that can be formed.

f. What will the dimensions of the paddock be to achieve its maximum area?

4. The equation for a support span is given by $h = \frac{1}{40}(x - 20)^2$ where h m is the distance below the base of a bridge and x m is the distance from the left side.

a. Determine the coordinates of the turning point of the graph.

b. Sketch a graph of the equation for suitable values of x.

c. What is the width of the support span?

d. What is the maximum height of the support span?

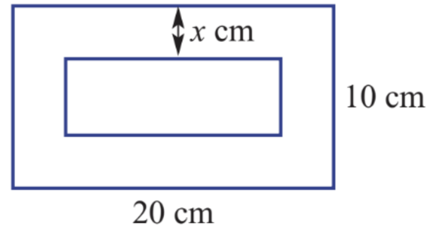
5. The height h metres of a flying kite is given by the rule $h = t^2 - 6t + 10$ for t seconds.

a. Find the minimum height of the kite during this time.

b. Does the kite ever hit the ground during this time? Give reasons.

6. A framed picture has a total length and width of 20 cm and 10 cm. The frame has width x cm.

a. Find the rule for the area (A cm^2) of the picture inside.



b. What are the minimum and maximum values of x?

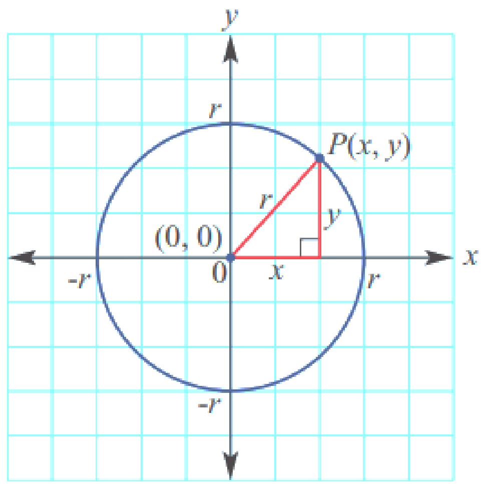
c. Sketch a graph of A vs x using suitable values of x.

d. Explain why there is no turning point for your graph using suitable values of x.

e. Find the width of the frame if the area of the picture is $144\ \mathrm{cm}^2$.

Chapter 7.7 Graphs of Circles

- The Cartesian equation of a circle with centre (0, 0) and radius r is given by $x^2 + y^2 = r^2$
- Making x or y the subject:
  - $y = \pm\sqrt{r^2 - x^2}$
  - $x = \pm\sqrt{r^2 - y^2}$



Using Pythagoras' theorem,
$a^2 + b^2 = c^2$ gives $x^2 + y^2 = r^2$

Practice Questions

1. A circle has equation $x^2 + y^2 = 9$. Complete the following.

   a. State the coordinates of the centre.

   b. State the radius.

   c. Find the values of y when $x = 2$.

d. Find the values of x when $y = \frac{3}{2}$.

e. Sketch a graph showing intercepts.

2. Give the radius of the circles with these equations.

a. $x^2 + y^2 = 144$

b. $x^2 + y^2 = 20$

3. Write the equation of a circle with centre $(0, 0)$ and the given radius.

a. 51

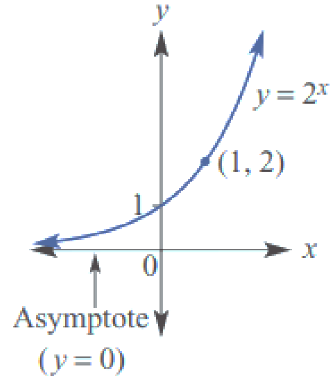
b. 0.5

4. Find the coordinates of the points where $x^2 + y^2 = 9$ intersects $y = x$. Sketch a graph showing the intersection points.

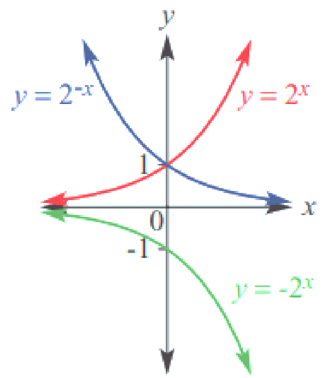
5. Determine the exact length of the chord formed by the intersection of $y = x - 1$ and $x^2 + y^2 = 5$. Sketch a graph showing the intersection points and the chord.

Chapter 7.8 Graphs of Exponentials

An **asymptote** is a line that a curve approaches by getting closer and closer to it but never reaching it.



- A simple **exponential** rule is of the form $y = a^x$ where $a > 0$ and $a \neq 1$.
  - y-intercept is (0, 1).
  - $y = 0$ is the equation of the asymptote.
- The graph of $y = -a^x$ is the reflection of the graph of $y = a^x$ in the x-axis. Note: $y = -a^x$ means $y = -1 \times a^x$.
- The graph of $y = a^{-x}$ is the reflection of the graph of $y = a^x$ in the y-axis.



Practice Questions

1. Sketch the graph of the following on the same set of axes, labelling the y-intercept and the point where $x = 1$.

   a. $y = 4^x$

   b. $y = 5^x$

2. Sketch the graph of the following on the same set of axes labelling the y-intercept and the point where $x = 1$.

a. $y = -5^x$

b. $y = -3^x$

3. Sketch the graph of the following on the same set of axes, labelling the y-intercept and the point where $x = 1$.

   a. $y = 3^{-x}$

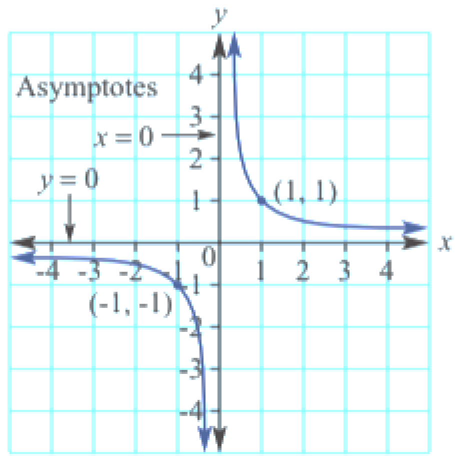
   b. $y = 6^{-x}$

4. a. Find the intersection of the graphs of $y = 2^x$ and $y = 4$.

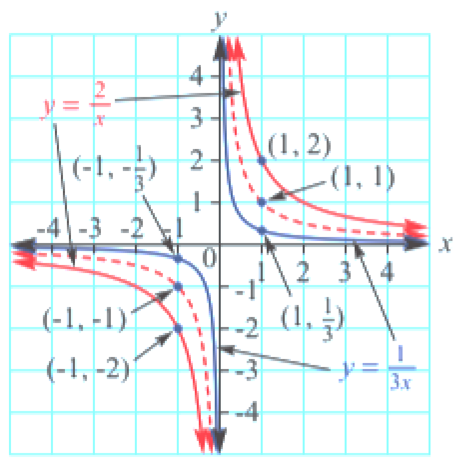
b. Find the intersection of the graphs of $y = 2^{-x}$ and $y = 8$.

Chapter 7.9 Graphs of Hyperbolas

- An **asymptote** is a straight line that a curve approaches more and more closely but never quite reaches.



- A **rectangular hyperbola** is the graph of the rule $y = \frac{a}{x}$, $a \neq 0$.
  - $y = \frac{1}{x}$ is the basic rectangular hyperbola.
  - $x = 0$ (y-axis) and $y = 0$ (x-axis) are its asymptotes.
  - For $a > 1$ the hyperbola will be further out from the asymptotes.
  - For $0 < a < 1$ the hyperbola will be closer in to the asymptotes
- The graph of $y = -\frac{a}{x}$ is a reflection of the graph of $y = \frac{a}{x}$ in the x (or y) axis.



Practice Questions

1. Sketch the graphs of these hyperbolas, labelling the points where $x = 1$ and $x = -1$.

   a. $y = \frac{2}{x}$

   b. $y = -\frac{2}{x}$

2. Find the coordinates on the graph of $y = \frac{2}{x}$ where:

   a. $x = -6$

3. Find the coordinates on the graph of $y = -\frac{5}{x}$ where:

   a. $x = 9$

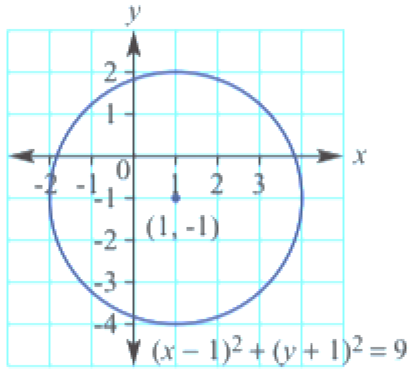
4. Find the coordinates on the graph of $y = \frac{3}{x}$ where:

   a. $y = -6$

5. Decide whether the point $(1, -5)$ lies on the hyperbola $y = -\frac{5}{x}$.

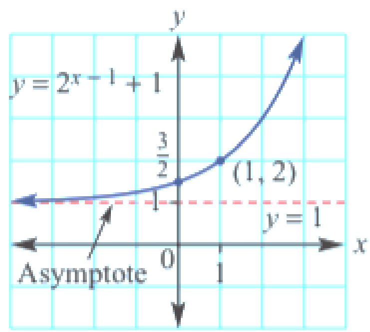
Chapter 7.10 Further Transformations of Graphs

- The equation of a **circle** in standard form is $(x-h)^2+(y-k)^2=r^2$.
  - (h, k) is the centre.
  - r is the radius.



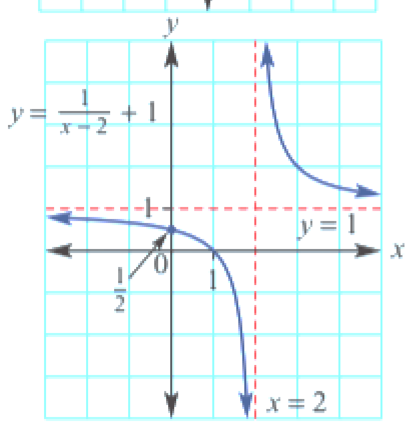
- For the graph of the **exponential** equation $y=a^{x-h}+k$ the graph of $y=a^x$ is:
  - translated h units to the right
  - translated k units up.

  The equation of the asymptote is $y=k$.



- For the graph of the **hyperbola** $y=\dfrac{1}{x-h}+k$ the graph of $y=\dfrac{1}{x}$ is:
  - translated h units to the right
  - translated k units up.

  The asymptotes are $x=h$ and $y=k$.



Practice Questions

1. Sketch the graph of the following circles. Label the coordinates of the centre and find the $x-$ and $y-$intercepts, if any.

   a. $(x - 1)^2 + (y + 3)^2 = 25$

   b. $x^2 + (y - 4)^2 = 36$

   c. $(x + 3)^2 + (y - 1)^2 = 5$

2. Sketch the graph of these exponentials. Label the asymptote and find the y-intercept.

a. $y = 2^x - 5$

b. $y = 2^{x+1}$

c. $y = 2^{x-3} - 4$

3. Sketch the graph of these hyperbolas. Label the asymptotes and find the $x-$ and $y-$ intercepts.

a. $y = \frac{1}{x+3}$

b. $y = \frac{1}{x-1} - 3$

c. $y = \frac{1}{x-5} + 6$

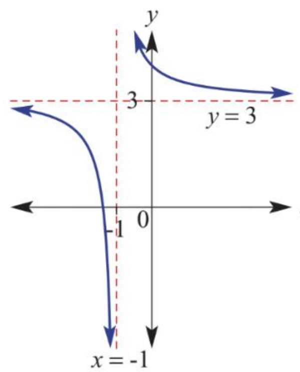
4. Sketch these hyperbolas labelling asymptotes and intercepts.

a. $y = \frac{-2}{x+2} - 1$

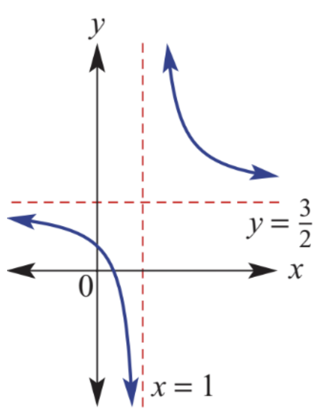
b. $y = \frac{-2}{x-3} - 2$

5. The following hyperbolas are of the form $y = \frac{1}{x-h} + k$. Write the rule for each graph.

a.



b.



6. Find the coordinates of the intersection of the graphs of these equations.

a. $y = \frac{1}{x-2} + 1$ and $y = x + 3$

b. $(x - 1)^2 + y^2 = 4$ and $y = 2x$

c. $x^2 + (y + 1)^2 = 10$ and $y = \frac{1}{3}x - 1$

CHAPTER 8 PROBABILITY

Table of Contents

Chapter 8.1 Review of Probability

Key Definitions:

- **Trial**: A single experiment or occurrence, such as one toss of a die.
- **Sample Space**: The set of all possible outcomes of an experiment.
- **Outcome**: A possible result of an experiment (e.g., getting a 3 on a die roll).
- **Event**: A set of favorable outcomes, which is a subset of the sample space.
- **Equally Likely Outcomes**: Outcomes that have the same chance of occurring in an experiment.

Probability Scale:

- **Probability** is a value between **0 and 1** that represents the likelihood of an event occurring:
    - **0** means the event is impossible.
    - **1** means the event is certain.
    - Values in between (like 0.5) show varying levels of likelihood.

Theoretical Probability:

For events where outcomes are equally likely, the probability is calculated as:

$$\Pr(\text{Event}) = \frac{\text{Number of favorable outcomes}}{\text{Total number of possible outcomes}}$$

Experimental Probability:

Experimental probability is calculated based on actual results from an experiment and is given by:

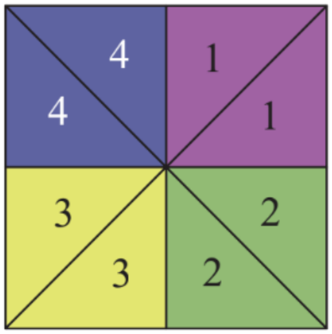
$$\Pr(\text{Event}) = \frac{\text{Number of favorable outcomes}}{\text{Total number of trials}}$$

The **long-run proportion** is an approximation of the probability obtained after a large number of trials.

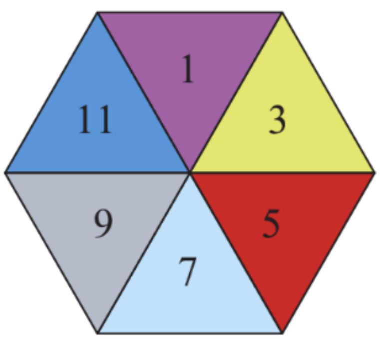
Practice Questions

1. For the following spinners, find the probability that the outcome will be a 4.

a.



b.



2. A letter is chosen from the word TEACHER. Find the probability that the letter is:

   a. an R

   b. an E

   c. not an E

   d. an R or an E

3. A 10-sided dice numbered 1 to 10 is tossed once. Find these probabilities.

a. $\Pr(8)$

b. $\Pr(\text{odd})$

c. $\Pr(\text{even})$

d. $\Pr(\text{less than } 6)$

e. $\Pr(\text{prime})$ (remember that 1 is not prime)

f. $\Pr(3 \text{ or } 8)$

g. $\Pr(8, 9 \text{ or } 10)$

4. An experiment involves tossing two dice and counting the number of sixes. Here are the results after running the experiment 100 times.

Number of sixes	0	1	2
Frequency	62	35	3

Find the experimental probability of obtaining:

a. 0 sixes

b. 2 sixes

c. fewer than 2 sixes

d. at least one six

5. Find the probability of choosing a red counter if a counter is chosen from a box that contains the following counters.

   a. 3 red and 5 yellow

   b. 5 red, 12 green and 7 orange

   c. 6 blue and 4 green

6. A card is chosen from a standard pack of 52 playing cards that includes 4 aces, 4 kings, 4 queens and 4 jacks. Find the following probabilities.

a. Pr(king)

b. Pr(heart or club)

c. Pr(heart or king)

d. Pr(neither a heart nor a king)

7. The probability of selecting a white chocolate from a box is $\frac{1}{5}$ and the probability of selecting a dark chocolate from the same box is $\frac{1}{3}$. The other chocolates are milk chocolates.

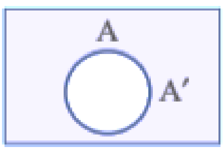
a. Find the probability of selecting a milk chocolate.

b. How many chocolates in total could be in the box? Give reasons. Is there more than one answer?

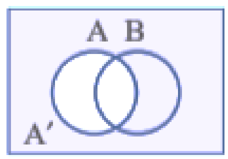
Chapter 8.2 Unions and Intersections

Set notation

- A **set** is a collection or group of elements that can include numbers, letters or other objects.
- The **sample space** denoted by S, Ω, U or ξ, is the set of all possible elements or objects considered in a particular situation. This is also called the **universal set**.
- A **Venn diagram** illustrates how all elements in the sample space are distributed among the events.



OR

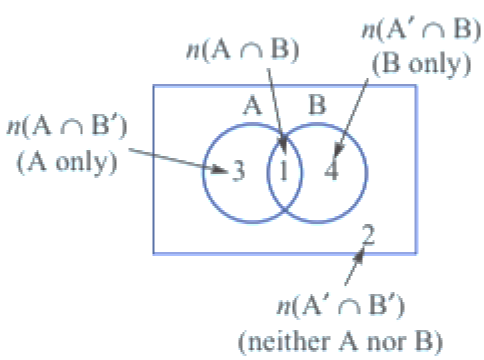


- A **null** or **empty set** is a set with no elements and is symbolised by { } or ∅.
- All elements that belong to both A *and* B make up the **intersection**: $A \cap B$.
- All elements that belong to either events A *or* B make up the **union**: $A \cup B$.
- Two sets A and B are **mutually exclusive** if they have no elements in common, meaning $A \cap B = \varnothing$.

- For an event A, the **complement** of A is A′ (or 'not A').
- $\Pr(A') = 1 - \Pr(A)$
- **A only** (or $A \cap B'$) is defined as all the elements in A but not in any other set.
- ***n*(A)** is the number of elements in set A.

Venn diagrams and two-way tables are useful tools when considering two or more events.

Venn diagram



Two-way table

	A	A′	
B	1	4	5
B′	3	2	5
	4	6	10

$n(A \cap B')$ $n(A \cap B)$ $n(A' \cap B)$

$n(B)$ $n(B')$ $n(\xi)$

$n(A)$ $n(A')$ $n(A' \cap B')$

Practice Questions

1. Consider the given events A and B, which involve numbers taken from the first 10 positive integers.

$$\mathrm{A} = \{1, 2, 4, 5, 7, 8, 10\} \qquad \mathrm{B} = \{2, 3, 5, 6, 8, 9\}$$

a. Represent events A and B in a Venn diagram.

b. List the following sets.

i. $\mathrm{A} \cap \mathrm{B}$

ii. $\mathrm{A} \cup \mathrm{B}$

c. If a number from the first 10 positive integers is randomly selected, find the probability that the following events occur.

i. A

ii. $A \cap B$

iii. $A \cup B$

d. Are the events A and B mutually exclusive? Why/why not?

2. From a group of 50 adults, 35 enjoy reading fiction (F), 20 enjoy reading non-fiction (N) and 10 enjoy reading both fiction and non-fiction.

   a. Illustrate the information in a Venn diagram.

   b. State the number of people who enjoy:

      i. fiction only

      ii. neither fiction nor non-fiction

c. Find the probability that a person chosen at random will enjoy reading:

i. non-fiction

ii. non-fiction only

iii. both fiction and non-fiction

3. At a show, 45 children have the choice of riding on the Ferris wheel (F) and/or the Big Dipper (B). Thirty-five of the children wish to ride on the Ferris wheel, 15 children want to ride on the Big Dipper and 10 children want to ride on both.

a. Illustrate the information in a Venn diagram.

b. Find:

i. $n(\mathrm{F\ only})$

ii. $n(\mathrm{neither\ F\ nor\ B})$

c. For a child chosen at random from the group, find the following probabilities.

i. $\Pr(\mathrm{F})$

ii. $\Pr(\mathrm{F} \cap \mathrm{B})$

iii. $\Pr(\mathrm{F} \cup \mathrm{B})$

iv. $\Pr(\mathrm{F}')$

v. $\Pr(\text{neither F nor B})$

4. From a total of 10 people, 5 like apples (A), 6 like bananas (B) and 4 like both apples and bananas.

   a. Draw a Venn diagram for the 10 people.

   b. Draw a two-way table.

c. Find:

i. $n(\mathrm{A}' \cap \mathrm{B})$

ii. $n(\mathrm{A}' \cap \mathrm{B}')$

iii. $\Pr(\mathrm{A} \cap \mathrm{B})$

iv. $\Pr(\mathrm{A} \cup \mathrm{B})$

5. Decide which of the elements would need to be removed from event A if the two events A and B described below are to become mutually exclusive.

a. $A = \{1, 2, 3, 4\}$
$B = \{4, 5, 6, 7\}$

b. $A = \{10, 12, 14, 16, 18\}$
$B = \{9, 10, 11, 12\}$

c. $A = \{1, 3, 5, 8, 10, 15, 20, 22, 23\}$
$B = \{7, 9, 14, 16, 19, 21, 26\}$

6. Complete the following two-way tables.

a.

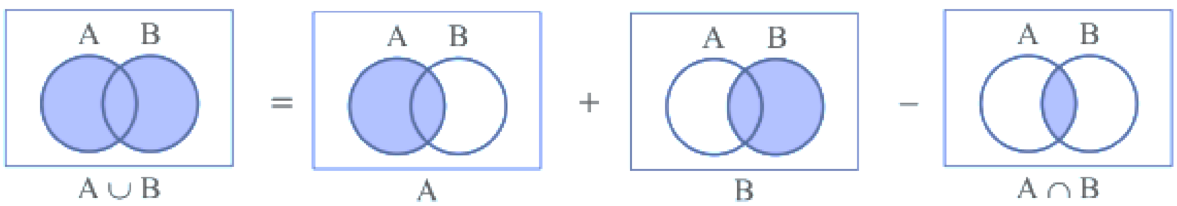
	A	A′	
B		3	6
B′			
		4	11

b.

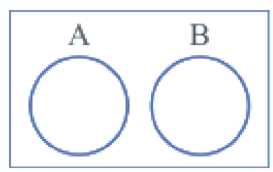
	A	A′	
B	2	7	
B′			3
	4		

Chapter 8.3 The Addition Rule

- The **addition rule** for two events A and B is:
  $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$



- If A and B are mutually exclusive then:
  - $\Pr(A \cap B) = 0$
  - $\Pr(A \cup B) = \Pr(A) + \Pr(B)$



Practice Questions

1. Substitute the given information into the addition rule to find $\Pr(A \cup B)$.

a. $\Pr(A) = 0.7, \Pr(B) = 0.5, \Pr(A \cap B) = 0.4$

b. $\Pr(A) = 0.6, \Pr(B) = 0.7, \Pr(A \cap B) = 0.5$

2. A card is selected from a pack of 52 playing cards.
   Let A be the event 'the card is a spade' and B be the event 'the card is an ace'.

   a. Find:
      i. $n(\mathrm{A})$

      ii. $n(\mathrm{B})$

      iii. $n(\mathrm{A} \cap \mathrm{B})$

b. Find:

i. $\Pr(A)$

ii. $\Pr(A')$

iii. $\Pr(A \cap B)$

c. Use the addition rule to find $\Pr(A \cup B)$.

d. Find the probability that the card is an ace and not a spade.

3. Two events A and B are such that $\Pr(A) = 0.3$, $\Pr(B) = 0.6$ and $\Pr(A \cup B) = 0.8$. Find:

a. $\Pr(A \cap B)$

b. $\Pr(A' \cap B')$

4. Of 32 cars at a show, 18 cars have four-wheel drive, 21 are sports cars and 27 have four-wheel drive or are sports cars.

a. Find the probability that a randomly selected car is both four-wheel drive and a sports car.

b. Find the probability that a randomly selected car is neither four-wheel drive nor a sports car.

5. A card is selected from a pack of 52 playing cards. Find the probability that the card is:

   a. a club or a queen

   b. a red card or a jack

   c. a king or not a heart

   d. a red card or not a 6

6. a. Find $\Pr(A \cap B')$, if $\Pr(A \cup B) = 0.8, \Pr(A) = 0.5$ and $\Pr(B) = 0.4$.

b. Find $\Pr(A' \cap B)$, if $\Pr(A \cup B) = 0.76, \Pr(A) = 0.31$ and $\Pr(B) = 0.59$.

Chapter 8.4 Conditional Probability

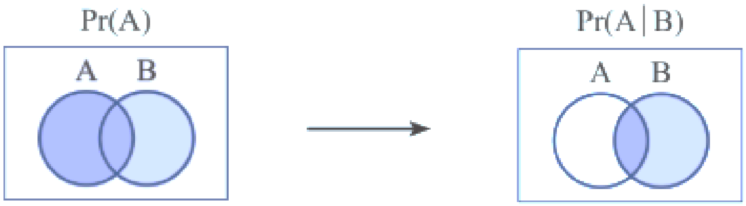
- The probability of event A occurring given that event B has occurred is denoted by $\Pr(A|B)$, which reads 'the probability of A given B'

  $\Pr(A|B) = \dfrac{\Pr(A \cap B)}{\Pr(B)}$ and $\Pr(B|A) = \dfrac{\Pr(A \cap B)}{\Pr(A)}$

- For problems in this section these rules can be simplified to:

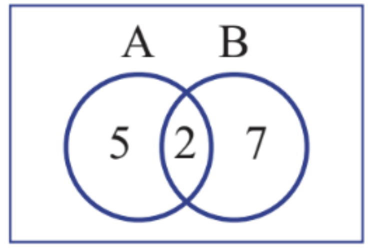
  $\Pr(A|B) = \dfrac{n(A \cap B)}{n(B)}$ and $\Pr(B|A) = \dfrac{n(A \cap B)}{n(A)}$

- $\Pr(A|B)$ differs from $\Pr(A)$ in that the sample space is reduced to the set B as shown in these Venn diagrams.



Practice Questions

1. Use this Venn diagram to answer these questions.



a. i Find $n(\mathrm{A} \cap \mathrm{B})$.

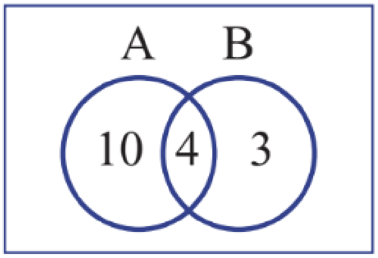
ii. Find $\mathrm{n}(B)$.

b. Find $\Pr(\mathrm{A}|\mathrm{B})$using $\Pr(\mathrm{A}|\mathrm{B}) = \frac{n(\mathrm{A}\cap\mathrm{B})}{n(\mathrm{B})}$ =

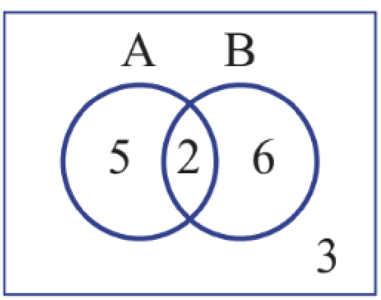
2. The following Venn diagrams display information about the number of elements associated with the events A and B. For each Venn diagram, find:

i. Pr(A) ii. Pr(A ∩ B) iii. Pr(A|B) iv. Pr(B|A)

a.



b.



3. The following two-way tables show information about the number of elements in the events A and B. For each two-way table find:

i. $\Pr(\mathrm{A}|\mathrm{B})$

ii. $\Pr(\mathrm{B}|\mathrm{A})$

a.

	A	A′	
B	1	4	5
B′	3	1	4
	4	5	9

b.

	A	A′	
B	4	2	6
B′	8	2	10
	12	4	16

4. Of a group of 20 English cricket fans at a match, 13 purchased a pie, 15 drank beer and 9 both purchased a pie and drank beer.

Let A be the event 'the fan purchases a pie'.
Let B be the event 'the fan drank beer'.

a. Represent the information in a two-way table.

b. Find the probability that a fan only purchased a pie (and did not drink beer).

c. Find the probability that a fan purchased a pie given that they drank beer.

d. Find the probability that a fan drank beer given that they purchased a pie.

5. Of 15 musicians surveyed to find out whether they play the violin or the piano, 5 play the violin, 8 play the piano and 2 play both instruments.

a. Represent the information in a Venn diagram.

b. How many of the people do not play either the violin or the piano?

c. Find the probability that one of the 15 people plays piano given they play the violin.

d. Find the probability that one of the 15 people plays the violin given they play piano.

6. A card is drawn from a pack of 52 playing cards. Find the probability that:

   a. the card is a king given that it is a heart

   b. the card is a jack given that it is a red card

Chapter 8.5 Multiple Events using Tables

When organizing the sample space for multiple events, we can use tables to list the possible outcomes. Here's how the situation changes depending on whether **replacement** is allowed or not:

1. With Replacement:

- If replacement is allowed, each outcome from the first selection can be followed by any outcome from the second selection, including the same outcome.
- This means that the same outcome can appear multiple times across different events.
- Example: If we select two digits from the set $\{1,2,3\}$ **with replacement**, the sample space would include repeated selections, like $(1, 1)$, $(2, 2)$, etc.

2. Without Replacement:

- If selections are made **without replacement**, each outcome can only appear once in the sample space.
- After one digit is selected, it cannot be selected again for the second selection.
- Example: If we select two digits from the set $\{1,2,3\}$ **without replacement**, the sample space would look like $(1, 2)$, $(1, 3)$, $(2, 1)$, $(2, 3)$, and $(3, 1)$, $(3, 2)$, without repeating any digits.

Practice Questions

1. Two digits are selected from the set $\{2, 3, 4\}$ to form a two-digit number. Find the number of two-digit numbers that can be formed if the digits are selected:

   a. with replacement

   b. without replacement

2. A 4-sided die is tossed twice.

a. List the sample space using a table.

b. State the total number of possible outcomes.

c. Find the probability of obtaining the outcome $(2, 4)$.

d. Find the probability of:

i. a double

ii. a sum of at least 5

iii. a sum not equal to 4

3. Two letters are chosen from the word SET without replacement.

   a. Show the sample space using a table.

   b. Find the probability of:

      i. obtaining the outcome (E, T)

      ii. selecting one T

iii. selecting at least one T

iv. selecting an S and a T

v. selecting an S or a T

4. A letter is chosen from the word LEVEL without replacement and then a second letter is chosen from the same word.

   a. Draw a table displaying the sample space for the pair of letters chosen.

   b. State the total number of outcomes possible.

   c. State the number of outcomes that contain exactly one of the following letters.

      i. V ii. L

d. Find the probability that the outcome will contain exactly one of the following letters.

i. V

ii. L

e. Find the probability that the two letters chosen will be the same.

Chapter 8.6 Using Tree Diagrams

Practice Questions

1. Two prizes are awarded to a group of 3 male (M) and 5 female (F) candidates.
   Copy and complete each tree diagram. Include the missing branch probabilities and outcome probabilities.

   a. with replacement

   b. without replacement

2. A 4-sided die is tossed twice and the pair of numbers is recorded.

   a. Use a tree diagram to list the sample space.

   b. State the total number of outcomes.

   c. Find the probability of obtaining:

      i. a 4 then a 1, i.e. the outcome (4, 1)

      ii. a double

d. Find the probability of obtaining a sum described by the following:

i. equal to 2

ii. equal to 5

iii. less than or equal to 5

3. Two students are selected from a group of 3 males (M) and 4 females (F) without replacement.

   a. Draw a tree diagram to find the probability of selecting:

      i. 2 males

      ii. 2 females

      iii. 1 male and 1 female

iv. 2 people either both male or both female

b. The experiment is repeated with replacement. Find the answers to each question in part **a**.

4. Two bottles of wine are randomly selected for tasting from a box containing two red and 2 white wines. Use a tree diagram to help answer the following.

   a. If the first bottle is replaced before the second is selected, find:

      i. $\Pr(2 \text{ red})$

      ii. $\Pr(\text{not two white})$

b. If the first bottle is not replaced before the second is selected, find:

i. $\Pr(1 \text{ red})$

ii. $\Pr(\text{at least one white})$

5. Cans of sliced apple produced by 'Just Apple' are sometimes underweight. A box of 10 cans is selected from the factory and then 2 cans from the 10 are tested without replacement. This particular box of 10 cans is known to have 2 cans that are underweight.

   a. State the probability that the first can chosen will be:
      i. underweight

      ii. not underweight

   b. Use a tree diagram to find the probability that:
      i. both cans are underweight

ii. one can is underweight

iii. at most one can is underweight

c. The factory passes the inspection if no cans are found to be underweight. Find the chance that this will occur and express your answer as a percentage rounded to one decimal place.

Chapter 8.7 Independent Events

Definitions of Independent Events:

Two events A and B are **independent** if the occurrence of one event does not affect the probability of the other event occurring.

Mathematical Expressions for Independent Events:

- **Conditional Probability:**
  If events A and B are independent, then the probability of A occurring given that B has occurred is the same as the probability of A. That is:

$$\Pr(A \mid B) = \Pr(A)$$

  Similarly,

$$\Pr(B \mid A) = \Pr(B)$$

- **Intersection Probability:**
  For independent events A and B, the probability of both events occurring together (the intersection) is the product of the individual probabilities:

$$\Pr(A \cap B) = \Pr(A) \times \Pr(B)$$

Independence in Different Selection Scenarios:

1. **With Replacement (Events are Independent):**
   - If selections are made **with replacement**, after each event, the sample space remains unchanged, and the probability of future events is not influenced by previous selections.
   - This means that each event is independent of the others.

Example:

If you draw a card from a deck, record it, and then put it back before drawing again, each draw is independent. The probability of drawing a heart on the first and second draw will be the same because the deck is restored to its original state after the first draw.

2. **Without Replacement (Events are Not Independent):**
    - If selections are made **without replacement**, the outcome of the first event influences the probability of the second event.
    - The sample space changes after each event, meaning that the events are dependent on each other.

Example:

If you draw a card from a deck and do not replace it before drawing again, the probability of drawing a heart on the second draw is influenced by whether the first card drawn was a heart or not. Hence, the events are dependent.

Practice Questions

1. A coin is tossed twice. Let A be the event ‘the first toss gives a tail’ and let $\underline{\mathrm{B}}$ be the event ‘the second toss gives a tail’.

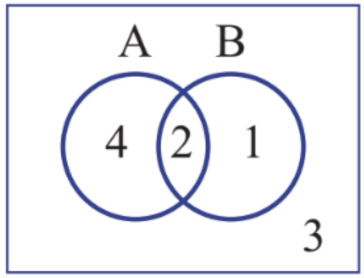
   a. Find

   i. $\Pr(\mathrm{A})$ ii. $\Pr(\mathrm{B})$

   b. Would you say that events A and B are independent?

   c. What is $\Pr(\mathrm{B}|\mathrm{A})$?

2. This Venn diagram shows the number of elements in events A and B.



a. Find

i. $\Pr(\mathrm{B})$

ii. Find $\Pr(\mathrm{B}|\mathrm{A})$

b. Is $\Pr(\mathrm{B}|\mathrm{A}) = \Pr(\mathrm{B})$?

c. Are the events A and B independent?

3. A selection of 8 offers for computer printers includes 3 with a free printer cartridge and 4 with a free box of paper, while 2 have both a free printer cartridge and a free box of paper.

Let A be the event 'choosing a printer with a free printer cartridge'.
Let B be the event 'choosing a printer with a free box of paper'.

a. Summarise the given information about the 8 computer printer offers in a Venn diagram.

b. i Find $\Pr(\mathrm{A})$.

ii Find $\Pr(\mathrm{A}|\mathrm{B})$.

c. State whether or not the events A and B are independent.

4. A selection of 6 different baby strollers includes 3 with a free rain cover, 4 with a free sun shade and 2 offer both a free rain cover and free sun shade.

   Let A be the event 'choosing a stroller with a free sun shade'.
   Let B be the event 'choosing a stroller with a free rain cover'.

   a. Summarise the given information about the six baby strollers in a Venn diagram.

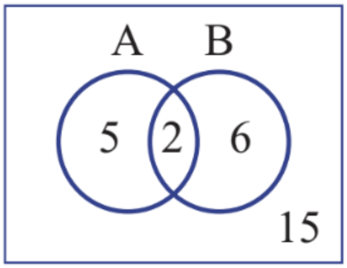
   b. i. Find $\Pr(A)$.

   ii. Find $\Pr(A|B)$.

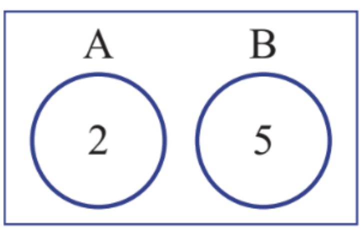
   c. State whether or not the events A and B are independent.

5. From events A and B in the given Venn diagrams:
   i. find $\Pr(A)$ and $\Pr(A|B)$
   ii. hence decide whether or not events A and B are independent

a.



b.



6. Of 17 leading accountants, 15 offer advice on tax (T) while 10 offer advice on investment (I). Eight of the accountants offer advice on both tax and investment. One of the 17 accountants is chosen at random.

   a. Use a Venn diagram or two-way table to help find:

   i. $\Pr(\mathrm{T})$

   ii. $\Pr(\mathrm{T\ only})$

iii. $\Pr(T|I)$

b. Are the events T and I independent?

7. A coin is tossed 5 times. Find the probability of obtaining:

   a. 5 heads

   b. at least one tail

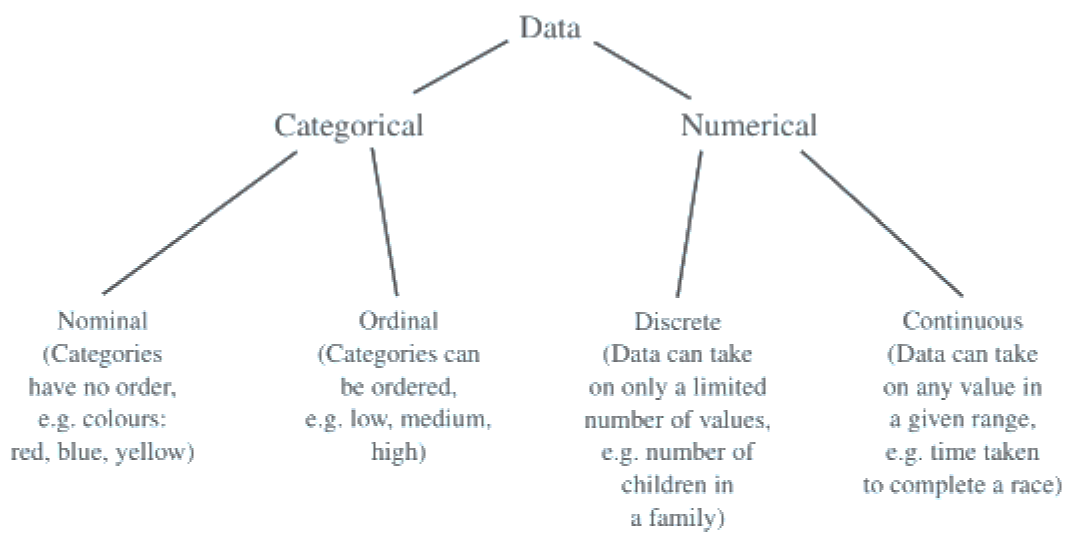
   c. at least one head

CHAPTER 9 STATISTICS

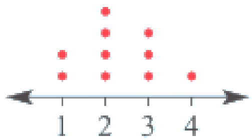
Table of Contents

Chapter 9.1 Review of Statistical Graphs

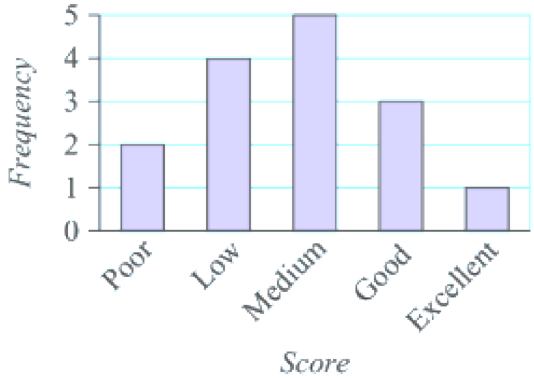
Statistical data can be divided into sub-groups.



- Graphs for a single set of categorical or discrete data
  - **Dot plot**



  - **Column graph**



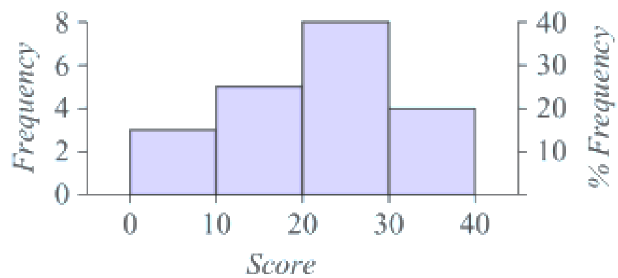
  - **Stem-and-leaf plot**

Stem	Leaf
0	1 3
1	2 5 9
2	1 4 6 7
3	0 4

2 | 4 means 24

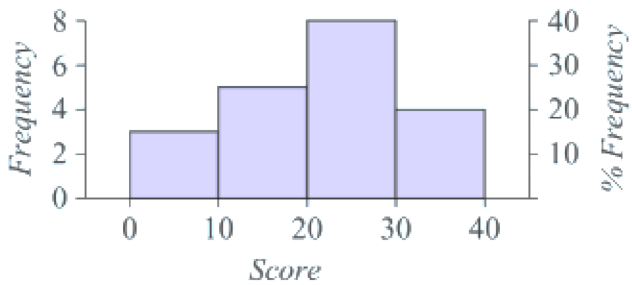
- **Histograms** can be used for grouped discrete or continuous numerical data.
  The interval 10– includes all numbers from 10 (including 10) to less than 20.

Class interval	Frequency	Percentage frequency
0–	3	15
10–	5	25
20–	8	40
30–40	4	20

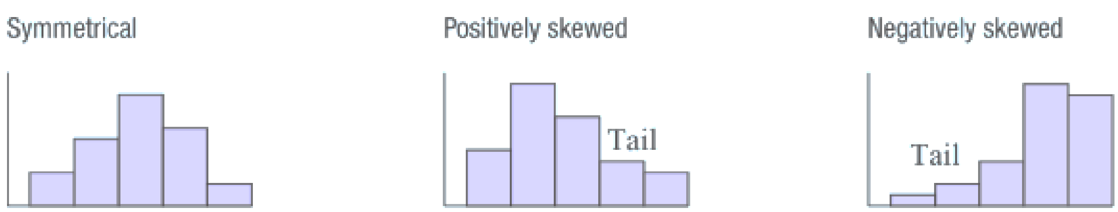


- **Histograms** can be used for grouped discrete or continuous numerical data.
  The interval 10– includes all numbers from 10 (including 10) to less than 20.

Class interval	Frequency	Percentage frequency
0–	3	15
10–	5	25
20–	8	40
30–40	4	20



- Measures of centre include:
  - **mean** $(\bar{x})$ $\bar{x} = \dfrac{\text{sum of all values}}{\text{number of scores}}$
  - **median** the middle value when data is placed in order
  - **mode** the most common value
- Data can be **symmetrical** or **skewed**.



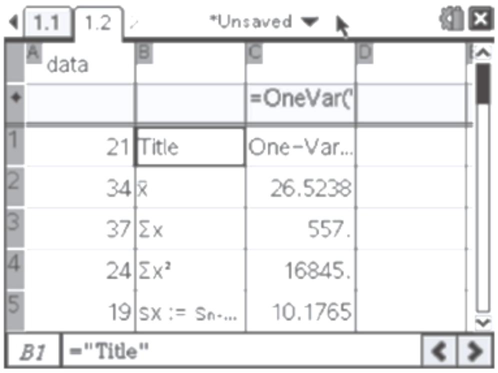
1 Enter this data in a list called *data* and find the mean and median.

21, 34, 37, 24, 19, 11, 15, 26, 43, 38, 25, 16, 9, 41, 36, 31, 24, 21, 30, 39, 17

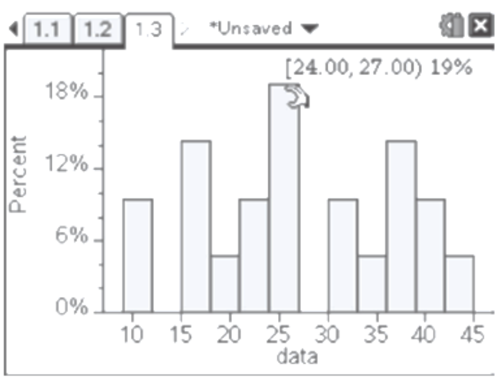
2 Construct a histogram using interval widths of 3 and percentage frequency for the above data.

Using the TI-Nspire:

1 Go to a **Lists and spreadsheets** page and enter the data into list A. Select **menu**, **Statistics**, **Stat Calculations**, **One-Variable Statistics**. Press **enter** to finish and scroll to view the statistics.

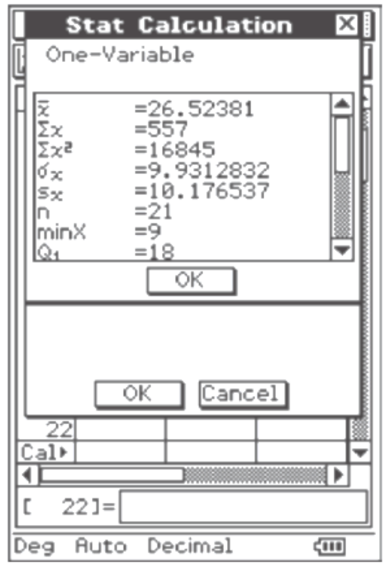


2 Go to a **Data and Statistics** page and select the *data* variable for the horizontal axis. Select **menu**, **Plot Type**, **Histogram**. Then select **menu**, **Plot Properties**, **Histogram Properties**, **Bin Settings**. Choose the **Width** to be 3 and **Alignment** to be 0. Drag the scale to suit. Select **menu**, **Plot Properties**, **Histogram Properties**, **Histogram Scale** to receive percentage frequecy.

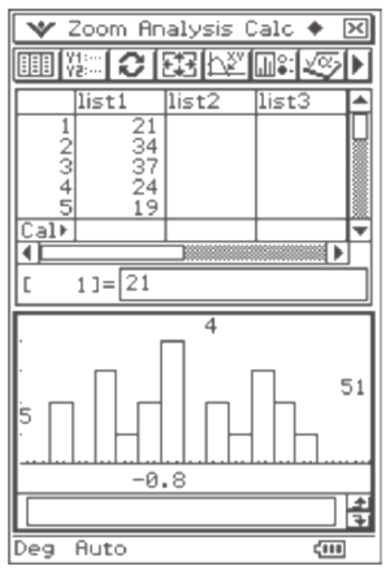


Using the ClassPad:

1 In the **Statistics** application enter the data into list1. Tap **Calc**, **One-Variable** then **OK**. Scroll to view the statistics.



2 Tap **SetGraph**, ensure StatGraph1 is ticked then tap **Setting**. Change the **Type** to **Histogram**, set **XList** to **list1**, **Freq** to **1** then tap on **Set**. Tap and set **HStart** to 9 and **HStep** to 3.



Practice Questions

1. The number of wins scored this season is given for 20 hockey teams. Here is the raw data.

$4, 8, 5, 12, 15, 9, 9, 7, 3, 7$
$10, 11, 1, 9, 13, 0, 6, 4, 12, 5$

a. Organise the data into a frequency table using class intervals of 5 and include a percentage frequency column.

b. Construct a histogram for the data showing both the frequency and percentage frequency on the one graph.

c. Construct a stem-and-leaf plot for the data.

d. Use your stem-and-leaf plot to find the median.

2. This frequency table displays the way in which 40 people travel to and from work.

Type of transport	Frequency	Percentage frequency
Car	16	
Train	6	
Tram	8	
Walking	5	
Bicycle	2	
Bus	3	
Total	**40**	

a. Complete the table.

b. Use the table to find:

i. the frequency of people who travel by train

ii. the most popular form of transport

iii. the percentage of people who travel by car

iv. the percentage of people who walk or cycle to work

v. the percentage of people who travel by public transport including trains, buses and trams.

3. For the data in these stem-and-leaf plots find:
   i. the mean (rounded to one decimal place) ii. the median iii. the mode

a.

Stem	Leaf
2	1 3 7
3	2 8 9 9
4	4 6

3 | 2 means 32

b.

Stem	Leaf
0	4
1	0 4 9
2	1 7 8
3	2

2 | 7 means 27

4. Two football players, Nick and Jack, compare their personal tallies of the number of goals scored for their team over a 12-match season. Their tallies are as follows.

Game	1	2	3	4	5	6	7	8	9	10	11	12
Nick	0	2	2	0	3	1	2	1	2	3	0	1
Jack	0	0	4	1	0	5	0	3	1	0	4	0

a. Draw a dot plot to display Nick's goal-scoring achievement.

b. Draw a dot plot to display Jack's goal-scoring achievement.

c. How would you describe Nick's scoring habits?

d. How would you describe Jack's scoring habits?

5. This tally records the number of mice that were weighed and categorised into particular mass intervals for a scientific experiment.

Mass (grams)	Tally
10–	III
15–	卌 I
20–	卌 卌 卌 I
25–	卌 卌 卌 卌 I
30–35	IIII

a. Construct a table using these column headings: Mass, Frequency and Percentage frequency.

b. Find the total number of mice weighed in the experiment.

c. State the percentage of mice that were in the 20 gram interval.

d. Which was the most common weight interval?

e. What percentage of mice were in the most common mass interval?

f. What percentage of mice had a mass of 15 grams or more?

Chapter 9.2 Quartiles

- **Five-figure summary**
  - **Minimum value** (Min) — the minimum value
  - **Lower quartile** (Q_1) — the number above 25% of the ordered data
  - **Median** (Q_2) — the middle value above 50% of the ordered data
  - **Upper quartile** (Q_3) — the number above 75% of the ordered data
  - **Maximum value** (Max) — the maximum value

Odd number

1 2 2 3 5 6 6 7 9

Q_1 (2) Q_2 (5) Q_3 (6.5)

IQR (4.5)

Even number

2 3 3 4 7 8 8 9 9 9

Q_1 (3) Q_2 (7.5) Q_3 (9)

IQR (6)

- **Measures of spread**
  - **Range** = Max value – Min value
  - **Interquartile range** (IQR)

    IQR = upper quartile – lower quartile

    $= Q_3 - Q_1$

- **Outliers** are data elements outside the vicinity of the rest of the data. A data point is an outlier if it is either:
  - less than $Q_1 - 1.5 \times \text{IQR}$ or
  - greater than $Q_3 + 1.5 \times \text{IQR}$

Practice Questions

1. Determine the range and IQR for these data sets by finding the five-figure summary.

    a. $10, 10, 11, 14, 14, 15, 16, 18$

    b. $41, 49, 53, 58, 59, 62, 62, 65, 66, 68$

2. Determine the median and mean of the following sets of data. Round to one decimal place where necessary.

    a. $2, 3, 4, 5, 5, 6$

    b. $4, 6, 3, 7, 3, 2, 5$

3. The following numbers of cars were counted on each day for 15 days, travelling on a quiet suburban street.

$$10, 9, 15, 14, 10, 17, 15, 0, 12, 14, 8, 15, 15, 11, 13$$

For the given data, find:

a. the minimum and maximum number of cars counted

b. the median

c. the lower and upper quartiles

d. the IQR

e. any outliers

f. a possible reason for the outlier

4. Twelve different calculators had the following numbers of buttons.

$$36, 48, 52, 43, 46, 53, 25, 60, 128, 32, 52, 40$$

a. For the given data, find:

i. the minimum and maximum number of buttons on the calculators

ii. the median

iii. the lower and upper quartiles

iv. the IQR

v. any outliers

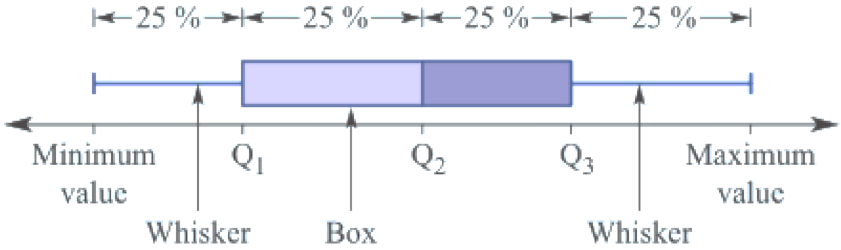
vi. the mean

b. Which is a better measure of the centre of the data, the mean or the median? Explain.

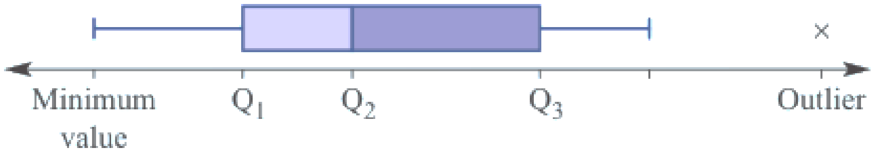
c. Can you give a possible reason why the outlier has occurred

Chapter 9.3 Boxplots

- A **boxplot** (also called a box-and-whisker plot) can be used to summarise a data set.



- An **outlier** is marked with a cross (×).
  - An outlier is greater than $Q_3 + 1.5 \times IQR$ or less than $Q_1 - 1.5 \times IQR$.
  - The whiskers stretch to the lowest and highest data values that are not outliers.



- **Parallel boxplots** are boxplots drawn on the same scale. They are used to compare data sets within the same context.

1 Type this data into lists and define as Test A and Test B.

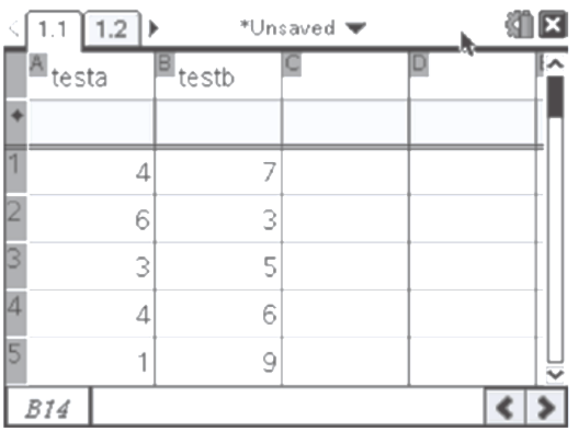
Test A: 4, 6, 3, 4, 1, 3, 6, 4, 5, 3, 4, 3

Test B: 7, 3, 5, 6, 9, 3, 6, 7, 4, 1, 4, 6

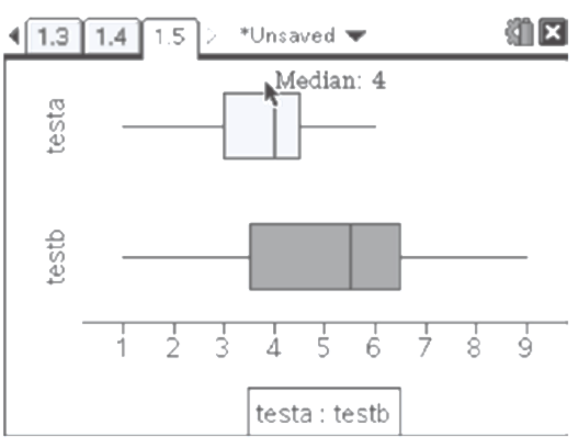
2 Draw parallel boxplots for the data.

Using the TI-Nspire:

1 Go to a new **Lists and spreadsheets** page and enter the data into the lists. Title each column.

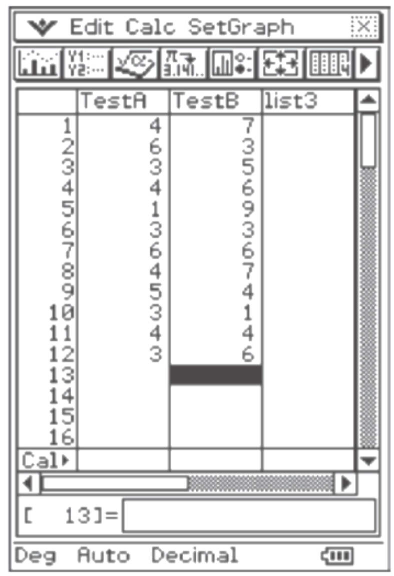


2 Go to a new **Data and Statistics** page and select the **testa** variable for the horizontal axis. Select **menu**, **Plot Type**, **Box Plot**. Trace to reveal the statistical measures. To show the boxplot for **testb**, go to **menu**, **Plot Properties**, **Add X Variable**. Click on **testb**.

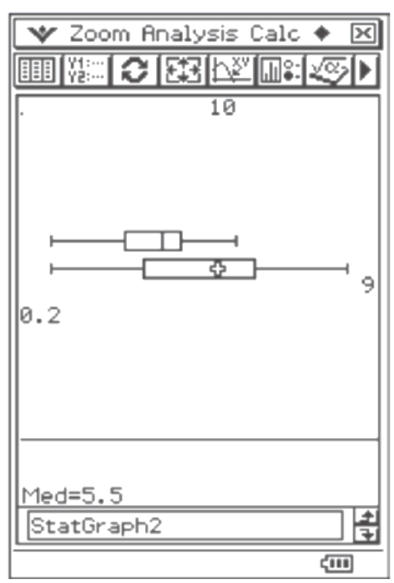


Using the ClassPad:

1 In the **Statistics** application enter the data into the lists. Title each column.



2 Tap . For graph 1 set **Draw** to **On**, **Type** to **MedBox**, **XList** to **main\TestA** and **Freq** to **1**. For graph 2 set **Draw** to **On**, **Type** to **MedBox**, **XList** to **main\TestB** and **Freq** to **1**. Tap **Set**. Tap .



Practice Questions

1. Consider the data sets below.
   i. Determine whether any outliers exist by first finding Q_1 and Q_3.
   ii.Draw a boxplot to summarise the data, marking outliers if they exist.

   a. $1.8, 1.7, 1.8, 1.9, 1.6, 1.8, 2.0, 1.1, 1.4, 1.9, 2.2$

   b. $21, 23, 18, 11, 16, 19, 24, 21, 23, 22, 20, 31, 26, 22$

2. The following masses, in kilograms, of 15 Madagascan lemurs were recorded as part of a conservation project.

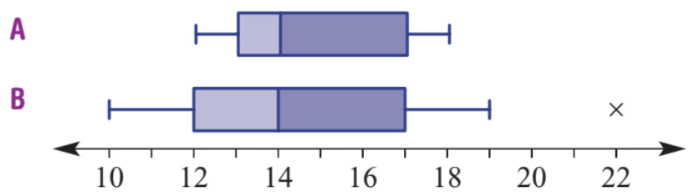
$14.4, 15.5, 17.3, 14.6, 14.7$
$15.0, 15.8, 16.2, 19.7, 15.3$
$13.8, 14.6, 15.4, 15.7, 14.9$

a. Find Q_1, Q_2 and Q_3.

b. Which masses, if any, would be considered outliers?

c. Draw a boxplot to summarise the lemur masses.

3. Two data sets can be compared using parallel boxplots on the same scale, as shown below.



a. What statistical measures do these boxplots have in common?

b. Which data set (A or B) has a wider range of values?

c. Find the IQR for:

i. data set A

ii. data set B

d. How would you describe the main difference between the two sets of data from which the parallel boxplots have been drawn?

Chapter 9.4 Time Series Data

Time Series Data:

Time series data consist of observations collected or recorded at regular intervals of time. These data are often used to track changes or patterns over time, such as stock prices, temperature measurements, or economic indicators.

Graphing Time Series Data:

When visualizing time series data:

- **Time** is plotted on the **horizontal axis (x-axis)**, representing the progression of time.
- **The variable of interest** (e.g., stock price, temperature) is plotted on the **vertical axis (y-axis)**.
- **Line segments** are drawn connecting the data points to show the trend over time.

This type of graph is often referred to as a **time series plot** or **line graph**.

Identifying Trends:

- If the points on the time series plot appear to follow a **straight line** or are closely grouped around a line, this suggests a **linear trend**.
- A **linear trend** means that the variable of interest is changing at a constant rate over time (e.g., increasing or decreasing steadily).

Key Points:

- Time series plots help to visualize trends, patterns, and fluctuations over time.
- A **linear trend** indicates that the data change at a constant rate.

Practice Questions

1. A company's share price over 12 months is recorded in this table.

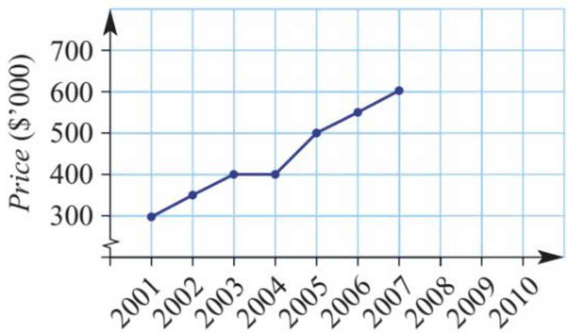
Month	J	F	M	A	M	J	J	A	S	O	N	D
Price ($)	1.30	1.32	1.35	1.34	1.40	1.43	1.40	1.38	1.30	1.25	1.22	1.23

a. Plot the time series graph. Break the y-axis to exclude values from $0 to $1.20.

b. Describe the way in which the share price has changed over the 12 months.

c. What is the difference between the maximum and minimum share price in the 12 months?

2. This time series plot shows the upwards trend of house prices in an Adelaide suburb over 7 years from 2001 to 2007.



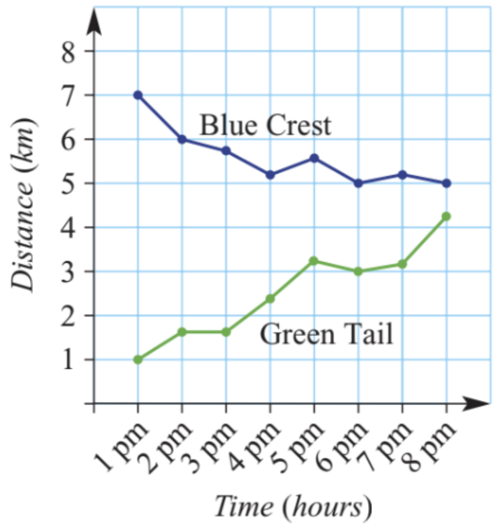
a. Would you say that the general trend in house prices is linear or non-linear?

b. Assuming the trend in house prices continues for this suburb, what would you expect the house price to be in:

i. 2008?

ii. 2010?

3. Two pigeons (Green Tail and Blue Crest) each have a beacon that communicates with a recording machine. The distance of each pigeon from the machine is recorded every hour for 8 hours.



a. State the distance from the machine at 3 pm of

i. Blue Crest

ii. Green Tail

b. Describe the trend in the distance from the recording machine for:

i. Blue Crest

ii. Green Tail

c. Assuming that the given trends continue, predict the time when the pigeons will be the same distance from the recording machine.

Chapter 9.5 Bivariate Data and Scatter Plots

Bivariate Data:

- **Bivariate data** consists of pairs of data values, where each pair includes two variables. For example, you might have data for **height** and **weight**, where each person's height is paired with their weight.

Scatter Plot:

- A **scatter plot** is a type of graph used to display bivariate data.
- In a scatter plot:
  - One variable is plotted along the x**-axis (horizontal)**, and the other variable is plotted along the y**-axis (vertical)**.
  - Each pair of data points (x, y) is represented as a dot on the graph.

Relationship, Correlation, and Association:

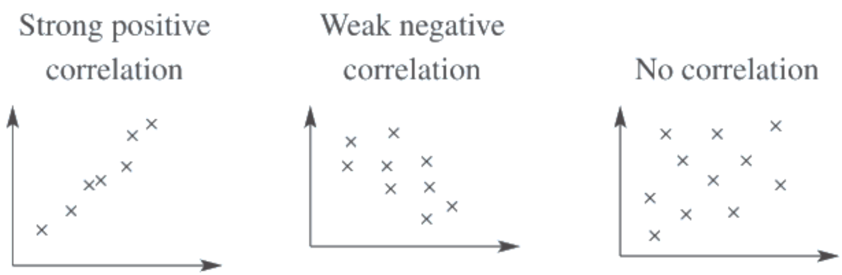
- **Relationship** refers to how the two variables are linked or connected.
- **Correlation** describes the strength and direction of the relationship between the variables.
  - A **positive correlation** means that as one variable increases, the other tends to increase.
  - A **negative correlation** means that as one variable increases, the other tends to decrease.
  - **No correlation** means there is no predictable pattern between the variables.
- **Association** is another term used to describe the connection or dependency between two variables.

Interpreting Scatter Plots:

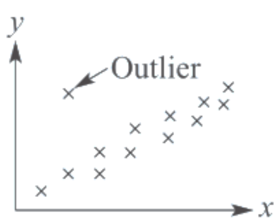
- A **strong correlation** will appear as a clear trend or pattern (such as a straight line or curve) in the scatter plot.
- A **weak correlation** will show scattered points with no clear pattern.

- Types of correlation:

  Examples



- An **outlier** can clearly be identified as a data point that is isolated from the rest of the data.

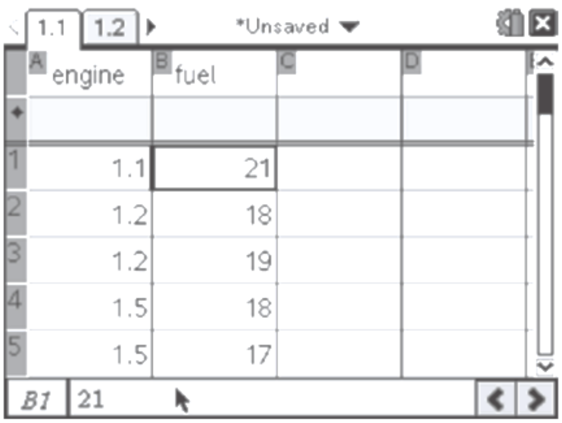


Type this data about car fuel economy into two lists and draw a scatter plot.

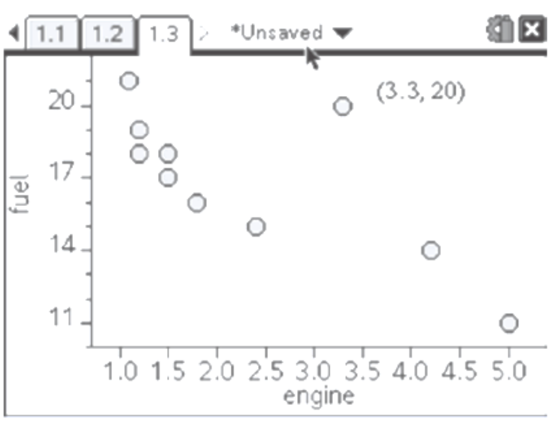
Car	A	B	C	D	E	F	G	H	I	J
Engine size	1.1	1.2	1.2	1.5	1.5	1.8	2.4	3.3	4.2	5.0
Fuel economy	21	18	19	18	17	16	15	20	14	11

Using the TI-Nspire:

1 Go to a new **Lists and spreadsheets** page and enter the data into the lists. Title each column.

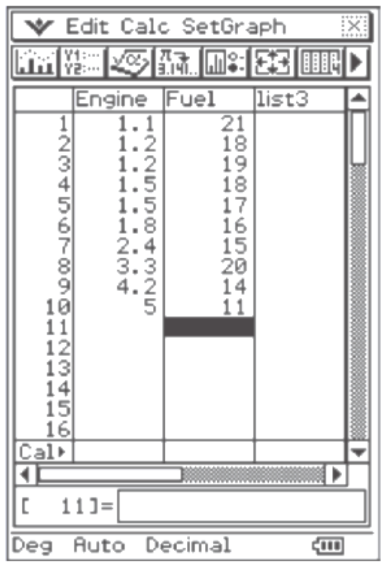


2 Go to a new **Data and Statistics** page and select the **engine** variable for the horizontal axis and **fuel** for the vertical axis. Hover over points to reveal coordinates.



Using the ClassPad:

1 In the **Statistics** application enter the data into the lists. Title each column.



2 Tap ▦. For graph 1 set **Draw** to **On**, **Type** to **Scatter**, **XList** to **main\Engine**, **YList** to **main\Fuel**, **Freq** to **1** and **Mark** to **square**. Tap **Set**. Tap ▦. Tap **Analysis**, **Trace** to reveal coordinates.

Practice Questions

1. Consider this simple bivariate data set. (Use technology to assist if desired.)

x	1	2	3	4	5	6	7	8
y	1.0	1.1	1.3	1.3	1.4	1.6	1.8	1.0

a. Draw a scatter plot for the data.

b. Describe the correlation between x and y as positive or negative.

c. Describe the correlation between x and y as strong or weak.

d. Identify any outliers.

2. By completing scatter plots (by hand or using technology) for each of the following data sets, describe the correlation between x and y as 'positive', 'negative' or 'none'.

a.

x	1.1	1.8	1.2	1.3	1.7	1.9	1.6	1.6	1.4	1.0	1.5
y	22	12	19	15	10	9	14	13	16	23	16

b.

x	4	3	1	7	8	10	6	9	5	5
y	115	105	105	135	145	145	125	140	120	130

3. A tomato grower experiments with a new organic fertiliser and sets up five separate garden beds: A, B, C, D and E. The grower applies different amounts of fertiliser to each bed and records the diameter of each tomato picked.

The average diameter of a tomato from each garden bed and the corresponding amount of fertiliser are recorded below.

Bed	A	B	C	D	E
Fertiliser (grams per week)	20	25	30	35	40
Average diameter (cm)	6.8	7.4	7.6	6.2	8.5

a. Draw a scatter plot for the data with 'Diameter' on the vertical axis and 'Fertiliser' on the horizontal axis. Label the points A, B, C, D and E.

b. Which garden bed appears to go against the trend?

c. According to the given results, would you be confident in saying that the amount of fertiliser fed to tomato plants does affect the size of the tomato produced?

4. On 14 consecutive days a local council measures the volume of sound heard from a freeway at various points in a local suburb. The volume (V) of sound is recorded against the distance (d m) between the freeway and the point in the suburb.

Distance (d)	200	350	500	150	1000	850	200	450	750	250	300	1500	700	1250
Volume (V)	4.3	3.7	2.9	4.5	2.1	2.3	4.4	3.3	2.8	4.1	3.6	1.7	3.0	2.2

a. Draw a scatter plot of V against d, plotting V on the vertical axis and d on the horizontal axis.

b. Describe the correlation between d and V as positive, negative or none.

c. Generally as d increases, does V increase or decrease?

Chapter 9.6 Line of Best Fit by Eye

Line of Best Fit (Trend Line):

- A **line of best fit**, also known as a **trend line**, is used to model the relationship between two variables in bivariate data.
- The line is drawn to balance the number of points above and below the line, and it minimizes the overall distance from the data points to the line.

Drawing the Line of Best Fit:

- You can position the **line of best fit** visually by ensuring that:
    - The number of points above the line roughly equals the number of points below the line.
    - The distance of each point from the line is minimized, giving the best representation of the data's trend.

Equation of the Line of Best Fit:

- The equation of the line of best fit is typically written as $\boldsymbol{y = mx + c}$, where:
    - $\boldsymbol{m}$ is the slope of the line, which represents how much y changes for a unit change in x.
    - $\boldsymbol{c}$ is the y-intercept, which is the value of y when $x = 0$.

Finding the Slope (m):

- The **slope (m)** can be calculated using two points $(\boldsymbol{x}_1, \boldsymbol{y}_1)$ and $(\boldsymbol{x}_2, \boldsymbol{y}_2)$ on the line of best fit:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

- The slope gives the rate of change of y with respect to x.

Finding the Y-Intercept (c):

- Once the slope $\boldsymbol{m}$ is known, you can find $\boldsymbol{c}$ (the y-intercept) by substituting the values of $\boldsymbol{m}, \boldsymbol{x}$, and $\boldsymbol{y}$ from one of the points (x_1, y_1) into the equation $\boldsymbol{y} = \boldsymbol{mx} + \boldsymbol{c}$:

$$c = y_1 - m(x_1)$$

Alternatively, you can use the point-slope form of the equation:

$$y - y_1 = m(x - x_1)$$

This formula can be used directly to create the equation once you have the slope $\boldsymbol{m}$ and a point $(\boldsymbol{x}_1, \boldsymbol{y}_1)$ on the line.

Uses of the Line of Best Fit:

- **Interpolation**: You can use the line of best fit to estimate values **within** the range of the given data. This is called **interpolation**.
    - Example: Using the line to predict the weight of someone of a specific height, assuming the data is within the observed range.

- **Extrapolation**: You can use the line of best fit to predict values **outside** the given data range. This is called **extrapolation**.
    - Example: Using the line to predict the weight of someone much taller than those in the data set, even if that height wasn't observed.

Practice Questions

1. Consider the variables x and y and the corresponding bivariate data.

x	1	2	3	4	5	6	7
y	2	2	3	4	4	5	5

a. Draw a scatter plot for the data.

b. Is there positive, negative or no correlation between x and y?

c. Fit a line of best fit by eye to the data on the scatter plot.

d. Use your line of best fit to estimate:

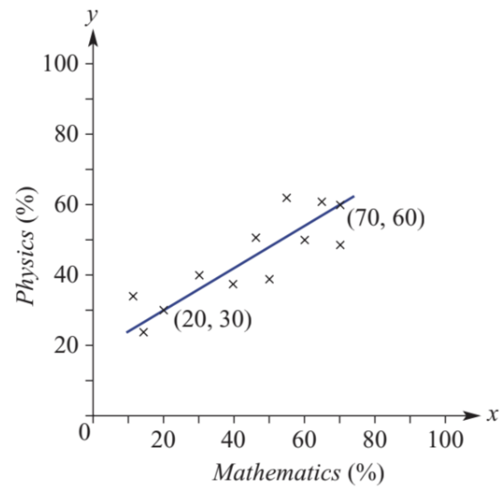
i. y when $x = 3.5$

ii. y when $x = 0$

iii. x when $y = 2$

iv. x when $y = 5.5$

2. This scatter plot shows a linear relationship between Mathematics marks and Physics marks in a small class of students. A trend line passes through $(20, 30)$ and $(70, 60)$.



a. Find the equation of the trend line.

b. Use your equation to find the Physics score if the Mathematics score is:

i. 40 	ii. 90

c. Use your equation to find the Mathematics score if the Physicsscore is:

i. 36 ii. 78

3. A line of best fit for a scatter plot relating the weight (kg) and length (cm) of a group of dogs, passes through the points $(15, 70)$ and $(25, 120)$. Assume weight is on the x-axis.

   a. Find the equation of the trend line.

b. Use your equation to estimate the length of a 18 kg dog.

c. Use your equation to estimate the weight of a dog that has a length of 100 cm.

Chapter 9.7 Standard Deviation

Standard Deviation

The **standard deviation** is a measure of how much the data in a set deviate (or spread out) from the mean. It provides insight into the **variability** or **dispersion** of data. The symbol for standard deviation is **s** (or **σ** for population data).

Steps to Calculate Standard Deviation:

1. **Find the Mean** ($\overline{x}$):
   - The **mean** is the average of all the data points.
   - Formula:
     - $\overline{x} = \frac{\sum x}{n}$
     - Where:
     - $\sum$ is the sum of all the data values.
     - n is the number of data points.

2. **Find the Deviation**:
   - The deviation is the difference between each data value and the mean.

$$\text{Deviation} = x_i - \overline{x}$$

   Where x_i is each data value and $\overline{x}$ is the mean.

3. **Square Each Deviation**:
   - Square each deviation to eliminate negative values and emphasize larger deviations.

$$(x_i - \overline{x})^2$$

4. **Sum the Squared Deviations**:

- Add up all the squared deviations.

$$\sum(x_i - \overline{x})^2$$

5. **Divide by the Number of Data Values (n)**:

- This step gives the **variance**.

$$\text{Variance} = \frac{\sum(x_i - \overline{x})^2}{n}$$

6. **Take the Square Root**:

- To return to the original units of measurement, take the square root of the variance. This gives the **standard deviation**.

$$s = \sqrt{\frac{\sum(x_i - \overline{x})^2}{n}}$$

Example:

Given the data set:

$$2,4,6,8,10$$

1. **Find the mean**:

$$\overline{x} = \frac{2+4+6+8+10}{5} = 6$$

2. **Find the deviations**:
   - $2-6=-4$
   - $4-6=-2$
   - $6-6=0$
   - $8-6=2$
   - $10-6=4$

3. **Square the deviations**:
   - $(-4)^2=16$
   - $(-2)^2=4$
   - $0^2=0$
   - $2^2=4$
   - $4^2=16$

4. **Sum the squared deviations**:

$$16+4+0+4+16=40$$

5. **Divide by the number of data values (5)**:

$$\frac{40}{5}=8$$

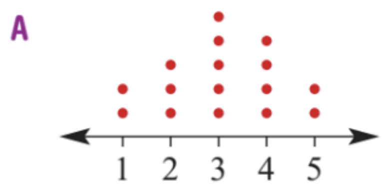
6. **Take the square root**:

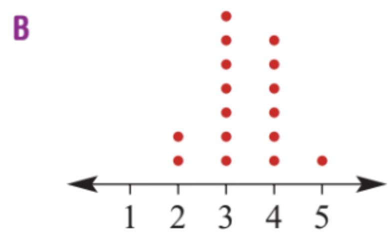
$$\sqrt{8} \approx 2.83$$

Thus, the **standard deviation** is approximately **2.83**.

Practice Questions

1. Here are two dot plots A and B.





a. Which data set (A or B) would have the higher mean?

b. Which data set (A or B) would have the higher standard deviation?

2. This back-to-back stem-and-leaf plot compares the number of trees or shrubs in the backyards of homes in the suburbs of Gum Heights and Oak Valley.

Gum Heights Leaf	Stem	Oak Valley Leaf
7 3 1	0	6
9 8 6 4 0	1	0 5
9 8 7 2	2	0 2 3 6 8 8 9
6 4	3	4 6 8
	4	3

2 | 8 means 28

a. Which suburb has the smaller mean number of trees or shrubs? Do not calculate the actual means.

b. Which suburb has the smaller standard deviation?

3. Calculate the mean and standard deviation for these small data sets. Round the standard deviation to one decimal place where necessary.

   a. $1, 1, 4, 5, 7$

   b. $28, 29, 32, 33, 36, 37$

4. This back-to-back stem-and-leaf plot shows the distribution of distances travelled by students at an inner city and an outer suburb school. The means and standard deviations are given.

Inner city Leaf	Stem	Outer suburb Leaf
9 6 4 3 1 1	0	3 4 9
9 4 2 0	1	2 8 8 9
7 1	2	1 3 4
	3	4
	4	1

Inner city
$\bar{x} = 10.6$
$\sigma = 8.0$
Outer suburb
$\bar{x} = 18.8$
$\sigma = 10.7$

2 | 4 means 24

Consider the position and spread of the data and then answer the following.

a. Why is the mean for the outer suburb school larger than that for the inner city school?

b. Why is the standard deviation for the inner city school smaller than that for the outer suburb school?

c. Give a practical reason for the difference in centre and spread for the two schools.

Chapter 9.8 Linear Regression with Technology

Linear Regression

Linear regression is a statistical method used to fit a straight line to **bivariate data**, i.e., data that consists of two variables. The goal is to find a straight line that best represents the relationship between these two variables, allowing for both **interpolation** (predicting values within the data range) and **extrapolation** (predicting values outside the data range).

Least Squares Regression Line

- The **least squares regression line** is the most commonly used method for fitting a straight line to data.
- The line is determined by minimizing the **sum of the squared deviations** (errors) between the actual data points and the points predicted by the line.
- This means the line is chosen such that the total distance between each data point and the line is as small as possible, with larger deviations having a larger impact due to squaring.

Equation of the Line:

The equation of the regression line is generally written as:

$$y = mx + c$$

Where:

- m is the slope (rate of change of y with respect to x),
- c is the y-intercept (the value of y when $x = 0$).

The slope m and intercept c are determined through the least squares method.

Impact of Outliers

- **Outliers** (data points that are significantly different from the rest of the data) can **distort the regression line** because they increase the squared deviations, making the line less representative of the overall trend.
- Since the least squares method uses the squares of the deviations, outliers have a disproportionately large effect on the position and slope of the regression line.

Median-Median Regression Line

- The **median-median regression line** is an alternative method for fitting a line to bivariate data. It is designed to be more **robust** to outliers compared to the least squares method.
- This method uses the **three medians** from the data divided into three groups:
    1. **Lower group**: The data values below the median.
    2. **Middle group**: The median itself.
    3. **Upper group**: The data values above the median.

- The median values are used to calculate the regression line. Outliers do not significantly alter the medians of the groups, so the line is not as affected by extreme values in the data.

Key Differences:

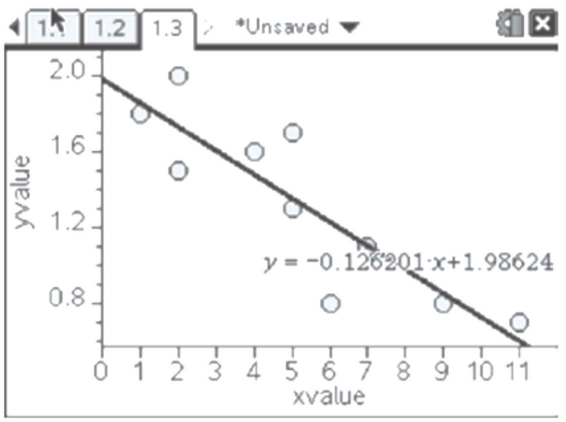
- **Least Squares Line**: The line that minimizes the sum of squared deviations, but sensitive to outliers.
- **Median-Median Line**: Uses medians from grouped data to reduce the influence of outliers, making it more robust.

Both methods have their use cases, with the least squares regression being more common when the data is relatively free of outliers and the median-median regression being preferred when the data has potential outliers that could distort the results.

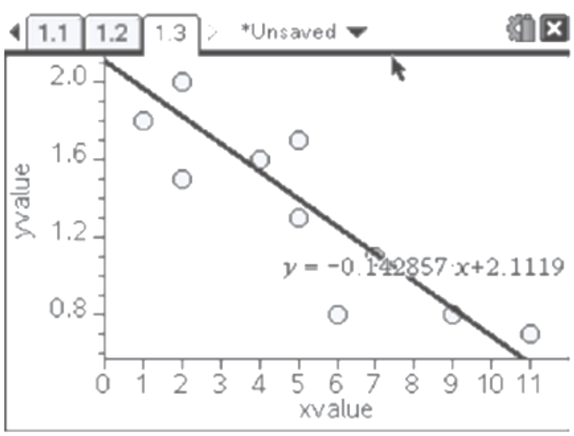
Using the TI-Nspire:

TI-nspire CAS Key strokes and screens

a, b, d Go to a new **Lists and spreadsheets** page and enter the data into the lists. Title each column **xvalue** and **yvalue**. Go to a new **Data and Statistics** page and select **xvalue** variable for the horizontal axis and **yvalue** for the vertical axis. To produce the least square regression line, go to **menu**, **Analyze**, **Regression**, **Show Linear (mx+b)**.



c, d To produce the median–median regression line, go to **menu**, **Analyze**, **Regression**, **Show Median-Median**.

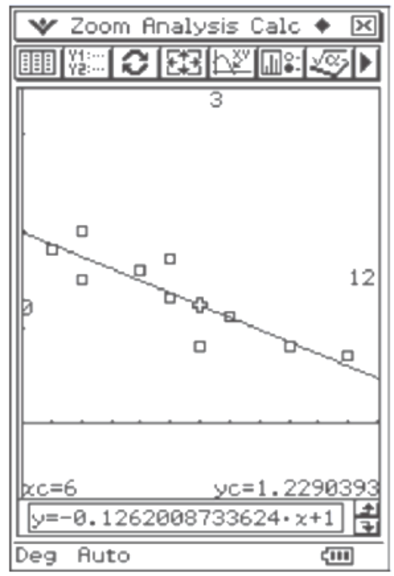


e i $y \approx 1.39$ **ii** $y \approx 0.06$

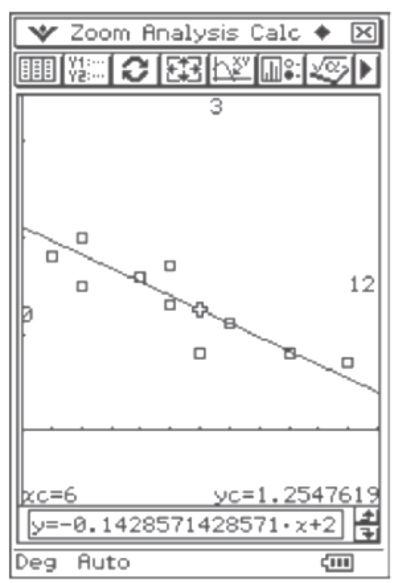
f i $y \approx 1.47$ **ii** $y \approx 0.03$

Using the ClassPad:

a, b, d In the **Statistics** application enter the data into the lists. Tap **Calc**, **Linear Reg** and set **XList** to **list1**, **YList** to **list2**, **Freq** to **1**, **Copy Formula** to **y1** and **Copy Residual** to **Off**. Tap **OK** to view the regression equation. Tap on OK again to view the regression line. Tap **Analysis**, **Trace** then scroll along the regression line.



c, d To produce the median–median regression line, tap **Calc**, **MedMed Line** and apply the settings previously stated.



Practice Questions

1. Consider the data in table and use a graphics or CAS calculator or software to help answer the following questions.

x	3	6	7	10	14	17	21	26
y	3.8	3.7	3.9	3.6	3.1	2.5	2.9	2.1

a. Construct a scatter plot for the data.

b. Find the equation of the least squares regression line.

c. Find the equation of the median–median regression line.

d. Sketch the graph of the regression lines on the scatter plot.

e. Estimate the value of y using the least squares regression line when x is:

i. 7 ii. 12

f. Estimate the value of y using the median–median regression line when x is:

i. 7 ii. 12

2. A factory that produces denim jackets does not have air-conditioning. It was suggested that high temperatures inside the factory were affecting the number of jackets able to be produced, so a study was completed and data collected on 14 consecutive days.

Max. daily temp. inside factory (°C)	28	32	36	27	24	25	29	31	34	38	41	40	38	31
Number of jackets produced	155	136	120	135	142	148	147	141	136	118	112	127	136	132

Use a graphics or CAS calculator to complete the following.

a. Draw a scatter plot for the data.

b. Find the equation of the least squares regression line.

c. Graph the line onto the scatter plot.

d. Use the regression line to estimate how many jackets would be able to be produced if the maximum daily temperature in the factory was:

i. 30°C

ii. 35°C

iii. 45°C

3. At a suburban sports club, the distance record for the hammer throw has increased over time. The first recorded value was $72.3\ \mathrm{m}$ in 1967 and the most recent record was $118.2\ \mathrm{m}$ in 1996. Further details are as follows.

Year	1967	1968	1969	1976	1978	1983	1987	1996
New record (m)	72.3	73.4	82.7	94.2	99.1	101.2	111.6	118.2

a. Draw a scatter plot for the data.

b. Find the equation of the median–median regression line.

c. Use your regression equation to estimate the distance record for the hammer throw for:

i. 2000 ii. 2015

d. Would you say that it is realistic to use your regression equation to estimate distance records beyond 2015? Why?

CHAPTER 10

LOGARITHMS AND POLYNOMIALS

Table of Contents

Chapter 10.1 Logarithms

Logarithms

- A **logarithm** is the exponent to which the base must be raised to produce a given number.

  Example: $\log_2 16 = 4$ because $2^4 = 16$.

- The base a is written as a subscript to the logarithm operator. For example: $\log_a$.

- **General formula:** If $a^x = y$, then $\log_a y = x$, where $a > 0$ and $y > 0$.

- This means: "The logarithm of y to the base a is x."

Practice Questions

1. Write the following in index form.

a. $\log_{10} 100 = 2$

b. $\log_3 27 = 3$

c. $\log_3 \frac{1}{9} = -2$

2. Write the following in logarithmic form.

a. $3^4 = 81$

b. $2^5 = 32$

c. $5^{-3} = \frac{1}{125}$

3. Evaluate the following logarithms.

a. $\log_3 27$

b. $\log_{10} 1000$

c. $\log_9 729$

d. $\log_1 1$

4. Evaluate the following.

a. $\log_{10} \frac{1}{1000}$

b. $\log_8 8$

c. $\log_2 0.5$

d. $\log_3 0.1$

5. Find the value of x in these equations.

a. $\log_5 625 = x$

b. $\log_3 x = 4$

c. $\log_7 x = -3$

d. $\log_x 64 = 2$

e. $\log_4 0.25 = x$

6. A single bacterial cell divides into two every minute.

Time (minutes)	0	1	2	3	4	5
Population	1	2				

a. Complete this cell population table.

b. Write a rule for the population P after t minutes.

c. Use your rule to find the population after 8 minutes.

d. Use trial and error to find the time (correct to the nearest minute) for the population to rise to 10000.

e. Write the exact answer to part **d** as a logarithm.

Chapter 10.2 Logarithm Laws

Logarithmic Laws

1. **Law 1**: $\log_a(xy) = \log_a x + \log_a y$
   - Relates to index law 1: $a^m \times a^n = a^{m+n}$.

2. **Law 2**: $\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$
   - Relates to index law 2: $\frac{a^m}{a^n} = a^{m-n}$.

3. **Law 3**: $\log_a(x^n) = n\log_a x$
   - Relates to index law 3: $(a^m)^n = a^{m\times n}$.

Other Properties of Logarithms:

- $\log_a 1 = 0$, because $a^0 = 1$
- $\log_a a = 1$, because $a^1 = a$
- $\log_a\left(\frac{1}{x}\right) = -\log_a x$, using Law 3.

Practice Questions

1. Simplify using the first logarithm law.

   a. $\log_a 5 + \log_a 3$

   b. $\log_a 7 + \log_a 4$

   c. $\log_b 1 + \log_b 17$

2. Simplify using the second logarithm law.

a. $\log_a 100 - \log_a 10$

b. $\log_b 7 - \log_b 5$

3. Simplify using the third logarithm law.

a. $2\log_a 5$

b. $3\log_a 3$

c. $3\log_a 10$

4. Evaluate:

a. $\log_x 1$

b. $\log_a a$

c. $\frac{1}{3} \log_7 7$

d. $\frac{\log_3 243}{10}$

5. Simplify and evaluate.

a. $\log_3 \frac{1}{27}$

b. $\log_3 \frac{1}{64}$

c. $\log_{10} \frac{1}{100\,000}$

6. Simplify and evaluate.

a. $\log_3 30 - \log_3 10$

b. $\log_8 16 + \log_8 4$

c. $\log_4 128 - \log_4 2$

d. $\log_{10} 50 + \log_{10} 2$

7. Simplify using a combination of log laws.

a. $4\log_{10} 2 + \log_{10} 3$

b. $5\log_7 2 - \log_7 16$

c. $\log_5 3 - \frac{1}{2}\log_5 9$

d. $\frac{1}{4}\log_5 16 + \frac{1}{5}\log_5 243$

Chapter 10.3 Exponential Equations using Logarithms

- Solving for x if $a^x = y$
  - Using the given base: $x = \log_a y$
  - Using base 10:

$$a^x = y$$

$$\log_{10} a^x = \log_{10} y \qquad \text{(taking } \log_{10} \text{ of both sides)}$$

$$x \log_{10} a = \log_{10} y \qquad \text{(using law 3)}$$

$$x = \frac{\log_{10} y}{\log_{10} a} \qquad \text{(dividing by } \log_{10} a\text{)}$$

- Most calculators can evaluate using log base 10, but only some calculators can work with any base.

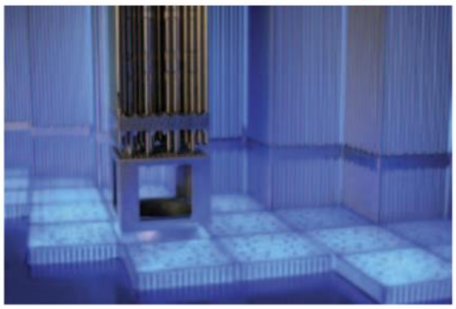
Practice Questions

1. An investment of $\$10000$ is expected to grow by 5% p.a. so the balance $\$A$ is given by the rule $A = 10000 \times 1.05^n$ where n is the number of years. Find the time (to two decimal places) for the investment to grow to:

   a. $\$32000$

   b. $\$100\ 000$

2. $50\ \mathrm{kg}$ of a radioactive isotope in a set of spent nuclear fuel rods is decaying at a rate of 1% per year. The mass of the isotope ($m\ \mathrm{kg}$) is therefore given by $m = 50 \times 0.99^n$ where n is the number of years. Find the time (to two decimal places) when the mass of the isotope reduces to:



a. $40\ \mathrm{kg}$

b. $20\ \mathrm{kg}$

3. The value of a bank balance increases by 10% per year. The initial amount is $\$2000$.

   a. Write a rule connecting the balance $\$A$ with the time (n years).

   b. Find the time, correct to the nearest year, when the balance is double the original amount.

4. The value of a Ferrari is expected to reduce by 8% per year. The original cost is $\$300000$.

a. Find a rule linking the value of the Ferrari ($\$F$) and the time ($n$ years).

b. Find the time it takes for the value of the Ferrari to reduce to $\$150000$. Round to one decimal place.

5. The half-life of a substance is the time it takes for the substance to reduce to half its original mass. Round answers to the nearest year.

   a. Find the half-life of a $10\ \mathrm{kg}$ rock if its mass reduces by 1% per year.

   b. Find the half-life of a $20\ \mathrm{g}$ crystal if its mass reduces by 0.05% per year.

Chapter 10.4 Polynomials

Polynomials

- A polynomial is an expression of the form:

  $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$

  where:

  - n is a whole number $\{0,1,2,\dots\}$
  - $a_n, a_{n-1}, \dots, a_0$ are coefficients
  - a_0 is the constant term (when x^0)
  - $a_n x^n$ is the leading term

Naming Polynomials by Degree:

- **Constant**: Degree 0 (e.g., 2)
- **Linear**: Degree 1 (e.g., $3x - 7$)
- **Quadratic**: Degree 2 (e.g., $2x^2 - 4x + 11$)
- **Cubic**: Degree 3 (e.g., $-4x^3 + 6x^2 - x + 3$)
- **Quartic**: Degree 4 (e.g., $x^4 - 4x^2 + 2$)
- **Degree 8**: (e.g., $3x^8 - 4x^5 + x - 3$)

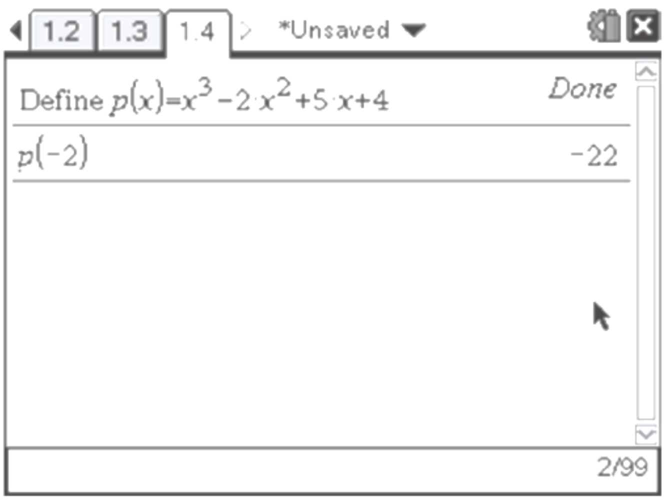
Function Notation:

- A polynomial in x can be written as $P(x)$.

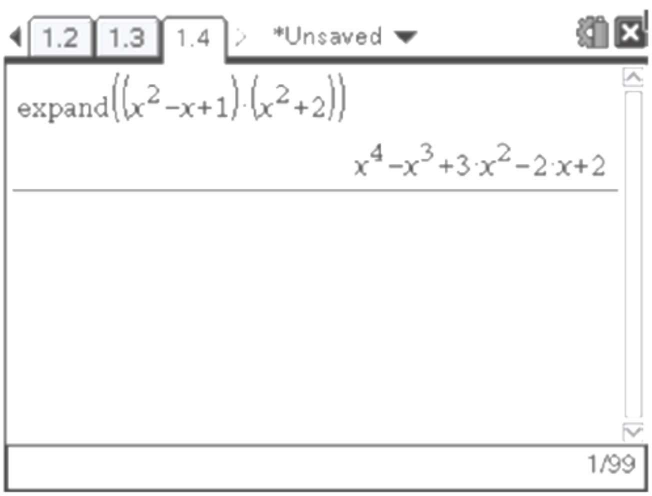
  For example: $P(x) = 2x^3 - x$ is a cubic polynomial.
- $P(k)$ is the value of the polynomial at $x = k$.
  - If $P(x) = 2x^3 - x$:
    - $P(3) = 2(3)^3 - 3 = 51$
    - $P(-1) = 2(-1)^3 - (-1) = -2 + 1 = -1$

Using the TI-Nspire:

1 In a **Calculator** page, press **menu** and select **Actions**, **Define**. Then type the polynomial. Evaluate by typing $p(-2)$.



2 Press **menu** and select **Algebra**, **Expand**. Then type and press **enter**.



Using the ClassPad:

1 In the **Main** application tap **Action**, **Command**, **Define**. Then type the polynomial. Evaluate by typing $p(-2)$.

2 Tap **Action**, **Transformation**, **expand**. Then type and press **EXE**.

Practice Questions

1. Decide if the following are polynomials.

a. $2x^4 - x^2 - 4$

b. $\frac{2}{x} - \frac{3}{x} + 2$

c. $x^4 - x^3 + \frac{2}{x^3}$

d. $4 - 7x^8$

e. $\sqrt[4]{x} + \sqrt[3]{x} + \sqrt{x}$

f. $x^3 + \frac{1}{\sqrt{x}}$

2. If $P(x) = x^3 - x^2$ and $Q(x) = 4 - 3x$ find:

a. $P(-2) - Q(-2)$

b. $\left(P(2)\right)^2 + \left(Q(1)\right)^2$

c. $\left(P(-1)\right)^3 - \left(Q(-1)\right)^3$

3. Evaluate $P(-2)$ for these polynomials.

a. $P(x) = (x - 2)(x + 3)(x + 1)$

b. $P(x) = x^2(x + 5)(x - 7)$

4. The height (P metres) of a roller coaster track above a platform is given by $P(x) = x^3 - 12x^2 + 35x$ where x metres is the horizontal distance from the beginning of the platform.

a. Find the height of the track using $x = 7$.

b. Does the track height ever fall below the level of the platform? If so find a value of x for which this occurs.

Chapter 10.5 Expanding and Simplifying Polynomials

Practice Questions

1. Expand and simplify.

   a. $2x^2(1 + 3x)$

   b. $-3x^2(x^4 - x)$

   c. $-4x^3(x^4 - 2x^7)$

2. Expand and simplify.

a. $(x^2 - x)(x^3 - 3x)$

b. $(x^3 - x^2)(x^2 - x + 4)$

c. $(x^4 - x^2 + 1)(x^4 + x - 3)$

3. $If\ P(x) = x^2 - 2x + 1$ and $Q(x) = x^3 + x - 1$, expand and simplify:

a. $P(x) \times Q(x)$

b. $\left(Q(x)\right)^2$

c. $\left(P(x)\right)^2$

4. If $P(x) = x^3 + 2x^2 - x - 4$ and $Q(x) = x^2 + x - 2$, expand and simplify:

a. $P(x) \times Q(x)$

b. $\left(Q(x)\right)^2$

c. $\left(P(x)\right)^2$

5. If $P(x) = x^2 - 5x + 1$ and $Q(x) = x^3 + x$, expand and simplify:

a. $Q(x) - P(x)$

b. $5P(x) + 2Q(x)$

c. $\left(P(x)\right)^2 - \left(Q(x)\right)^2$

6. Expand and simplify.

a. $x^3(x + 3)(x - 1)$

b. $(x + 4)(2x - 1)(3x + 1)$

c. $(x^2 + 1)(x^2 - 2)(x + 3)$

Chapter 10.6 Division of Polynomials

dividend $\rightarrow \dfrac{x^3 - x^2 + x - 1}{x + 2} = x^2 - 3x + 7 - \dfrac{15}{x + 2}$ remainder

divisor quotient

We can write this as:

$x^3 - x^2 + x - 1 = (x + 2)(x^2 - 3x + 7) - 15$

dividend divisor quotient remainder

Practice Questions

1. Divide $P(x) = x^3 + x^2 - 2x + 3$ by $(x - 1)$ and write in the form $P(x) = (x - 1)Q(x) + R$, where R is the remainder.

2. For each of the following express in this form:
   $\text{Dividend} = \text{divisor} \times \text{quotient} + \text{remainder}$ (as in the examples)

   a. $(2x^3 + 2x^2 - x - 3) \div (x + 2)$

b. $(-x^3 + x^2 - 10x + 4) \div (x - 4)$

c. $(-5x^3 + 11x^2 - 2x - 20) \div (x - 3)$

3. Divide the following and express in the usual form.

a. $(x^3 + x^2 - 3) \div (x - 1)$

b. $(x^4 - x^2) \div (x - 4)$

4. There are three values of k for which $P(x) = x^3 - 2x^2 - x + 2$ divided by $(x - k)$ gives a remainder of zero. Find the three values of k.

5. Find the remainder when $P(x) = -3x^4 - x^3 - 2x^2 - x - 1$ is divided by these expressions.

a. $2x + 3$

b. $-3x - 2$

Chapter 10.7 The Remainder and Factor Theorems

Remainder Theorem

The Remainder Theorem states that when a polynomial $P(x)$ is divided by $(x - a)$, the remainder is equal to $P(a)$.

Key Points:

- When dividing by $(x - 3)$, the remainder is $P(3)$.
- When dividing by $(x + 2)$, the remainder is $P(-2)$.

Example: If $P(x) = x^3 - 3x^2 - 3x + 10$, then:

- When dividing by $(x - 2)$, the remainder is $P(2)$.

Factor Theorem

The Factor Theorem is a special case of the Remainder Theorem. It states that if a polynomial $P(x)$ is divided by $(x - a)$ and the remainder is zero (i.e., $P(a) = 0$, then $(x - a)$ is a factor of $P(x)$.

Key Points:

- If $P(2) = 0$, then $(x - 2)$ is a factor of $P(x)$.
- Example: $P(x) = x^3 - 3x^2 - 3x + 10$, where $P(2) = 0$, so $(x - 2)$ is a factor.

Example: Given $P(x) = x3 - 3x2 - 3x + 10$ and $P(2) = 0$, we can factor $P(x)$ as:

$$P(x) = (x - 2)(x^2 - x - 5)$$

Thus, $(x - 2)$ is a factor of $P(x)$.

Practice Questions

1. Find the remainder when $P(x) = x^3 - 2x^2 + 7x - 3$ is divided by:

   a. $x - 4$

   b. $x + 3$

2. Decide which of the following are factors of $P(x) = x^3 - 4x^2 + x + 6$.

a. $x + 2$

b. $x + 4$

3. Decide which of the following are factors of $P(x) = x^4 - 2x^3 - 25x^2 + 26x + 120$.

a. $x - 3$

b. $x + 5$

4. Use the factor theorem and trial and error to find a linear factor of these polynomials.

a. $P(x) = x^3 + 2x^2 - x - 2$

b. $P(x) = x^3 - 2x - 4$

5. Use the factor theorem to find all three linear factors of these polynomials.

a. $P(x) = x^3 - 2x^2 - 5x + 6$

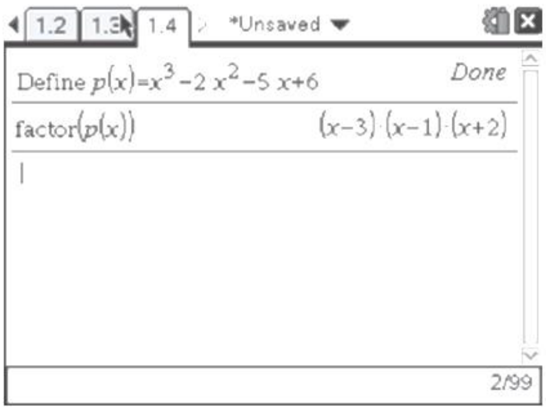
b. $P(x) = x^3 - 2x^2 - 19x + 20$

Chapter 10.8 Solving Polynomial Equations

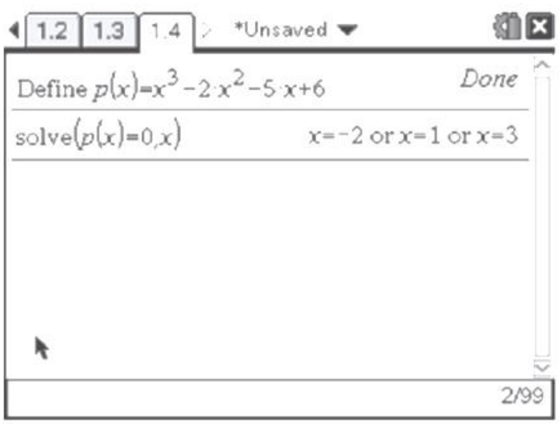
Solve polynomials using CAS:

Using the TI-Nspire:

1 In a **Calculator** page press **menu** and select **Actions**, **Define**. Then type the polynomial. Press **menu** and select **Algebra**, **Factor**.



2 Press **menu** and select **Algebra**, **Solve**. Then type $p(x) = 0,x$).



Using the ClassPad:

1 In the **Main** application tap **Action**, **Command**, **Define**. Then type the polynomial. Tap **Action**, **Transformation**, **factor** then type $p(x)$).

2 Tap **Action**, **Advanced**, **solve**. Then type $p(x) = 0,x$).

Practice Questions

1. Solve for x using the Null Factor Law.

a. $(x + 2)(x + 7)(x - 1) = 0$

b. $\left(x + \frac{1}{2}\right)(x - 3)\left(x + \frac{1}{3}\right) = 0$

c. $(4x - 1)(5x - 2)(7x + 2) = 0$

d. $(5x + 3)(19x + 2)\left(x - \frac{1}{2}\right) = 0$

2. For each of the following cubic equations follow these steps.
   - Use the factor theorem to find a factor.
   - Use long division to find the quotient.
   - Factorise the quotient.
   - Write the polynomial in a fully factorised form.
   - Use the Null Factor Law to solve for x.

   a. $x^3 + 6x^2 + 11x + 6 = 0$

   b. $x^3 - 8x^2 + 19x - 12 = 0$

   c. $x^3 + 6x^2 - x - 30 = 0$

3. Solve by first taking out a common factor.

a. $2x^3 - 14x^2 + 14x + 30 = 0$

b. $3x^3 + 12x^2 + 3x - 18 = 0$

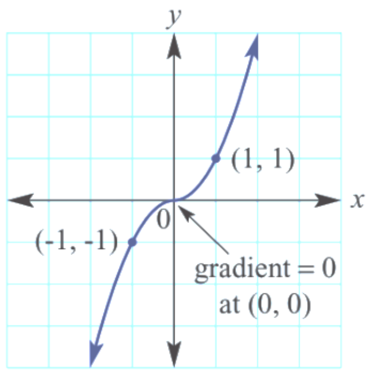
4. Solve for x.

a. $x^3 - 13x + 12 = 0$

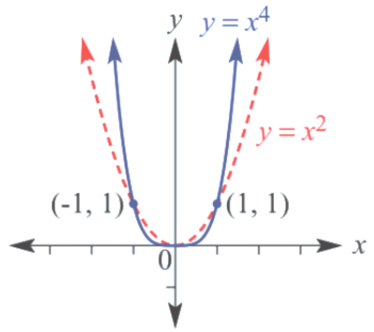
b. $x^3 - 7x - 6 = 0$

Chapter 10.9 Graphs of Polynomials

- Graphs of basic polynomials
  - $y = x^3$

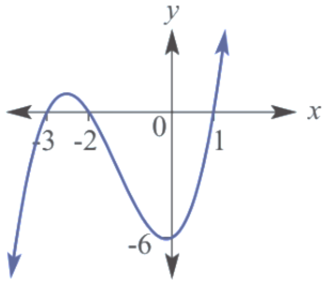


  - $y = x^4$

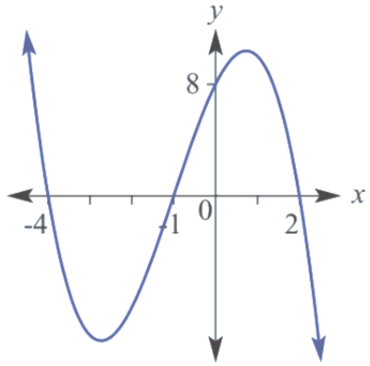


- To sketch cubic graphs in factorised form with three factors:
  - find the three x-intercepts using the Null Nactor Law.
  - find the y-intercept.
  - connect to sketch a positive or negative cubic graph.

Positive cubic
(The coefficient of x^3 is positive)
$y = (x - 1)(x + 2)(x + 3)$

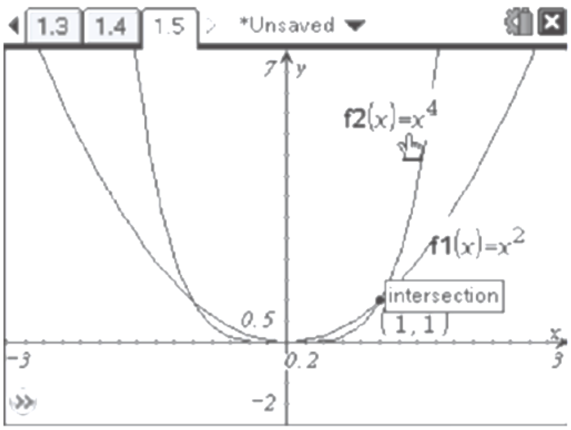


Negative cubic
(The coefficient of x^3 is negative)
$y = -(x + 4)(x - 2)(x + 1)$

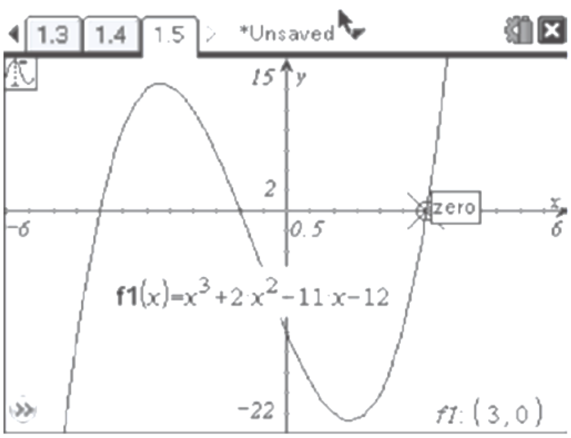


Using the TI-Nspire:

1 In a **Graphs and geometry** page enter the rules $y = x^2$ and $y = x^4$. Adjust the scale using **Window Settings** and find their intersection point using **Analyze**, **Intersection**.

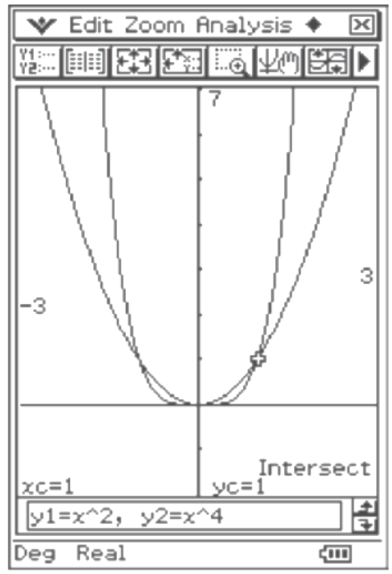


2 Enter the rule $P(x) = x^3 + 2x^2 - 11x - 12$. Adjust the scale using **Window Settings**. Find the x-intercepts using **Graph Trace**.

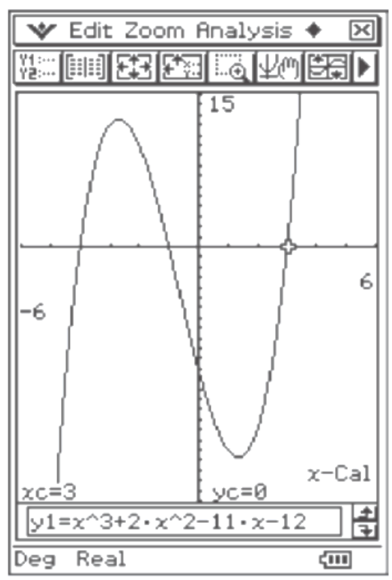


Using the ClassPad:

1 In the **Graph&Table** application enter the rules $y1 = x^2$ and $y2 = x^4$. Tap to adjust the scale. Tap to see the graph. Tap **Analysis**, **G-Solve**, **Intersect**.



2 Enter the rule $y1 = x^3 + 2x^2 - 11x - 12$. Tap to see the graph. Adjust the scale by tapping on . Tap **Analysis**, **G-Solve**, **x-Cal** and enter a y value of **0** to find the x-intercepts.



Practice Questions

1. Sketch the graphs of the following by finding $x-$ and $y-$intercepts.

a. $y = (x - 3)(x - 4)(x + 1)$

b. $y = \frac{1}{2}(x + 3)(x - 2)(x - 1)$

c. $y = x(x - 5)(x + 1)$

d. $y = -\frac{1}{3}x(x + 1)(x - 3)$

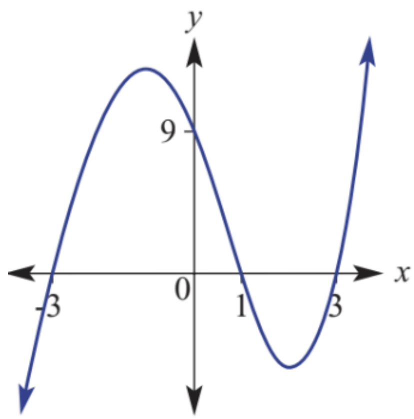
2. Sketch the graph of:

a. $y = -x^3$

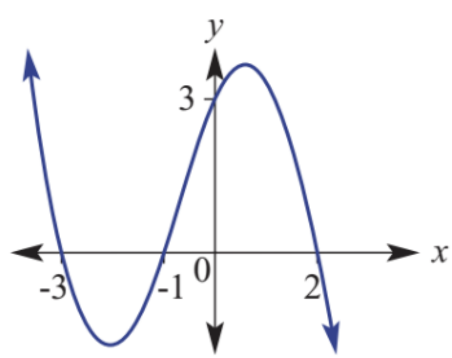
b. $y = -x^4$

3. Find a cubic rule for these graphs.

a.



b.



4. Sketch these quartics.

a. $y = (x - 5)(x - 3)(x + 1)(x + 2)$

b. $y = -x(x + 4)(x + 1)(x - 4)$