Quaternionic Harmonic Analysis and Classical Special Functions

PhD thesis

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ANALYSE HARMONIQUE QUATERNIONIQUE ET FONCTIONS SPÉCIALES CLASSIQUES

JURY

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To my wife, my children,
my parents and my brothers

Contents

Preface

This work is an abridged and corrected version of my PhD thesis "Analyse Harmonique Quaternionique et Fonctions Spéciales Classiques" (Université de Reims Champagne-Ardenne, 2017). The parts that I have taken out of the original version are just those parts that were written in French so as to comply with regulations in French universities (translated introduction, summaries of all chapters and conclusion). Many corrections have been made, but they are all minor corrections and they have no effect on results.

I am deeply endebted to my PhD supervisor Pr. Michael Pevzner (Reims) for his guidance, kindness and enthusiasm. One of the many good memories I'll keep of my PhD years is the time we spent in Moscow at the Interdisciplinary Scientific Center J.-V. Poncelet to discuss future works whilst I was still finishing my dissertation.

As a PhD student, I had the opportunity to travel and meet people who, each in their own way, would give me some insight on Lie theory. In particular, I met Pr. Toshiyuki Kobayashi (University of Tokyo) and Pr. Pierre Clare (College of William & Mary) on several occasions and I am extremely grateful to them for many enlightening and helpful discussions.

Finally, a more "straight to the point" presentation of this work can be found in [44].

Grégory Mendousse, Reims, January 2023

Notation

The choices made here apply to all integers $n \geq 1$.

The standard Euclidean inner product of $\mathbb{R}^n$ is denoted by a simple dot and defined for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ by the usual sum:

$$x \cdot y = \sum_{i=1}^{n} x_i y_i \ .$$

Given any two vectors $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n)$ of $\mathbb{C}^n$, we set:

$$\langle z, w \rangle = \sum_{i=1}^{n} z_i w_i \ .$$

We point out that this does not correspond to the standard Hermitian inner product of $\mathbb{C}^n$.

We identify real, complex and quaternionic coordinates (in Chapter 2, we explain all we need to know about quaternions and quaternionic linear algebra). The identifications work as follows:

- a vector $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \simeq \mathbb{R}^{2n}$ will correspond to the vector $z = x + iy \in \mathbb{C}^n$;
- a vector $(z, w) \in \mathbb{C}^n \times \mathbb{C}^n \simeq \mathbb{C}^{2n}$ will correspond to the vector $h = z + jw \in \mathbb{H}^n$.

Denote by $\mathbb{K}$ the field $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ and consider any integers $r \geq 1$ and $s \geq 1$. Then:

- We write $\mathbb{K}^\times = \mathbb{K} \setminus \{0\}$.

- $\mathrm{M}_{r,s}(\mathbb{K})$ denotes the $\mathbb{K}$-vector space of $r \times s$ matrices (meaning matrices with r rows and s columns) whose coefficients all belong to $\mathbb{K}$.

- Given $g \in \mathrm{M}_{r,s}(\mathbb{K})$, ${}^t g \in \mathrm{M}_{s,r}(\mathbb{K})$ denotes the transpose of g.

- $\mathrm{M}(n, \mathbb{K})$ denotes the $\mathbb{K}$-vector space of $n \times n$ matrices whose coefficients all belong to $\mathbb{K}$.

- $\mathrm{GL}(n, \mathbb{K})$ denotes the group of invertible elements of $\mathrm{M}(n, \mathbb{K})$.

We always identify matrices with the linear maps they correspond to with respect to the canonical basis of the underlying vector space.

All zero matrices are simply denoted by 0. The $n \times n$ identity matrix is denoted by I_n.

Whatever kind of matrices we are working with, $E_{r,s}$ refers to the matrix whose terms are all 0 except for one term which is equal to 1 and sits in Row r and Column s; such matrices are called *elementary matrices.*

We denote by $d\sigma$ the standard Euclidean measure on the unit sphere S^{n-1}.

Because $\mathbb{R}^{4n} \simeq \mathbb{C}^{2n} \simeq \mathbb{H}^n$, the unit sphere S^{4n-1} of $\mathbb{R}^{4n}$ will also be denoted by S^{4n-1} when it is seen as a subset of $\mathbb{C}^{2n}$ or $\mathbb{H}^n$.

Given a measurable space X and a measure μ on X:

- $L^1(X, \mu)$ (or just $L^1(X)$ if μ is understood) denotes the complex Banach space of equivalence classes of integrable complex-valued functions on X with respect to μ.

- $L^2(X, \mu)$ (or just $L^2(X)$ if μ is understood) denotes the complex Hilbert space of equivalence classes of square-integrable complex-valued functions on X with respect to μ.

Given a topological space X, $C^0(X)$ denotes the complex vector space of continuous complex-valued functions on X.

We now fix once and for all an integer $n \geq 2$ and we set:

$$m = n - 1 \ ; \ N = 2n.$$

As for the bibliography, the references in no way mean to cover the huge literature connected with this work; they only appear if explicitly mentioned at some point.

Chapter 1

Introduction

In broad terms, the point of representation theory is to give tangible pictures of abstract groups by identifying their elements with operators that one can visualise. This procedure helps understand the structure of the given groups, but, more importantly perhaps, uses the knowledge we have about these groups to solve important and difficult problems. For instance, one might look for solutions of a certain differential equation; these solutions might form a vector space on which some relevant group acts by linear operators, thereby defining a representation; one might then try to break the vector space into explicit irreducible invariant subspaces in order to write down the desired solutions as combinations of elements of these subspaces.

These ideas appear in Fourier's works. In his famous book [14], Fourier studies the heat equation and decomposes its solutions into sums of trigonometric functions. Of course, to refer to a single person when discussing pioneering work is unfair, in the sense that ideas always depend on existing knowledge (for example, Euler had already worked on trigonometric series). To summarise two centuries of mathematical research in a few pages is obviously impossible, so many important names and contributions will unjustly be left out of this introduction. But our intention here is merely to outline a few important stages that have led to representation theory as we know it today, in order to see how our own work fits in the global picture of this theory. Though we have had to choose which facts and people we mention as part of this exciting mathematical story, we have nonetheless tried to be as accurate as possible. The first pages of this introduction are based on Mackey's historical survey [41] and also on [24].

The ideas of Fourier were developed by Cauchy, Poisson, Dirichlet, Riemann, Cayley and others, leading for instance to spherical harmonics in arbitrary dimensions and more applications to physics. In the late 19th century, group theory, which had appeared in the works of Galois and Abel, was placed at the heart of analysis and geometry. Klein exposed this point of view in a famous lecture given at the University of Erlangen (see [26]). Lie studied the actions of continuous groups. Frobenius decided to concentrate on representations of finite groups, showing that they are unitarisable and completely reducible, studying the connections between characters and functions on the group, looking at the regular representation and so forth. Weyl studied representations of compact groups, obtaining similar results to those of Frobenius, but using more sophisticated notions, in particular the theory of integration made available by Borel, Lebesgue and Haar. With his student Peter, he proved in [48] the famous Peter-Weyl theorem, which is a generalisation of the Plancherel formula for Fourier series.

The next step involved Lie algebras. Lie had studied continuous transformation groups from an infinitesimal point of view. Infinitesimal generators of one-parameter subgroups gave rise to what would be called the Lie algebra of the given group. Lie algebras were then studied in their own right by Killing and E. Cartan, whose works led to the complete classification of finite-dimensional simple Lie algebras. E. Cartan studied representations of Lie algebras and classified the irreducible finite-dimensional ones in terms of highest weights. This gave new tools to Weyl to study representations of compact groups in further detail, obtaining for instance the formulas known as the Weyl character formula and the Weyl dimension formula.

Many scientists contributed to quantum physics in the early 20th century (Schrödinger, Planck, Heisenberg, Von Neumann and so on). Their works built up the setting we know today: a physical system corresponds to a certain Hilbert space, whose one-dimensional subspaces define the (pure) states of the system; observables correspond to self-adjoint operators, which define probability distributions. The role of Hilbert spaces here justified the general interest in unitary representations: Weyl used the exponential map to assign to a self-adjoint operator a family of unitary operators, thereby defining a unitary representation of the additive group of the real line; diagonalising the self-adjoint operator meant decomposing the unitary representation.

The study of unitary representations of arbitrary groups (including non-abelian and non-compact groups) began in the forties, with the works of Gel'fand, amongst others. The general idea arose that there existed a correspondance between groups and the set of their unitary representations (see for instance [16]) and that one could study actions of groups on topological spaces X in terms of representations of these groups on spaces of functions on X. Chevalley published an account of Lie theory (see [7]), extending the theory and introducing new ideas (for instance the notion of analytic subgroup). In the fifties, Harish-Chandra set to study the irreducible representations of general semisimple Lie groups on Banach spaces, while Mackey developed the theory of induced representations (see [38], [39] and [40]). From the contributions over the years of numerous mathematicians and physicists, two guidelines emerged:

- the orbit method of A. A. Kirillov (see [25]), which associates unitary irreducible representations of a large class of groups with orbits of their co-adjoint actions;

- the Langlands classification theorem, which says that, given a minimal parabolic subgroup of a linear connected reductive group G, there is a bijection between equivalence classes of irreducible admissible representations of G and certain triples; those triples are called the Langlands parameters (see [27], Section 15 of Chapter VIII); our work comes under this approach.

A great many general results are well known, such as:

- Irreducible unitary representations of compact topological groups are finite-dimensional; this result is part of the Peter-Weyl theorem (see [27], Chapter I).

- Given a compact linear connected reductive group, the equivalence classes of its irreducible representations are parametrised, as Cartan showed, by specific linear forms (the highest weights; see [27], Chapter IV).

- Non-trivial irreducible unitary representations of non-compact linear semisimple groups are infinite-dimensional (see [23], Section 11.1).

- An irreducible unitary representation of a linear connected reductive group, when restricted to a maximal compact subgroup, decomposes into a Hilbert sum of subrepresentations, each of which consists of a finite direct sum of equivalent irreducible representations which are finite-dimensional (this result is due to the Peter-Weyl theorem and to the works of Harish-Chandra - see [27], Section 2 of Chapter VIII); in other words, irreducibility plus unitarity imply admissibility.

- Harish-Chandra introduced the notion of $(\mathfrak{g}, K)$-module (see [53], Chapter 2) and obtained a remarkable theorem, of which we give a stronger version (the subrepresentation theorem) proved by Casselman and Miličić in [5] (Theorem 8.21): consider the Lie algebra $\mathfrak{g}$ of a connected semisimple Lie group whose center is finite (the class of groups this theorem applies to is in fact larger); consider a maximal compact subgroup K and a minimal parabolic subgroup P; then an irreducible admissible $(\mathfrak{g}, K)$-module is always embedded in a representation which is induced from some irreducible finite-dimensional representation of P.

In fact, so much is known that one could almost feel that representation theory is near to complete. Which of course is not true. For one thing, understanding an object requires assumptions; changing these assumptions changes the theory; one could decide to work with other spaces than Hilbert spaces, with fields of positive characteristic and so forth (we will not go in these directions). For another thing, there are many aspects of representations one can wish to investigate. For instance:

- Give tangible examples of representations: the Stone von Neumann theorem (in the thirties) classifies unitary representations of the Heisenberg group, those of $\mathrm{SL}(2,\mathbb{R})$ were studied by Gel'fand, Naĭmark and Bargmann in the forties (see [1] an [15]), actions of symplectic groups were studied in the sixties and led to the metaplectic representation (one can find an account of this in [13], along with representations of the Heisenberg group); more examples will be discussed further on. The point of examples is of course to justify one's interest in a theory but also to give some insight on various areas it is connected to; it so happens that representation theory is connected to many.

- Study the way representations break into irreducibles and what representations become when restricted to various subgroups. Along those lines, one has Kostant's general branching theorem, Howe's study of certain representations of $\mathrm{O}(p,q)$, $\mathrm{U}(p,q)$ and $\mathrm{Sp}(p,q)$, works of T. Kobayashi and Pevzner in terms of differential operators (see [36]) and so forth.

- Amongst the representations given by the Langlands classification, determine the ones that are unitary; many people have worked in this direction (Zuckermann, Adams, Vogan...).

- Study connections that exist between representations and other objects. The present work investigates the appearance of special functions in representation theory. The fact that such functions do appear is hardly a discovery. They can emerge from the computation of matrix coefficients: for example, the role of Bessel functions is shown in Chapter 4 of [51] (Section 4.1) for some action of $\mathrm{ISO}(2)$ on smooth functions on the circle. They can appear as spherical functions of Riemannian symmetric spaces. Also, on an infinitesimal level, enveloping algebras correspond to differential operators that lead to differential equations which sometimes characterise well known special functions: for instance, we use

K-types and the Casimir operator to obtain a hypergeometric equation. Special functions also play a key role in the construction of symmetry breaking operators (see [29] and [36]).

The interesting thing about special functions is that they are solutions of particular differential equations, that they can be expressed in various ways (as series, integrals and through recursive methods) and that they sometimes define orthonormal bases of functional Hilbert spaces.

Bearing in mind the role of special functions in representation theory, here are a number of considerations that guided us towards the setting we have chosen for our work.

First, the subrepresentation theorem justified, to us, the choice of induced representations.

Then, to study the appearance of explicit objects, it seems appropriate to select concrete groups: the notion of reductive group is a generalisation of a number of matrix groups that are preserved by some involution; why not choose straight away one of those groups explicitly? Which brings us now to the actual choice of a matrix group.

The unitary duals of $\mathrm{GL}(n,\mathbb{R})$ and $\mathrm{GL}(n,\mathbb{C})$ have been studied in great detail. For instance, it is shown in [50] how to use unitary characters of $\mathbb{R}^\times$ (resp. $\mathbb{C}^\times$) and, in the real case, certain irreducible representations of $\mathrm{GL}(2,\mathbb{R})$, to construct all irreducible unitarisable representations of $\mathrm{GL}(n,\mathbb{R})$ (resp. $\mathrm{GL}(n,\mathbb{C})$) via tensor products and parabolic induction. Looking at subgroups of $\mathrm{GL}(n,\mathbb{R})$ and $\mathrm{GL}(n,\mathbb{C})$, the classical ones are defined with respect to various non-degenerate bilinear or sesquilinear forms they preserve. Representations of $\mathrm{O}(n)$, $\mathrm{U}(n)$ and $\mathrm{Sp}(n)$ are thoroughly understood, in terms of highest weights, because these groups are compact (see for instance [54] and more specifically [45] for $\mathrm{Sp}(n)$); we shall come back to them in a moment. Representations of $\mathrm{O}(p,q)$ and $\mathrm{U}(p,q)$ have been studied extensively. Submodules of some representations of these groups on spaces of homogeneous functions are described in [22]. In a series of papers ([32], [33] and [34]), Kobayashi and Ørsted study the minimal unitary representation of $\mathrm{O}(p,q)$, describing the minimal K-types in terms of special functions. One can also look towards symplectic groups, which preserve alternating non-degenerate bilinear forms – see for example [22], [35] and [8]. These works are based on the K-type structure, which comes from the admissibility assumption of the given representations. Therefore, not only are the irreducible representations of $\mathrm{O}(n)$, $\mathrm{U}(n)$ and $\mathrm{Sp}(n)$ well understood, they are essential ingredients in the general study of representations of reductive groups. Added to which it is well known that they are related to special functions via spherical harmonics.

All these considerations have drawn us to study K-types of irreducible unitary parabolically-induced representations of classical linear groups. But why focus on $\mathrm{Sp}(n,\mathbb{C})$?

In [35] and [8], the authors study degenerate principal series of $\mathrm{Sp}(n,\mathbb{R})$ and $\mathrm{Sp}(n,\mathbb{C})$ that are geometrically realised on spaces of functions defined on $\mathbb{R}^{2n}\setminus\{0\}$ and $\mathbb{C}^{2n}\setminus\{0\}$. Restricting those functions to the unit sphere connects them to spherical harmonics. The cases of $\mathrm{Sp}(n,\mathbb{R})$ and $\mathrm{Sp}(n,\mathbb{C})$ are different but share a same feature: scalar multiplication by numbers of modulus 1 guides us to irreducibles. To be more specific, any invariant subspace under the action of a subgroup that contains K (K denotes a maximal compact subgroup) is a step forward towards a K-type decomposition. The direct sums below illustrate this. In these sums, the parameters correspond to the characters of $\mathrm{O}(1)$ (resp. $\mathrm{U}(1)$) and the summands are invariant subspaces under the left action of $\mathrm{O}(4n)$ (resp. $\mathrm{U}(2n)$):

- $L^2(S^{4n-1}) = L^2_{\text{even}}(S^{4n-1}) \oplus L^2_{\text{odd}}(S^{4n-1})$;
- $L^2(S^{4n-1}) = \widehat{\bigoplus_{\delta\in\mathbb{Z}}} L^2_\delta(S^{4n-1})$, where each space $L^2_\delta(S^{4n-1})$ is the Hilbert sum of all subspaces $\mathcal{Y}^{\alpha,\beta}$ of spherical harmonics of homogeneous degrees (α,β) such that $\delta=\beta-\alpha$.

So two actions seem to rule the decomposition of $L^2(S^{4n-1})$ we are looking for: left action of matrices and scalar multiplication by numbers of modulus 1. This diagram captures the situation:

$$\begin{array}{ccccc} \mathrm{O}(4n) & \curvearrowright & L^2(S^{4n-1}) & \curvearrowleft & \mathrm{O}(1) \\ \cup & & & & \cap \\ \mathrm{U}(2n) & \curvearrowright & L^2(S^{4n-1}) & \curvearrowleft & \mathrm{U}(1). \end{array}$$

One instinctively wants to add a line with:

- the group of linear isometries of $\mathbb{H}^n$, which is isomorphic to $\mathrm{Sp}(n)$ and thus embedded in $\mathrm{U}(2n)$;
- the group of unit quaternions, which is isomorphic to the group $\mathrm{Sp}(1)$ and contains an embedding of $\mathrm{U}(1)$.

In other words, the line would be:

$$\mathrm{Sp}(n) \quad \curvearrowright \quad L^2(S^{4n-1}) \quad \curvearrowleft \quad \mathrm{Sp}(1).$$

This suggests an interesting interaction between the left action of $\mathrm{Sp}(n)$ and a scalar kind of action of $\mathrm{Sp}(1)$. It must be pointed out that this scalar action is not as simple as one might expect: non-commutativity of the field of quaternions requires a choice of sides for scalar multiplication: we choose right multiplication. This multiplication in the quaternionic setting must then be translated back into the complex setting. We call this scalar action the *right action* of $\mathrm{Sp}(1)$.

Putting all these considerations together, the central question of this work is:

> *Via the left action of* $\mathrm{Sp}(n)$ *and the right action of* $\mathrm{Sp}(1)$*, how do special functions connect to* K*-types of degenerate principal series of* $\mathrm{Sp}(n,\mathbb{C})$*?*

Remark: what we are trying to do is generalise spherical harmonics by changing fields, choosing the non-commutative field $\mathbb{H}$ instead of $\mathbb{R}$ or $\mathbb{C}$.

Let us point out that principal series of classical groups over the field of quaternions are discussed in [9], [18] and [47].

Generally speaking, works such as [20], [21], [30], [31] and [35] serve as guidelines to obtain special functions: we make use of differential operators and Fourier transforms.

Throughout this work, we consider an integer $n \geq 2$: the minimal value 2 comes from the fact that we wish to work with a non-minimal parabolic subgroup.

Chapter 2 introduces the background of this work: standard definitions and properties of Lie groups, Lie algebras and their representations, quaternions, quaternionic linear algebra and various computations that are required further on.

Chapter 3 introduces the actual degenerate principal series we are interested in. This series is obtained by parabolic induction which we make explicit: choice of parabolic subgroup (its nilradical is isomorphic to the complex Heisenberg group), choice of character and geometric realisations. We obtain a two parameter family of representations of $G = \mathrm{Sp}(n, \mathbb{C})$, denoted by $\pi_{i\lambda,\delta}$ with $(\lambda, \delta) \in \mathbb{R} \times \mathbb{Z}$ and defined as follows (denoting by $V_{i\lambda,\delta}$ the carrying space of $\pi_{i\lambda,\delta}$):

- Consider the space $V^0_{i\lambda,\delta}$ of all functions $f \in C^0(\mathbb{C}^N \setminus \{0\})$ that are covariant, by which we mean that for all $c \in \mathbb{C}^\times$ and $x \in \mathbb{C}^N \setminus \{0\}$:

  $$f(cx) = \left(\frac{c}{|c|}\right)^{-\delta} |c|^{-i\lambda-N} f(x).$$

  This equality will be refered to as the covariance property. Consider the left action of G on this space, defined by

  $$\pi_{i\lambda,\delta}(g)f(x) = f(g^{-1}x)$$

  for all $(g, f, x) \in G \times V^0_{i\lambda,\delta} \times (\mathbb{C}^N \setminus \{0\})$.

- $V_{i\lambda,\delta}$ and $\pi_{i\lambda,\delta}$ are then obtained by completion of $V^0_{i\lambda,\delta}$ with respect to the norm $\|\cdot\|$ defined by:

$$\|f\|^2 = \int_{S^{4n-1}} |f(x)|^2 d\sigma(x).$$

Representations $\pi_{i\lambda,\delta}$ are unitary. They are also irreducible if and only if $(\lambda,\delta) \neq (0,0)$.

The above description of $\pi_{i\lambda,\delta}$ is called the induced picture. By changing the carrying space in a suitable way, one obtains other descriptions. In particular, the covariance property enables one to change the carrying space $V_{i\lambda,\delta}$ into some subspace of $L^2(S^{4n-1})$; this gives the so-called compact picture of $\pi_{i\lambda,\delta}$ (see Chapter 3) and leads us to study the whole space $L^2(S^{4n-1})$.

In Chapter 4, we make the structure of $L^2(S^{4n-1})$ explicit with respect to both the left action of $\mathrm{Sp}(n)$ and the right action of $\mathrm{Sp}(1)$. Although this double K-type structure appears in [22], the credit one can give to the present work is to offer a personal treatment of the subject, totally explicit and self-contained. Moreover, we highlight polynomials which are invariant under both actions and we show how to compute them. They are the ingredient that takes us to hypergeometric equations. The main results of Chapter 4 are gathered in Theorems A and B.

Theorem A.

- *With respect to the left action of* $\mathrm{Sp}(n)$, *the space* $L^2(S^{4n-1})$ *decomposes into the following Hilbert sum:*

$$L^2(S^{4n-1}) = \widehat{\bigoplus_{k\in\mathbb{N}}} \bigoplus_{\substack{(\alpha,\beta)\in\mathbb{N}\times\mathbb{N}\\ \alpha+\beta=k}} \bigoplus_{\gamma=0}^{\min(\alpha,\beta)} V_\gamma^{\alpha,\beta}.$$

  In this sum, $V_\gamma^{\alpha,\beta}$ *is the irreducible invariant subspace generated by the left translates of the restriction to* S^{4n-1} *of the polynomial* $P_\gamma^{\alpha,\beta}$ *which is defined by:*

$$P_\gamma^{\alpha,\beta}(z,w) = w_1^{\alpha-\gamma}\overline{z_1}^{\beta-\gamma}(w_2\overline{z_1} - w_1\overline{z_2})^\gamma.$$

Here, (z,w) denote the coordinates on $\mathbb{C}^n \times \mathbb{C}^n \simeq \mathbb{C}^{2n}$, taking $z=(z_1,\ldots,z_n)$ and $w=(w_1,\ldots,w_n)$.

- *With respect to the right action of* $\mathrm{Sp}(1)$, *the space* $L^2(S^{4n-1})$ *decomposes into the Hilbert sum*

$$L^2(S^{4n-1}) = \widehat{\bigoplus_{k\in\mathbb{N}}} \bigoplus_{\gamma=0}^{\mathbb{E}\left(\frac{k}{2}\right)} d_\gamma^k W_\gamma^k,$$

where $\mathbb{E}\left(\frac{k}{2}\right)$ *denotes the integer part of* $\frac{k}{2}$. *In this sum,* W_γ^k *is a finite-dimensional irreducible invariant subspace which contains the restriction* $P_\gamma^{k-\gamma,\gamma}|_{S^{4n-1}}$ *as a highest weight vector;* d_γ^k *is a positive integer and there are* d_γ^k *invariant subspaces that are equivalent to* W_γ^k.

Theorem B. *Consider an even integer $k\in\mathbb{N}$ and set $\alpha=\frac{k}{2}$. Denote by* $\mathrm{H}^k(\mathbb{R}^{4n})$ *the space of harmonic polynomials of $4n$ real variables with complex coefficients and which are homogeneous of degree k. Denote by* $1\times\mathrm{Sp}(n-1)$ *the group of matrices of* $\mathrm{Sp}(n)$ *that can be written, in the quaternionic setting, as* $\begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}$ *with* $A\in\mathrm{Sp}(n-1)$. *Then there exists a unique (up to a constant) element of $H^k(\mathbb{R}^{4n})$ (we see this element as a polynomial of $2n$ complex variables and their conjugates) which is invariant under the left action of* $1\times\mathrm{Sp}(n-1)$ *and also under the right action of* $\mathrm{Sp}(1)$*; such a polynomial is said to be bi-invariant (and we show how to compute it). The restriction of this polynomial to S^{4n-1} (also said to be bi-invariant) belongs to $V_\alpha^{\alpha,\alpha}$.*

As mentioned earlier on, our interest lies in the link one can establish between representations and special functions. What makes the choice of unitary groups all the more appropriate for us is that it is well known how to associate special functions to irreducible spaces of spherical harmonics by adding an additional invariance constraint (defining zonal functions) and using the Laplace operator (see Chapter 9 of [12]). To use similar methods with symplectic groups requires a slight adjustment of the additional constraint; this is precisely the point of the right action of Sp(1). The special functions we end up with, as for the orthogonal groups, are solutions of a hypergeometric equation, but this only works for specific K-types. These methods are developed in Chapter 5; here is a partial statement of the main result:

Theorem C (Compact picture and hypergeometric equation)**.** *Consider any integer $\alpha \in \mathbb{N}$. Then the unique (up to a constant) bi-invariant function of $V_\alpha^{\alpha,\alpha}$ (given by Theorem B) can, after a suitable reduction of variables, be written as a solution of the following hypergeometric equation:*

$$u(1-u)\varphi'' + 2(1-nu)\varphi' + (\alpha^2 + (2n-1)\alpha)\varphi = 0.$$

Another common description of parabolically induced representations is the so-called non-compact picture (see Chapter 3). A way to obtain special functions can then be to apply Fourier transforms to this picture (this leads to the so-called non-standard picture), simply because special functions can be expressed in many ways, some of which involve integrals. In [35], the authors consider the group $\mathrm{Sp}(n, \mathbb{R})$ and an analogue of our degenerate principal series $\pi_{i\lambda,\delta}$; a specific partial Fourier transform enables them to use Bessel functions to write down interesting formulas for elements of minimal K-types (when the parameter λ is equal to 0).

In Chapter 6, we generalise this, not only by adapting it to a complex setting, but also by finding explicit formulas of highest weight vectors in terms of modified Bessel functions for all pairs (λ, δ) and for a much wider set of K-types. Before we actually give the formulas, let us define on $L^2(\mathbb{C}^{2m+1})$ the partial Fourier transform $\mathcal{F}$ on which is based the non-standard picture. It is defined for $f \in L^1(\mathbb{C}^{2m+1}) \cap L^2(\mathbb{C}^{2m+1})$ by

$$\mathcal{F}(f)(s,u,v) = \int_{\mathbb{C}\times\mathbb{C}^m} f(\tau,u,\xi)\, e^{-2i\pi \mathrm{Re}(s\tau + \langle v,\xi\rangle)}\, d\tau d\xi,$$

where (s,u,v) denote the coordinates of $\mathbb{C}\times\mathbb{C}^m\times\mathbb{C}^m \simeq \mathbb{C}^{2m+1}$. We can now state our main theorem, namely Theorem 6.11, but in a slightly different and lighter way:

Theorem D (Non-standard picture and Bessel functions)**.** *Let $n \in \mathbb{N}$ be such that $n \geq 2$; set $N = 2n$ and $m = n-1$. Let $k \in \mathbb{N}$ and*

$(\alpha, \beta) \in \mathbb{N}^2$ *be such that* $\alpha + \beta = k$ *; set* $\delta = \beta - \alpha$*. For* $\lambda \in \mathbb{R}$*, consider the representation* $\pi_{i\lambda,\delta}$ *of* $\mathrm{Sp}(n, \mathbb{C})$ *and the function*

$$\begin{array}{rcccc} f & : & \mathbb{C}^n \times \mathbb{C}^n \setminus \{(0,0)\} & \longrightarrow & \mathbb{C} \\ & & (z, w) & \longmapsto & \left(\|z\|^2 + \|w\|^2\right)^{-\frac{i\lambda+k}{2}-n} \, w_1^{\alpha} \, \overline{z_1}^{\beta}, \end{array}$$

where z_1 *(resp.* w_1*) denotes the first coordinate of* z *(resp.* w*). Then* f *generates a finite-dimensional subspace under the action of* $\pi_{i\lambda,\delta}\big|_{\mathrm{Sp}(n)}$*;* f *is a highest weight vector of this subspace and the non-standard form of* f*, meaning the function* $\mathcal{F}\Big((\tau, u, \xi) \longmapsto f(1, u, 2\tau, \xi)\Big)$*, assigns to all triples* $(s, u, v) \in \mathbb{C} \times \mathbb{C}^m \times \mathbb{C}^m$ *such that* $v \neq 0$ *and* $s \neq 0$ *the value*

$$R(s,u,v) \; K_{\frac{i\lambda+\delta}{2}} \left(\pi \sqrt{1 + \|u\|^2} \sqrt{|s|^2 + 4\|v\|^2} \right),$$

where we set

$$R(s,u,v) = \frac{(-i\,\overline{s})^{\alpha} \, \pi^{i\lambda+\beta+n}}{2^{\frac{i\lambda+k}{2}+1} \, \Gamma(\frac{i\lambda+k}{2}+n)} \left(\frac{\sqrt{|s|^2 + 4\|v\|^2}}{\pi\sqrt{1+\|u\|^2}} \right)^{\frac{i\lambda+\delta}{2}}.$$

Chapter 6 ends with two interesting observations:

- there exists a simple differential operator that connects the non-standard forms of certain highest weight vectors to one another (Theorem 6.12);
- the formula given in Theorem D is linked to differential equations known as Emden-Fowler equations (Section 6.7.2).

Chapter 2

Preliminaries: Lie theory and quaternions

2.1 Basic definitions and fundamental properties

Given a set X, denote by $\mathrm{Bij}(X)$ the set of bijections from X onto X; it is a group with respect to composition. An *action* of a group G on X is a group homomorphism from G into $\mathrm{Bij}(X)$.

Given a vector space V, a *representation* of a group G on V is an action π of G on V such that all maps $\pi(g)$ are linear isomorphisms of V; one also says that (π, V) is a representation of G. The space V is the *carrying space* of π; the *dimension* of π is the dimension of V (possibly infinite). A subspace W of V is *invariant* (or *stable*) under the action of π if $\pi(g)W \subseteq W$ for all $g \in G$; in this case, the restriction $\pi|_W$ is a *subrepresentation* of π. The representation π is *algebraically irreducible* (or *irreducible*, for short) if it has no invariant subspaces other than $\{0\}$ and V itself.

Take $\mathbb{K}$ to be the field $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ and denote by V the vector space $\mathbb{K}^n$. Consider a subgroup G of $\mathrm{GL}(n, \mathbb{K})$. The *natural action* of G on V is simply defined by standard matrix multiplication: a matrix $g \in G$ assigns to an element $x \in V$ the element $gx \in V$. Now consider a subset S of V which is stable under the natural action of G and then consider the complex vector space $\mathcal{F}(S,\mathbb{C}) = \{f : S \longrightarrow \mathbb{C}\}$. Because S is stable under the natural action of G, given $f \in \mathcal{F}(S,\mathbb{C})$ and $g \in G$ one can define a new element of $\mathcal{F}(S,\mathbb{C})$:

$$\begin{array}{rccc} L(g)f: & S & \longrightarrow & \mathbb{C} \\ & x & \longmapsto & f\left(g^{-1}x\right). \end{array}$$

By varying f, we define a bijection:

$$\begin{array}{rccc} L(g): & \mathcal{F}(S,\mathbb{C}) & \longrightarrow & \mathcal{F}(S,\mathbb{C}) \\ & f & \longmapsto & L(g)f. \end{array}$$

This induces another map:

$$\begin{array}{rccc} L: & G & \longrightarrow & \mathrm{Bij}\left(\mathcal{F}(S,\mathbb{C})\right) \\ & g & \longmapsto & L(g). \end{array}$$

This map L is a representation of G on $\mathcal{F}(S,\mathbb{C})$; we will simply call it the *left action of* G *on* $\mathcal{F}(S,\mathbb{C})$. We say that an element f of $\mathcal{F}(S,\mathbb{C})$ is *left-invariant* if f is invariant under the left action of G, meaning that $L(g)f = f$ for all $g \in G$.

Given a subspace $\mathcal{F}'$ of $\mathcal{F}(S,\mathbb{C})$ which is stable under the left action of G, the map

$$\begin{array}{ccc} G & \longrightarrow & \mathrm{Bij}\left(\mathcal{F}'\right) \\ g & \longmapsto & L(g)\big|_{\mathcal{F}'} \end{array}$$

is of course also a representation; it will be referred to as the *left action of* G *on* $\mathcal{F}'$ and, for simplicity, also denoted by L.

According to the kind of spaces one is interested in, some additional structure is necessary to be technically able to study group representations on those spaces. Our setting will be that of Lie groups acting on complex Hilbert spaces.

A *Lie group* G is a group that has the structure of a smooth manifold such that multiplication and inversion are smooth. For instance, the group $\mathrm{GL}(n,\mathbb{K})$ and all its closed subgroups are Lie groups (see

[28], Theorem 0.15); they are in fact called *linear Lie groups*; when connected and stable under conjugate transpose, taking $\mathbb{K}$ to be $\mathbb{R}$ or $\mathbb{C}$, they are called *linear connected reductive groups* and if their center is also finite they are *linear connected semisimple groups.* Being a manifold, G has tangent spaces at all of its elements. In particular, denote by $\mathfrak{g}$, or $\mathrm{Lie}(G)$, the tangent space at the identity element. It can be shown that $\mathfrak{g}$ is a *Lie algebra* with respect to a bilinear skew-symmetric product, called the *Lie bracket*, which involves left-invariant vector fields and relates to G via the so-called exponential map. We do not give the details here, because all we need to know in this work is that when G is a linear Lie group:

- The Lie bracket of its Lie algebra $\mathfrak{g}$ is just the usual commutator $[\cdot,\cdot]$ of matrices, defined by $[X,Y] = XY - YX$; this in fact defines the *adjoint representation* ad of $\mathfrak{g}$: $\mathrm{ad}(X)Y = [X,Y]$.

- The exponential map $\exp : \mathfrak{g} \longrightarrow G$ just assigns to each $X \in \mathfrak{g}$ the usual matrix $\exp X = I + X + \frac{X^2}{2} + \frac{X^3}{3!} + \frac{X^4}{4!} + \ldots$.

The choice of Hilbert spaces is not a random choice: the scalar product enables one to use orthogonality to study invariant subspaces, from the corresponding norm arise continuity and differentiability, unitarity of operators has led to extensive results, quantum theory is based on Hilbert spaces, finite-dimensional spaces can always be turned into Hilbert spaces and so on.

Consider a Lie group G. From now on, when considering a representation π of G on a Hilbert space H, we automatically assume that π is *continuous*, by which we mean that π satisfies the following *continuity property*: the map

$$\begin{array}{ccc} G \times H & \longrightarrow & H \\ (g,x) & \longmapsto & \pi(g)x \end{array}$$

is continuous. In particular, this implies that each linear isomorphism $\pi(g)$ is continuous; the bounded inverse theorem (which is a consequence of the open mapping theorem) then implies that the inverse map $(\pi(g))^{-1} = \pi(g^{-1})$ is also continuous.

Consider a Hilbert space H. Let us slightly adjust the notion of reducibility: π is said to be irreducible if there is no closed invariant subspace of H other than $\{0\}$ and H. Also, π is said to be *unitary* if all operators $\pi(g)$ are unitary operators of H. If one considers another representation π' of G, on a Hilbert space H', then π and π' are *equivalent* if there exists a bounded linear isomorphism $f : H \longrightarrow H'$ such that for all $g \in G$ the following diagram commutes:

$$\begin{array}{ccc} H & \xrightarrow{\pi(g)} & H \\ \Big\downarrow f & & \Big\downarrow f \\ H' & \xrightarrow[\pi'(g)]{} & H'. \end{array}$$

The set of equivalence classes of all irreducible unitary representations of G is called the *unitary dual* of G and denoted by $\widehat{G}$. We identify $\widehat{G}$ with any set of representatives of its equivalence classes: an equivalence class denoted by $\tau \in \widehat{G}$ implicitely means that τ is a representation of G that belongs to this equivalence class; conversely, to say that a representation τ of G is irreducible and unitary we just write $\tau \in \widehat{G}$, denoting its equivalence class also by τ.

One says that a vector $v \in H$ is a *smooth vector* (or a C^∞-vector) of a representation π if the map $g \longrightarrow \pi(g)v$ is C^∞ on G. We denote by $C^\infty(\pi)$ the set of such vectors. It can be shown that it is a dense subspace of H (see [27], Theorem 3.15). One can define on this subspace $C^\infty(\pi)$ the differential $d\pi$ of π at the identity element by

$$d\pi(X)v = \frac{d}{dt}\bigg|_{t=0} \Big(\pi(\exp tX)v\Big)$$

for all $(X, v) \in \mathfrak{g} \times C^\infty(\pi)$. Elements X are thereby seen as first order differential operators. In our work, we will in fact not need to worry about C^∞-vectors, because calculations will always be done for restrictions of representations to invariant subspaces which are finite-dimensional and whose elements are therefore all C^∞-vectors (this follows from Theorem 3.15 of [27]).

We finish this section with perhaps the most important example, then a historical and founding theorem and finally a major consequence; they all revolve around compactness.

This proposition follows [27] (Chapter I, Section 3, fourth example):

Proposition 2.1 (Important example). *Consider a compact Lie group K, a left-invariant measure μ of K, the Hilbert space $L^2(K,\mu)$ and the left action of K on $L^2(K,\mu)$, defined (as we saw earlier on) by*

$$L(k)f(x) = f(k^{-1}x)$$

for all $(k,f,x) \in K \times L^2(K,\mu) \times K$. Then L is a continuous unitary representation of K on $L^2(K,\mu)$, called the left regular representation of K.

Then comes a theorem that is also proved in [27] (Theorem 1.12):

Theorem 2.2 (Peter-Weyl Theorem). *Let K be a compact Lie group and π a unitary representation of K on a Hilbert space H. Then:*

1. *For each $\tau \in \widehat{K}$, let H_τ denote the sum of all invariant subspaces of H that define irreducible subrepresentations that are equivalent to τ (if there are none, just set $H_\tau = \{0\}$). There exists a countable number of such subspaces that add up to H_τ; we will denote by $n_\tau \in \mathbb{N} \cup \{\infty\}$ the lowest of such numbers (if $H_\tau = \{0\}$, one just sets $n_\tau = 0$).*

2. *All representations $\tau \in \widehat{K}$ are finite-dimensional.*

3. *Given $\tau \in \widehat{K}$, we denote by E_τ the orthogonal projection on H_τ; elements $h \in H$ can be written $h = \sum_{\tau \in \widehat{K}} E_\tau(h)$. If two representations τ and τ' of $\widehat{K}$ are inequivalent, then:*

$$E_\tau E_{\tau'} = E_{\tau'} E_\tau = 0.$$

Another way to state Theorem 2.2 is to say that π decomposes into a sum over $\tau \in \hat{K}$ of subrepresentations $\pi|_{H_\tau}$ such that each $\pi|_{H_\tau}$ decomposes into n_τ finite-dimensional irreducible unitary subrepresentations that are equivalent to τ. This kind of decomposition is written:

$$\pi \cong \sum_{\tau \in \widehat{K}}^{\oplus} n_\tau \tau.$$

Theorem 2.2 can also be used to study representations of groups that are not compact, simply by restricting to compact subgroups. Under certain assumptions, multiplicities are finite:

Theorem 2.3. *Suppose G is a linear connected reductive group and let K be a maximal compact subgroup of G. Consider an element π of $\widehat{G}$. Apply Theorem 2.2 to the restriction of π to the subgroup K:*

$$\pi|_K \cong \sum_{\tau \in \widehat{K}}^{\oplus} n_\tau \tau.$$

Then each n_τ is finite. More accurately:

$$n_\tau \leq \dim \tau < \infty.$$

The sum in Theorem 2.3 is called the *isotypic decomposition* of π. An element τ of $\hat{K}$ for which $n_\tau \neq 0$ is called a *K-type* of π; the number n_τ is its *multiplicity*.

Remarks:

- This theorem is part of Harish-Chandra's work. For a proof, one can read Section 2 of Chapter VIII in [27] (Theorem 8.1).
- We point out that all maximal compact subgroups of G are conjugate (see for instance [4], Chapter VII, Theorem 1.2).

2.2 Symplectic groups

First of all, it will be convenient to split matrices into blocks. A $2n \times 2n$ complex matrix g, that is, an element of $\mathrm{M}(2n, \mathbb{C})$, will be split into four $n \times n$ complex matrices A, B, C, D:

$$g = \begin{pmatrix} A & C \\ B & D \end{pmatrix}.$$

In particular, we shall be interested in the matrix:

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.$$

We denote by ω the *standard symplectic form* on $\mathbb{C}^{2n}$, defined by:

$$\omega(X, X') = {}^tXJX' = \langle y, x' \rangle - \langle x, y' \rangle ,$$

where $X = (x, y)$ and $X' = (x', y')$ belong to $\mathbb{C}^n \times \mathbb{C}^n$.

A $2n \times 2n$ complex matrix g is said to be *symplectic* if

$${}^tgJg = J.$$

This is equivalent to saying that g preserves ω, meaning:

$$\forall (X, X') \in \mathbb{C}^{2n} \times \mathbb{C}^{2n} : \ \omega(gX, gX') = \omega(X, X').$$

The group of such matrices is the *complex symplectic group* (or just *symplectic group*)

$$G = \mathrm{Sp}(n, \mathbb{C}).$$

It is straightforward to check that a $2n \times 2n$ complex matrix $\begin{pmatrix} A & C \\ B & D \end{pmatrix}$ is symplectic if and only if:

$${}^tAB = {}^tBA,\ {}^tCD = {}^tDC \text{ and } {}^tAD - {}^tBC = I. \tag{2.1}$$

The group G is a closed subgroup of $\mathrm{GL}(n, \mathbb{C})$ and is therefore a Lie group. It is in fact a linear connected semisimple group. Its Lie algebra is the *complex symplectic algebra* (or just *symplectic algebra*) $\mathfrak{sp}(n, \mathbb{C})$, that is, the complex vector space

$$\mathfrak{g} = \mathfrak{sp}(n, \mathbb{C}) = \Big\{ X \in \mathrm{M}(2n, \mathbb{C}) \ / \ {}^tXJ + JX = 0 \Big\}$$

or, expressed otherwise:

$$\mathfrak{g} = \left\{ X = \begin{pmatrix} A & C \\ B & -{}^tA \end{pmatrix} \in \mathrm{M}(2n, \mathbb{C}) \ / \ B \text{ and } C \text{ are symmetric} \right\}.$$

Define the compact group $K = \mathrm{Sp}(n) = \mathrm{Sp}(n, \mathbb{C}) \cap \mathrm{U}(2n)$ (as usual, $\mathrm{U}(2n)$ denotes the group of unitary complex $2n \times 2n$ matrices). Unitarity and preservation of the standard symplectic form imply immediately that K consists of all matrices of $\mathrm{U}(2n)$ that can be written

$$\begin{pmatrix} A & -\overline{B} \\ B & \overline{A} \end{pmatrix},$$

where of course $\overline{M}$ denotes the (complex) conjugate of a given matrix M. It is well known (see for instance Theorem 6.5.2 in [42]) that K is simply connected (and therefore connected). It can be shown that K is a maximal compact subgroup of G, via what is called the Cartan decomposition of $\mathfrak{g}$ (which we say nothing of, just referring the reader to [27], Section 1 of Chapter I). Being a closed subgroup of $\mathrm{GL}(n, \mathbb{C})$, K is itself a Lie group; denote by $\mathfrak{k} = \mathfrak{sp}(n)$ its Lie algebra (it is a Lie subalgebra of $\mathfrak{g}$). This Lie algebra is the real vector space that consists of all skew-hermitian (skew, for short) elements of $\mathfrak{g}$:

$$\mathfrak{k} = \left\{ X = \begin{pmatrix} A & -\overline{B} \\ B & \overline{A} \end{pmatrix} \in \mathrm{M}(2n, \mathbb{C}) \ / \ A \text{ is skew and } B \text{ is symmetric} \right\}.$$

Because K is connected, $\exp \mathfrak{k}$ generates K (see [28], Corollary 0.20).

2.3 Lie theory applied to $\mathrm{Sp}(n)$ and $\mathfrak{sp}(n)$

Let us point out that the complexification of $\mathfrak{k}$ is $\mathfrak{g}$ and that $\mathfrak{g}$ is a complex *semisimple* Lie algebra (for a definition of semisimplicity, we refer the reader to [49], Sections 1.6 and 1.10).

In Sections 2.3.1 and 2.3.2 we summarise some well known facts, applying the general theory of complex semisimple Lie algebras to the particular case of $\mathfrak{g}$. We follow [27] (Chapter IV) and [49] (Chapter 2), using the explicit setting of $\mathfrak{k} = \mathfrak{sp}(n)$ and $\mathfrak{g} = \mathfrak{sp}(n, \mathbb{C})$.

2.3.1 Roots

Let $\mathfrak{h}$ be the usual *Cartan subalgebra* of $\mathfrak{g}$ consisting of diagonal elements of $\mathfrak{g}$. If r belongs to $\{1, ..., n\}$, denote by L_r the linear form that assigns to an element of $\mathfrak{h}$ its r^{th} diagonal term. Denote by Δ_K the set of *roots* of $\mathfrak{g}$ with respect to $\mathfrak{h}$; if ν is a root, the corresponding *root space* is (by definition of a root, this space is not reduced to $\{0\}$):

$$\mathfrak{g}_\nu = \{X \in \mathfrak{g} \,/\, \forall H \in \mathfrak{h} : \ \mathrm{ad}(H)(X) = \nu(H)X\} \,.$$

The set Δ_K consists of the following linear forms, where r and s denote integers that belong to $\{1, ..., n\}$:

- $L_r - L_s$ with $r < s$;
- $-L_r + L_s$ with $r < s$;
- $L_r + L_s$ with $r < s$;
- $-L_r - L_s$ with $r < s$;
- $2L_r$;
- $-2L_r$.

Because $\mathfrak{g}$ is semisimple, the root spaces are complex one-dimensional subspaces. One can check that the root spaces of the above roots are respectively generated by the following *root elements*:

- $U^+_{r,s} = E_{r,s} - E_{n+s,n+r}$ (with $r < s$);
- $U^-_{r,s} = E_{s,r} - E_{n+r,n+s}$ (with $r < s$);
- $V^+_{r,s} = E_{r,n+s} + E_{s,n+r}$ (with $r < s$);
- $V^-_{r,s} = E_{n+s,r} + E_{n+r,s}$ (with $r < s$);
- $D^+_r = E_{r,n+r}$;
- $D^-_r = E_{n+r,r}$.

As settled in the notation section, each $E_{r,s}$ refers to an elementary matrix. We will write

$$H_r = E_{r,r} - E_{n+r,n+r}$$

for each integer $r \in \{1, ..., n\}$. Let us point out that

$$\{H_r\}_{r\in\{1,...,n\}}$$

is a $\mathbb{C}$-basis of $\mathfrak{h}$ and that

$$\{H_r, D_r^+, D_r^-\}_{r\in\{1,...,n\}} \cup \{U_{r,s}^+, U_{r,s}^-, V_{r,s}^+, V_{r,s}^-\}_{r,s\in\{1,...,n\},r<s} \tag{2.2}$$

is a $\mathbb{C}$-basis of $\mathfrak{g}$.

Denote by $\mathfrak{h}_{\mathbb{R}}$ the real form of $\mathfrak{h}$ that consists of real matrices of $\mathfrak{h}$. Denote by $\mathfrak{h}'_{\mathbb{R}}$ the space of real-valued linear forms defined on $\mathfrak{h}_{\mathbb{R}}$. Any element σ of $\mathfrak{h}'_{\mathbb{R}}$ can be written $\sigma = \sum_{r=1}^{n} l_r L_r$, where each l_r is a real number; we then identify σ with its coordinates $(l_1, l_2, \cdots, l_n)$. The space $\mathfrak{h}'_{\mathbb{R}}$ is endowed with a natural Euclidean product $< \cdot, \cdot >$ defined by:

$$< \sigma, \sigma' >= \sum_{r=1}^{n} l_r l'_r.$$

Remark: the inner product used in [27] (Chapter IV, Section 2) corresponds to $\frac{1}{2} < \cdot, \cdot >$. The fact that we have omitted the coefficient $\frac{1}{2}$ has no consequence in what we discuss further on: for instance, in the Weyl dimension formula (see Section 2.3.2), all the coefficients $\frac{1}{2}$ just cancel out.

The restrictions of the various roots to $\mathfrak{h}_{\mathbb{R}}$ are real-valued and therefore belong to $\mathfrak{h}'_{\mathbb{R}}$; they also completely determine the corresponding roots. This is why we shall think of roots as elements of $\mathfrak{h}'_{\mathbb{R}}$. If r belongs to $\{1, ..., n\}$, we denote the restriction $L_r\big|_{\mathfrak{h}_{\mathbb{R}}}$ again by L_r, for simplicity.

Consider the lexicographical order on $\mathfrak{h}'_{\mathbb{R}}$ (with respect to the coordinates introduced above). Then the *positive roots* are the following linear forms:

- $L_r - L_s$ with $r < s$;
- $L_r + L_s$ with $r < s$;
- $2L_r$.

Denote by Δ_K^+ the set of positive roots and by ρ_K their half sum. It is straightforward to check the formula

$$\rho_K = nL_1 + (n-1)L_2 + ... + L_n,$$

which, in coordinates, reads:

$$\rho_K = (n, n-1, \ldots, 1). \tag{2.3}$$

2.3.2 Highest weights

Consider a representation π of K on a finite-dimensional complex vector space V. One can study its infinitesimal action, that is, the $\mathbb{R}$-linear representation $d\pi$ of $\mathfrak{k}$. It can always be thought of as a $\mathbb{C}$-linear representation ϖ of $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{k}$ on V by extending it to $\mathfrak{g}$ in the following natural way:

$$\varpi(X + iY) := d\pi(X) + id\pi(Y)$$

for all $X, Y \in \mathfrak{k}$. One refers to ϖ as the *complexification* of $d\pi$ and says that $d\pi$ has been *complexified* (one often simply writes $d\pi$ instead of ϖ). The correspondance between the initial representation $d\pi$ and its complexification ϖ is clearly one-to-one.

One defines *weights* similarly to roots: weights are linear forms on $\mathfrak{h}$ that give the corresponding eigenvalues of joint eigenvectors of all linear maps $\varpi(H)$, where H runs through $\mathfrak{h}$; if σ is a weight, the corresponding *weight space* is (one also adds in the definition of a weight that this space must not be reduced to $\{0\}$):

$$V_\sigma = \{v \in V / \forall H \in \mathfrak{h} : \ \varpi(H)(v) = \sigma(H)v\}.$$

The restrictions of the various weights to $\mathfrak{h}_\mathbb{R}$ are real-valued. This is why we can think of weights as elements of $\mathfrak{h}'_\mathbb{R}$.

Consider a highest weight vector v of (highest) weight σ, that is, a vector v that belongs to V_σ and that is cancelled by the action under ϖ of all elements of

$$\mathfrak{n}^+ = \mathrm{Vect}_\mathbb{C}\{U_{r,s}^+, V_{r,s}^+\}_{1\leq r<s\leq n} \oplus \mathrm{Vect}_\mathbb{C}\{D_r^+\}_{1\leq r\leq n}$$

(see Section 2.3.1 for notation). Applying ϖ to v for all $X \in \mathfrak{g}$ then generates an irreducible invariant subspace $V_{irr}(v)$ of V; the linear form σ is called a highest weight because basic Lie theory shows that if a certain weight vector belongs to $V_{irr}(v)$, its weight cannot be larger than σ. Basic Lie theory also assures us that applying $\pi(k)$ to v for all $k \in K$ generates the same subspace $V_{irr}(v)$. Though it is in fact defined with respect to ϖ, we shall say that the linear form σ (resp. the vector v) is a *highest weight* (resp. *highest weight vector*) of π.

The *highest weight theorem* (see [27], Theorem 4.28) applied to K sets a correspondence between irreducible representations of K and their highest weights; these are the forms $\sigma = \sum_{r=1}^{n} l_r L_r$ such that $(l_1, ..., l_n) \in \mathbb{N}^n$ and $l_1 \geq l_2 \geq ... \geq l_n$; as mentioned previously, we identify σ with $(l_1, ..., l_n)$.

If σ is the highest weight of an irreducible representation of K, then the dimension d_σ of this representation is given by *Weyl's dimension formula* (see Theorem 4.48 of [27]):

$$d_\sigma = \frac{\prod_{\alpha \in \Delta_K^+} < \sigma + \rho_K, \alpha >}{\prod_{\alpha \in \Delta_K^+} < \rho_K, \alpha >}. \tag{2.4}$$

2.3.3 Casimir operator

As we have seen, the Lie subalgebra $\mathfrak{k}$ consists of complex $N \times N$ matrices

$$\begin{pmatrix} A & -\overline{B} \\ B & \overline{A} \end{pmatrix}$$

such that A is a skew-Hermitian $n \times n$ matrix and B a symmetric $n \times n$ matrix.

Define (r and s denote integers that belong to $\{1, ..., n\}$):

- $A_r = iE_{r,r} - iE_{n+r,n+r}$;
- $B_{r,s} = E_{r,s} - E_{s,r} + E_{n+r,n+s} - E_{n+s,n+r}$ (when $r \neq s$);
- $C_{r,s} = iE_{r,s} + iE_{s,r} - iE_{n+r,n+s} - iE_{n+s,n+r}$ (when $r \neq s$);

- $D_r = E_{n+r,r} - E_{r,n+r}$;
- $E_r = iE_{n+r,r} + iE_{r,n+r}$;
- $F_{r,s} = E_{n+r,s} + E_{n+s,r} - E_{r,n+s} - E_{s,n+r}$ (when $r \neq s$);
- $G_{r,s} = iE_{n+r,s} + iE_{n+s,r} + iE_{r,n+s} + iE_{s,n+r}$ (when $r \neq s$).

One easily sees that:

- If $n \geq 2$ then

$$\{A_r, D_r, E_r\}_{r\in\{1,...,n\}} \cup \{B_{r,s}, C_{r,s}, F_{r,s}, G_{r,s}\}_{r,s\in\{1,...,n\},r<s}$$

  is a basis of $\mathfrak{k}$ over $\mathbb{R}$.

- If $n = 1$ then $\{A_1, D_1, E_1\}$ is a basis of $\mathfrak{k}$ over $\mathbb{R}$.

Elements of $\mathfrak{k}$ correspond to elements of K via the exponential map. One easily obtains the following table for $n, r, s \in \{1, ..., n\}$:

Matrix M	M^2	M^3
A_r	$-I_r$	$-A_r$
D_r	$-I_r$	$-D_r$
E_r	$-I_r$	$-E_r$
$B_{r,s}$	$-K_{r,s}$	$-B_{r,s}$
$C_{r,s}$	$-K_{r,s}$	$-C_{r,s}$
$F_{r,s}$	$-K_{r,s}$	$-F_{r,s}$
$G_{r,s}$	$-K_{r,s}$	$-G_{r,s}$

where we denote by I_r the matrix $E_{r,r} + E_{n+r,n+r}$ and by $K_{r,s}$ the matrix $E_{r,r} + E_{s,s} + E_{n+r,n+r} + E_{n+s,n+s}$.

From this table we obtain the following exponentials for $t \in \mathbb{R}$:

Lemma 2.4.

- *For $r \in \{1, 2, ..., n\}$ and $M \in \{A_r, D_r, E_r\}$:*

$$\exp(-tM) = I - (\sin t)M + (\cos t - 1)I_r.$$

- *For $r, s \in \{1, ..., n\}$ such as $r < s$ (here, we assume that $n \geq 2$) and for $M \in \{B_{r,s}, C_{r,s}, F_{r,s}, G_{r,s}\}$:*

$$\exp(-tM) = I - (\sin t)M + (\cos t - 1)K_{r,s}.$$

Define on $\mathfrak{k}$ the bilinear form β by setting

$$\beta(X, Y) = -\frac{1}{2}\operatorname{Tr}(XY)$$

for all $(X, Y) \in \mathfrak{k} \times \mathfrak{k}$, with Tr denoting the matrix trace form. One easily checks:

Lemma 2.5.

1. *The basis*

$$\mathcal{B}_{\mathfrak{k}} = \{A_r, D_r, E_r\}_{r\in\{1,\dots,n\}} \cup \left\{ \frac{B_{r,s}}{\sqrt{2}}, \frac{C_{r,s}}{\sqrt{2}}, \frac{F_{r,s}}{\sqrt{2}}, \frac{G_{r,s}}{\sqrt{2}} \right\}_{r,s\in\{1,\dots,n\}, r<s}$$

   of $\mathfrak{k}$ is orthonormal with respect to β.

2. *The symmetric bilinear form β is real-valued and positive-definite; in other words it is an inner-product on $\mathfrak{k}$.*

3. *β is* ad*-invariant, meaning invariant under the adjoint action:*

$$\forall X, Y, Z \in \mathfrak{g} : \ \beta([X, Y], Z) = -\beta(Y, [X, Z]).$$

Let us consider a finite-dimensional representation σ of K. Because β is a nondegenerate ad-invariant bilinear form on $\mathfrak{k}$ and because $\mathcal{B}_{\mathfrak{k}}$ is an orthonormal basis of $\mathfrak{k}$, we can define (as in [12], see Section 7 of Chapter VI) the Casimir operator of σ:

$$\Omega_\sigma = \sum_{r=1}^{dim(\mathfrak{k})} (d\sigma(X_r))^2, \tag{2.5}$$

where, for the time being, we denote by X_r the elements of $\mathcal{B}_{\mathfrak{k}}$. Thinking of this operator as an element of the enveloping algebra of $\mathfrak{k}$ makes (2.5) somewhat lighter:

$$\Omega_\sigma = \sum_{r=1}^{dim(\mathfrak{k})} X_r^2. \tag{2.6}$$

As we explained in Section 2.3.2, we can always extend $d\sigma$ to obtain, in a one-to-one fashion, a $\mathbb{C}$-linear representation of $\mathfrak{g}$. This enables us to write down elements of $\mathcal{B}_{\mathfrak{k}}$ as combinations of elements of the basis (2.2) of $\mathfrak{g}$ and consider the action on these combinations of the complexification of $d\sigma$, which, in turn, enables us to use information we have on weights, in particular highest weights. The formulas for the Casimir operator become quite simple when applied to a highest weight vector. If the representation σ is irreducible, then Schur's lemma, combined with the well known fact (see for instance [12], Proposition 6.7.1) that Ω_σ commutes with $d\sigma$, implies that Ω_σ is a scalar multiple of the identity map. This scalar is easy to determine when considering a highest weight vector; the aim of this section is precisely to compute it.

One easily checks that for all integers r and s in $\{1, ..., n\}$ ($U^{\pm}_{r,s}$, $V^{\pm}_{r,s}$ and $D^{\pm}_r$ denote the root elements defined in Section 2.3.1):

- $A_r = iH_r$;
- $B_{r,s} = U^+_{r,s} - U^-_{r,s}$ $(r < s)$;
- $C_{r,s} = iU^+_{r,s} + iU^-_{r,s}$ $(r < s)$;
- $D_r = D^-_r - D^+_r$;
- $E_r = iD^-_r + iD^+_r$;
- $F_{r,s} = V^-_{r,s} - V^+_{r,s}$ $(r < s)$;
- $G_{r,s} = iV^-_{r,s} + iV^+_{r,s}$ $(r < s)$.

Proposition 2.6. *If σ is a finite-dimensional irreducible representation of K on a complex vector space and if λ is its highest weight, then:*

$$\Omega_\sigma = -\left(\sum_{r=1}^{n}\left(\lambda^2(H_r) + 2\lambda(H_r)\right) + \sum_{r=1}^{n-1} 2(n-r)\lambda(H_r)\right) Id,$$

where Id denotes the identity map.

Proof:

By definition:

$$\Omega_\sigma = \sum_{r=1}^{n} \left(A_r^2 + D_r^2 + E_r^2\right) + \frac{1}{2} \sum_{1\le r<s\le n} \left(B_{r,s}^2 + C_{r,s}^2 + F_{r,s}^2 + G_{r,s}^2\right). \quad (2.7)$$

Using the formulas given just before this proposition, multiplying the brackets out (one must be careful: the operators do not commute) and cancelling out various terms, we get:

$$\Omega_\sigma = -\sum_{r=1}^{n} \left(H_r^2 + 2D_r^- D_r^+ + 2D_r^+ D_r^-\right) - \sum_{1\le r<s\le n} \left(U_{r,s}^+ U_{r,s}^- + U_{r,s}^- U_{r,s}^+ + V_{r,s}^+ V_{r,s}^- + V_{r,s}^- V_{r,s}^+\right). \quad (2.8)$$

Apply Ω_σ to a highest weight vector v. Then, by definition of a highest weight vector, all terms such as $D_r^+(v)$, $U_{r,s}^+(v)$ and $V_{r,s}^+(v)$ are equal to 0. So (2.8) applied to v becomes:

$$\Omega_\sigma(v) = -\sum_{r=1}^{n} \left(H_r^2(v) + 2D_r^+ D_r^-(v)\right) - \sum_{1\le r<s\le n} \left(U_{r,s}^+ U_{r,s}^-(v) + V_{r,s}^+ V_{r,s}^-(v)\right). \quad (2.9)$$

One can check the following commutation rules:

$$[D_r^+, D_r^-] = H_r \quad ; \quad [U_{r,s}^+, U_{r,s}^-] = H_r - H_s \quad ; \quad [V_{r,s}^+, V_{r,s}^-] = H_r + H_s.$$

Using these rules in (2.9) we get

$$\Omega_\sigma(v) = -\sum_{r=1}^{n} \left(H_r^2(v) + 2H_r(v)\right) - \sum_{1\le r<s\le n} \left(H_r(v) - H_s(v) + H_r(v) + H_s(v)\right),$$

which can be written:

$$\Omega_\sigma(v) = -\sum_{r=1}^{n}\Big(H_r^2(v) + 2H_r(v)\Big) - \sum_{r=1}^{n-1} 2(n-r)H_r(v). \tag{2.10}$$

Because λ is a weight and v a corresponding weight vector, (2.10) finally gives:

$$\Omega_\sigma(v) = -\left(\sum_{r=1}^{n}\Big(\lambda^2(H_r) + 2\lambda(H_r)\Big) + \sum_{r=1}^{n-1} 2(n-r)\lambda(H_r)\right)(v).$$

One concludes with Schur's lemma.

End of proof.

2.4 Quaternions

2.4.1 One way to define quaternions

For the time being, we regard the set of *quaternions* $\mathbb{H}$ as the set $\mathbb{C}^2$. We write $(1,0) = 1_{\mathbb{H}}$ and $(0,1) = j_{\mathbb{H}}$. We endow $\mathbb{C}^2$ with the usual complex vector space structure, denoting scalar multiplication simply by a dot. A pair $(u,v) \in \mathbb{C}^2$ corresponds to the element $u \cdot 1_{\mathbb{H}} + v \cdot j_{\mathbb{H}}$. The following rules define a multiplication $\odot$ on $\mathbb{H}$ (we call it *quaternionic multiplication*):

- (i) $\forall (u,v) \in \mathbb{C}^2 : \ (u \cdot 1_{\mathbb{H}}) \odot (v \cdot 1_{\mathbb{H}}) = (uv) \cdot 1_{\mathbb{H}}$.
- (ii) $\forall (u,v) \in \mathbb{C}^2 : \ (u \cdot 1_{\mathbb{H}}) \odot (v \cdot j_{\mathbb{H}}) = (uv) \cdot j_{\mathbb{H}}$.
- (iii) $\forall (u,v) \in \mathbb{C}^2 : \ (u \cdot j_{\mathbb{H}}) \odot (v \cdot 1_{\mathbb{H}}) = (u\overline{v}) \cdot j_{\mathbb{H}}$. In particular, this implies the fundamental formula for all $v \in \mathbb{C}$:

$$j_{\mathbb{H}} \odot (v \cdot 1_{\mathbb{H}}) = \overline{v} \cdot j_{\mathbb{H}}.$$

- (iv) $\forall (u,v) \in \mathbb{C}^2 : \ (u \cdot j_{\mathbb{H}}) \odot (v \cdot j_{\mathbb{H}}) = (-u\overline{v}) \cdot 1_{\mathbb{H}}$. In particular, this implies that $j_{\mathbb{H}} \odot j_{\mathbb{H}} = -1 \cdot 1_{\mathbb{H}}$, which we of course just write $j_{\mathbb{H}}^2 = -1_{\mathbb{H}}$.

- (v) $\forall (h, h', h'') \in \mathbb{H}^3$:
    - $h \odot (h' + h'') = (h \odot h') + (h \odot h'')$.
    - $(h + h') \odot h'' = (h \odot h'') + (h' \odot h'')$.

These rules clearly imply that $1_{\mathbb{H}}$ is the left and right identity element for quaternionic multiplication. Let us also point out that quaternionic multiplication is not commutative, precisely because of rule (iii), but is associative:

$$\forall (h, h', h'') \in \mathbb{H}^3 : (h \odot h') \odot h'' = h \odot (h' \odot h'').$$

Moreover, we obviously have:

$$\forall (u, h) \in \mathbb{C} \times \mathbb{H} : \; u \cdot h = (u \cdot 1_{\mathbb{H}}) \odot h.$$

In this sense, if we identify complex numbers u with quaternions $u \cdot 1_{\mathbb{H}}$, we can say that quaternionic multiplication has absorbed complex scalar multiplication, which is therefore no longer needed: all the structure of $\mathbb{H}$ lies in addition and quaternionic multiplication. For simplicity, we now:

- just write u instead of $u \cdot 1_{\mathbb{H}}$, given $u \in \mathbb{C}$;
- just write j instead of $j_{\mathbb{H}}$;
- omit all multiplication symbols.

Rule (iii) implies that any quaternion h can be written in two ways for suitable $(u, v) \in \mathbb{C}^2$: either $u + jv$ or $u + wj$ (with $w = \overline{v}$). In other words, we could choose to write the second coefficients of quaternions either on the right or on the left of j. We choose the first possibility because we find it more convenient for quaternionic linear algebra. Finally, as we shall soon see, every non-zero quaternion has a unique inverse, which is both a left and right one. Now let us summarise the facts and definitions we have seen so far:

- The set of quaternions is $\mathbb{H} = \left\{ u + jv \;/\; (u, v) \in \mathbb{C}^2 \right\}$.
- $\mathbb{H}$ is a skew (non-commutative) field, with respect to addition and quaternionic multiplication.

- $\forall v \in \mathbb{C} : \; jv = \overline{v}j$.
- $j^2 = -1$.

Given a quaternion $h = u + jv$, one defines its *quaternionic conjugate* h^* (or *conjugate* for short):

$$h^* = \overline{u} - jv.$$

Lemma 2.7. *Given any quaternions h, h_1 and h_2:*

1. $hh^* = h^*h$;
2. $(h_1 + h_2)^* = h_1^* + h_2^*$;
3. $(h_1 h_2)^* = h_2^* h_1^*$.

The *modulus* of a quaternion h is then defined by:

$$|h| = \sqrt{hh^*} = \sqrt{|u|^2 + |v|^2}.$$

This definition shows that every non-zero quaternion h, as announced previously, has a unique left inverse, a unique right inverse, and that both of these inverses coincide:

$$h^{-1} = \frac{h^*}{|h|^2}.$$

Also, one obviously has:

Lemma 2.8. *Given any quaternions h_1 and h_2:*

1. $h_1 = 0 \Longleftrightarrow |h_1| = 0$;
2. $|h_1 + h_2| \leq |h_1| + |h_2|$;
3. $|h_1 h_2| = |h_1||h_2|$.

A quaternion h will be called a *unit quaternion* if $|h| = 1$. We point out that the set of unit quaternions is a group with respect to quaternionic multiplication; we shall denote it by $\mathrm{U}_{\mathbb{H}}$; it is clearly diffeomorphic to the three-dimensionnal sphere.

Remark: this complex version of quaternions corresponds exactly to the traditional and historical real version (used by W. R. Hamilton himself - see [19]). In the real version, quaternions are usually written

$$h = a + bi + cj + dk,$$

with $k = ij$ and $(a, b, c, d) \in \mathbb{R}^4$, their conjugates becoming

$$a - bi - cj - dk$$

and their moduli

$$\sqrt{|a|^2 + |b|^2 + |c|^2 + |d|^2}.$$

The correspondance between the real and complex versions works via the identifications $u = a + ib$ and $v = c - id$.

2.4.2 Quaternionic linear algebra

Given a positive integer p, we endow $\mathbb{H}^p$ with the usual addition and scalar multiplication (by quaternions). Multiplication can apply either on the left or on the right of coordinates; the only difference, compared to $\mathbb{R}^p$ or $\mathbb{C}^p$, is that left and right multiplication usually give different results ($\mathbb{H}$ is non-commutative). Considering left multiplication, we see that all axioms of a vector space are satisfied, so we say that $\mathbb{H}^p$ is a *left quaternionic vector space*; similarly, one can focus on right multiplication and say that $\mathbb{H}^p$ is a *right quaternionic vector space*. In a way, which point of view one chooses has no importance, as long as one's choice is clear. However, linearity of functions must be specified accordingly: left linearity does not coincide with right linearity, again, because $\mathbb{H}$ is not commutative. *We shall always work with right linearity*: for us, a *linear map* from some quaternionic vector space to another will always mean a right-linear map; the reason for this choice lies in matrix multiplication.

Given positive integers r and s, we endow $\mathrm{M}_{r,s}(\mathbb{H})$ with the usual matrix addition. Scalar multiplication (by quaternions) can be defined on the left or on the right, so $\mathrm{M}_{r,s}(\mathbb{H})$ is both a left and a right quaternionic vector space. Multiplying matrices by matrices of suitable sizes is defined as usual, but one must respect the order in which the coefficients come.

Coming back to a positive integer p, a matrix $A \in \mathrm{M}(p, \mathbb{H})$ is *invertible* if for some matrix $B \in \mathrm{M}(p, \mathbb{H})$ one has $AB = BA = I_p$, in which case B is the *inverse* of A and is denoted by A^{-1}. The set of invertible $p \times p$ matrices is of course a group with respect to matrix multiplication and is denoted by $\mathrm{GL}(p, \mathbb{H})$.

2.4.3 Complex and quaternionic coordinates: identifications

Again, consider any positive integer p. As we have seen, $\mathbb{C}^{2p}$ identifies with $\mathbb{H}^p$: $(u, v) \in \mathbb{C}^p \times \mathbb{C}^p$ identifies with $u + jv \in \mathbb{H}^p$. Given an element x of $\mathbb{H}^p$, we shall write:

- $x = (x_1, ..., x_p) \in \mathbb{H}^p$;
- $x_r = u_r + jv_r$, for all $r \in \{1, \cdots, p\}$, with $(u_r, v_r) \in \mathbb{C}^2$;
- $u = (u_1, \cdots, u_p) \in \mathbb{C}^p$ and $v = (v_1, \cdots, v_p) \in \mathbb{C}^p$;
- $x = u + jv$.

We define for all vectors x and y in $\mathbb{H}^p$:

- $\langle x, y \rangle_{\mathbb{H}} = \sum_{r=1}^{p} x_r y_r^* \in \mathbb{H}$;
- $\|x\|^2 = \langle x, x \rangle_{\mathbb{H}} = \sum_{r=1}^{p} x_r x_r^* = \sum_{r=1}^{p} |x_r|^2$.

Lemma 2.9. *Given any vectors $x, y \in \mathbb{H}^p$ and any quaternion h:*

- $x = 0 \iff \|x\| = 0$*;*
- $\|hx\| = \|xh\| = |h|\|x\|$*;*
- $\|x + y\| \leq \|x\| + \|y\|$*.*

Therefore $\| \cdot \|$ is a norm on $\mathbb{H}^p$. It corresponds to the standard norm of $\mathbb{R}^{4p} \simeq \mathbb{H}^p$, turning $\mathbb{H}^p$ into a real Banach space as well as a complex Banach space.

A matrix $M \in \mathrm{M}(p, \mathbb{H})$ will be decomposed as $M = A + jB$, with A and B two complex $p \times p$ matrices; M can then be identified with the following element of $\mathrm{M}(2p, \mathbb{C})$:

$$M' = \begin{pmatrix} A & -\overline{B} \\ B & \overline{A} \end{pmatrix}.$$

We denote by $\mathrm{M}_{\mathbb{H}}(2p, \mathbb{C})$ the vector space that consists of all elements of $\mathrm{M}(2p, \mathbb{C})$ that can be written like M'. This procedure defines a map:

$$\begin{array}{rccc} E_{\mathbb{H}}^{\mathbb{C}} : & \mathrm{M}(p, \mathbb{H}) & \longrightarrow & \mathrm{M}_{\mathbb{H}}(2p, \mathbb{C}) \\ & M & \longmapsto & M'. \end{array}$$

This map is bijective and respects matrix multiplication, meaning that, given any two elements M_1 and M_2 of $\mathrm{M}(p, \mathbb{H})$:

$$E_{\mathbb{H}}^{\mathbb{C}}(M_1 M_2) = E_{\mathbb{H}}^{\mathbb{C}}(M_1) E_{\mathbb{H}}^{\mathbb{C}}(M_2).$$

This implies that if M_2 is a right inverse of M_1, then $M_2' = E_{\mathbb{H}}^{\mathbb{C}}(M_2)$ is a right inverse of $M_1' = E_{\mathbb{H}}^{\mathbb{C}}(M_1)$. The same goes for left inverses and, because left and right inverses coincide in $\mathrm{GL}(2p, \mathbb{C})$:

Lemma 2.10. *If a matrix M of $\mathrm{M}(p, \mathbb{H})$ has a right (resp. left) inverse M', then M' is also a left (resp. right) inverse of M.*

This is why we grouped both kinds of inverses in a single definition when we introduced invertible matrices earlier on. We denote by $\mathrm{GL}_{\mathbb{H}}(2p, \mathbb{C})$ the image of $\mathrm{GL}(p, \mathbb{H})$ under the map $E_{\mathbb{H}}^{\mathbb{C}}$; it is obviously a subgroup of $\mathrm{GL}(2p, \mathbb{C})$ and the restriction of $E_{\mathbb{H}}^{\mathbb{C}}$ to $\mathrm{GL}(p, \mathbb{H})$ is an isomorphism onto $\mathrm{GL}_{\mathbb{H}}(2p, \mathbb{C})$.

Applying a matrix $M \in \mathrm{M}(p, \mathbb{H})$ to a vector $u + jv \in \mathbb{H}^p$ corresponds to applying $E_{\mathbb{H}}^{\mathbb{C}}(M) = M'$ to the vector $(u, v) \in \mathbb{C}^{2p}$.

If one looks back at the expression of elements of the compact group K given in section 2.2, one cannot miss the similarity with matrices such as the matrix M' above. This tells us that, via the map $E_{\mathbb{H}}^{\mathbb{C}}$, the group $\mathrm{Sp}(p)$ corresponds to the group of elements of $\mathrm{GL}(p, \mathbb{H})$ that preserve the norms of vectors of $\mathbb{H}^p$, that is, the group of linear isometries of

$\mathbb{H}^p$. We again denote this group by $\mathrm{Sp}(p)$. In particular: $\mathrm{Sp}(1) \simeq \mathrm{U}_{\mathbb{H}}$.

Left and right multiplication by quaternions in $\mathbb{H}^p$ can also be read in $\mathbb{C}^{2p}$, but not as simple scalar multiplications: they combine the complex coordinates in a more subtle way. Indeed, multiplying a quaternion $h = u + jv \in \mathbb{H}$ by another quaternion $q = a + jb \in \mathbb{H}$ on the right gives:

$$hq = (ua - \overline{v}b) + j(va + \overline{u}b).$$

This formula also shows how left multiplication operates: h and q play symmetric roles. Let us summarise how the actions of quaternionic matrices and quaternionic scalars read in the complex setting:

Lemma 2.11. *Given a vector $x = u+jv \in \mathbb{H}^p$, a scalar $q = a+jb \in \mathbb{H}$ and a matrix $M = A + jB \in \mathrm{M}(p, \mathbb{H})$:*

1. *$Mx \in \mathbb{H}^n$ corresponds to $\begin{pmatrix} A & -\overline{B} \\ B & \overline{A} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{C}^{2p}$;*
2. *$xq \in \mathbb{H}^p$ corresponds to*

$$(u_1 a - \overline{v}_1 b, \ldots, u_p a - \overline{v}_p b, v_1 a + \overline{u_1} b, \ldots, v_p a + \overline{u}_p b) \in \mathbb{C}^{2p}.$$

Consider a quaternionic matrix $M \in \mathrm{M}(p, \mathbb{H})$ and denote by m_{rs} its coefficients. Write $m_{rs} = u_{rs} + jv_{rs}$ and define:

$$\|M\|_{\mathrm{op}} = max\{|u_{rs}|, |v_{rs}|\}_{1 \leq r,s \leq n}.$$

Obviously, $\|M\|_{\mathrm{op}}$ is equal to the standard norm of $E_{\mathbb{H}}^{\mathbb{C}}(M)$ in $\mathrm{M}(2p, \mathbb{C})$ which hands out the highest modulus of the coefficients of $E_{\mathbb{H}}^{\mathbb{C}}(M)$.

Lemma 2.12. *Given any matrices $M, M' \in \mathrm{M}(p, \mathbb{H})$ and any quaternion h:*

- $\|M\|_{\mathrm{op}} \geq 0$*;*
- $M = 0 \Longleftrightarrow \|M\|_{\mathrm{op}} = 0$*;*
- $\|hM\|_{\mathrm{op}} = \|Mh\|_{\mathrm{op}} = |h| \|M\|_{\mathrm{op}}$*;*
- $\|M + M'\|_{\mathrm{op}} \leq \|M\|_{\mathrm{op}} + \|M'\|_{\mathrm{op}}$*.*

In other words, $\|\cdot\|_{\mathrm{op}}$ is a norm on $\mathrm{M}(p,\mathbb{H})$. It turns $\mathrm{M}(p,\mathbb{H})$ into a Banach space and the map $E_{\mathbb{H}}^{\mathbb{C}}$ into a homeomorphism, allowing one to define the exponential map exp on $\mathrm{M}(p,\mathbb{H})$ by the usual formula:

$$\exp(M) = \sum_{r=0}^{\infty} \frac{M^r}{r!}.$$

One easily checks that the maps $E_{\mathbb{H}}^{\mathbb{C}}$ and exp commute; in other words, we have the following commutative diagram:

$$\begin{tikzcd}
\mathrm{GL}(p,\mathbb{H}) \arrow[r, "E_{\mathbb{H}}^{\mathbb{C}}"] & \mathrm{GL}_{\mathbb{H}}(2p,\mathbb{C}) \\
\mathrm{M}(p,\mathbb{H}) \arrow[u, "\exp"] \arrow[r, "E_{\mathbb{H}}^{\mathbb{C}}"'] & \mathrm{M}_{\mathbb{H}}(2p,\mathbb{C}). \arrow[u, "\exp"']
\end{tikzcd}$$

2.4.4 Symplectic matrices seen as quaternionic matrices

Elements of $\mathrm{Sp}(n)$ (see Section 2.4.3) can be seen, via the embedding map $E_{\mathbb{H}}^{\mathbb{C}}$, as quaternionic matrices; in this process of identification, for simplicity, we will not change notation, writing in particular:

- $A_r = iE_{r,r}$;
- $B_{r,s} = E_{r,s} - E_{s,r}$ (when $n \geq 2$ and $r \neq s$);
- $C_{r,s} = iE_{r,s} + iE_{s,r}$ (when $n \geq 2$ and $r \neq s$);
- $D_r = jE_{r,r}$;
- $E_r = jiE_{r,r}$;
- $F_{r,s} = jE_{r,s} + jE_{s,r}$ (when $n \geq 2$ and $r \neq s$);
- $G_{r,s} = jiE_{r,s} + jiE_{s,r}$ (when $n \geq 2$ and $r \neq s$).

Then $\{A_r, D_r, E_r\}_{r\in\{1,\ldots,n\}} \cup \{B_{r,s}, C_{r,s}, F_{r,s}, G_{r,s}\}_{r,s\in\{1,\ldots,n\},r<s}$ is a basis over $\mathbb{R}$ of $\mathfrak{k}$ (seen as a subspace of $\mathrm{M}(n,\mathbb{H})$). Applying the exponential map and Lemma 2.4, given $t \in \mathbb{R}$, we have:

Lemma 2.13.

- *For* $r \in \{1, 2, ..., n\}$ *and* $M \in \{A_r, D_r, E_r\}$*:*

$$\exp(-tM) = I - (\sin t)M + (\cos t - 1)E_{r,r}.$$

- *For* $r, s \in \{1, ..., n\}$ *such as* $r < s$ *(here it is assumed that* $n \geq 2$*) and for any* $M \in \{B_{r,s}, C_{r,s}, F_{r,s}, G_{r,s}\}$*:*

$$\exp(-tM) = I - (\sin t)M + (\cos t - 1)(E_{r,r} + E_{s,s}).$$

One can transport the inner product β defined in section 2.3.3 to matrices of $\mathfrak{k}$ seen as quaternionic matrices: for such matrices X and Y, one sets $\beta(X, Y) = \beta\left(E_{\mathbb{H}}^{\mathbb{C}}(X), E_{\mathbb{H}}^{\mathbb{C}}(Y)\right)$. Again:

- β is an inner product on $\mathfrak{k}$ (seen as a subspace of $\mathrm{M}(n, \mathbb{H})$).
- The set

  $$\mathcal{B}_{\mathfrak{k}} = \{A_r, D_r, E_r\}_{r \in \{1,...,n\}} \cup \left\{\frac{B_{r,s}}{\sqrt{2}}, \frac{C_{r,s}}{\sqrt{2}}, \frac{F_{r,s}}{\sqrt{2}}, \frac{G_{r,s}}{\sqrt{2}}\right\}_{r,s \in \{1,...,n\}, r<s}$$

  is an orthonormal basis of $\mathfrak{k}$ (seen as a subspace of $\mathrm{M}(n, \mathbb{H})$) with respect to β.

The following proposition will come out useful when studying stabilisers of actions of K:

Proposition 2.14.

1. *Consider a quaternionic matrix* $M = \begin{pmatrix} h & L \\ 0 & T \end{pmatrix}$, *with* $h \in \mathbb{H}$, $L \in \mathrm{M}_{1,n-1}(\mathbb{H})$ *and* $T \in \mathrm{M}(n-1, \mathbb{H})$. *Then* M *belongs to* K *if and only if* $L = 0$, $h \in \mathrm{Sp}(1)$ *and* $T \in \mathrm{Sp}(n-1)$.

2. *Consider a quaternionic matrix* $M = \begin{pmatrix} h & 0 \\ C & T \end{pmatrix}$, *with* $h \in \mathbb{H}$, $C \in \mathrm{M}_{n-1,1}(\mathbb{H})$ *and* $T \in \mathrm{M}(n-1, \mathbb{H})$. *Then* M *belongs to* K *if and only if* $C = 0$, $h \in \mathrm{Sp}(1)$ *and* $T \in \mathrm{Sp}(n-1)$.

Proof:

Both items work similarly, so we just prove Item 1 (in fact, each item implies the other via matrix transpose).

Suppose M belongs to K. One can decompose M into $M = U + jV$, with

$$U = \begin{pmatrix} h_1 & L_1 \\ 0 & T_1 \end{pmatrix} \in \mathrm{M}_n(\mathbb{C})$$

and

$$V = \begin{pmatrix} h_2 & L_2 \\ 0 & T_2 \end{pmatrix} \in \mathrm{M}_n(\mathbb{C}).$$

Recalling the definition of the map $E_{\mathbb{H}}^{\mathbb{C}}$, we have:

$$E_{\mathbb{H}}^{\mathbb{C}}(M) = \begin{pmatrix} h_1 & L_1 & -\overline{h_2} & -\overline{L_2} \\ 0 & T_1 & 0 & -\overline{T_2} \\ h_2 & L_2 & \overline{h_1} & \overline{L_1} \\ 0 & T_2 & 0 & \overline{T_1} \end{pmatrix}.$$

The matrix M belongs to K, so $E_{\mathbb{H}}^{\mathbb{C}}(M)$ is unitary:

$$^t\overline{E_{\mathbb{H}}^{\mathbb{C}}(M)}E_{\mathbb{H}}^{\mathbb{C}}(M) = E_{\mathbb{H}}^{\mathbb{C}}(M)\,^t\overline{E_{\mathbb{H}}^{\mathbb{C}}(M)} = I_{2n}.$$

By looking at the top-left coefficients of these products, one finds (here, the symbol $\|\cdot\|$ denotes the norm of $\mathbb{C}^{n-1}$ and we identify row matrices with vectors):

$$|h_1|^2 + \|L_1\|^2 + |h_2|^2 + \|L_2\|^2 = 1$$

and

$$|h_1|^2 + |h_2|^2 = 1.$$

This implies:

$$\|L_1\|^2 + \|L_2\|^2 = 0.$$

Consequently, $L = 0$. From this it follows that h and T must belong respectively to $\mathrm{Sp}(1)$ and $\mathrm{Sp}(n-1)$.

This proves one implication; the converse is straightforward.

End of proof.

Chapter 3

Degenerate principal series of $\mathrm{Sp}(n,\mathbb{C})$

The heart of our work is the study of certain induced representations of $G = \mathrm{Sp}(n,\mathbb{C})$. Why it is interesting to concentrate on such representations was discussed in the introduction. They are based on the choice of a parabolic subgroup. The theory (usually refered to as *structure theory* of reductive Lie groups) that defines such subgroups would take too long to introduce here, so we will just make the specific parabolic subgroup we work with explicit, along with the corresponding induced representations, referring the reader to [27] (Sections 2 and 5 of Chapter V and Section 1 of Chapter VII) for a thorough lecture on the theory that underlies Section 3.2.1.

3.1 A specific parabolic subgroup

We write $2n \times 2n$ matrices in the following way:

$$g = \begin{pmatrix} * & * & * & * \\ * & E & * & G \\ * & * & * & * \\ * & F & * & H \end{pmatrix},$$

where E, F, G, H are $m \times m$ matrices and where the stars denote suitable numbers, row matrices or column matrices. We list below the subgroups of G that we need to define our induced representations.

- We denote by M the subgroup of G consisting of all matrices of the following type:

$$m = \left(\begin{array}{cc|cc} e^{i\theta} & 0 & 0 & 0 \\ 0 & A & 0 & C \\ \hline 0 & 0 & e^{-i\theta} & 0 \\ 0 & B & 0 & D \end{array}\right),$$

with $\theta \in \mathbb{R}$ and $\left(\begin{array}{c|c} A & C \\ \hline B & D \end{array}\right) \in \mathrm{Sp}(m, \mathbb{C})$. We then write $\theta = \theta(m)$.

- We denote by A the subgroup of G consisting of all matrices of the following type:

$$a = \left(\begin{array}{cc|cc} \alpha & 0 & 0 & 0 \\ 0 & I_m & 0 & 0 \\ \hline 0 & 0 & \alpha^{-1} & 0 \\ 0 & 0 & 0 & I_m \end{array}\right),$$

with $\alpha \in]0, \infty[$. We then write $\alpha = \alpha(a)$.

- We denote by N the subgroup of G consisting of all matrices of the following type:

$$n = \left(\begin{array}{cc|cc} 1 & {}^t u & 2s & {}^t v \\ 0 & I_m & v & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & -u & I_m \end{array}\right),$$

with $(s, u, v) \in \mathbb{C} \times \mathbb{C}^m \times \mathbb{C}^m$. The coefficient 2 in front of s is not really necessary in the definition, but we keep it, as customary, to signify the isomorphism between N and the so-called complex Heisenberg group H^{2m+1}: matrices n as above correspond to elements (s, u, v) of H^{2m+1}. Though we will say nothing more of it, this isomorphism underlies the non-compact picture to come.

- We define the *parabolic subgroup* Q as the group of elements q of G that can be written as $q = man$ for some triple (m, a, n)

of $M \times A \times N$. We write $Q = MAN$, calling this equality the *Langlands decomposition* of Q (it defines a diffeomorphism of Q onto $M \times A \times N$).

- We denote by $\overline{N}$ the subgroup of G that consists of all matrices ${}^t n$ for $n \in N$.

Remark: we are aware that symbols m, n and N also refer to dimensions, but with context there can be no confusion. We also point out that N and $\overline{N}$ are swapped in [35]; we have not followed the choice of the authors of [35] in order to stay close to Chapter VII of [27].

When defining induced representations, one can wish to change the carrying spaces. In order to do so, the following proposition will prove useful:

Proposition 3.1. *Q is the subgroup of G consisting of all matrices of G in which the entries of the first column are all 0 except the one at the top.*

Proof:

If q belongs to MAN then one can write:

$$q = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & A & 0 & C \\ 0 & 0 & \alpha^{-1} & 0 \\ 0 & B & 0 & D \end{pmatrix} \begin{pmatrix} 1 & {}^t u & s & {}^t v \\ 0 & I & v & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -u & I \end{pmatrix},$$

where $\alpha \in \mathbb{C} \setminus \{0\}$, $\begin{pmatrix} A & C \\ B & D \end{pmatrix}$ belongs to $\mathrm{Sp}(m, \mathbb{C})$ and where (s, u, v) belongs to $\mathbb{C} \times \mathbb{C}^m \times \mathbb{C}^m$. This gives the following form of q (we'll say that a matrix of this form is a Q-form matrix):

$$q = \begin{pmatrix} \alpha & \alpha {}^t u & \alpha s & \alpha {}^t v \\ 0 & A & Av - Cu & C \\ 0 & 0 & \alpha^{-1} & 0 \\ 0 & B & Bv - Du & D \end{pmatrix}.$$

We see that the first column of q satisfies the required condition.

Suppose now that the first column of a matrix q of G has all its entries equal to 0 except the one at the top; let us write:

$$q = \begin{pmatrix} \alpha & {}^tk & t & {}^tl \\ 0 & A & x & C \\ 0 & {}^te & b & {}^tf \\ 0 & B & y & D \end{pmatrix},$$

where A, B, C and D are $m \times m$ matrices, where α is non zero, (t, k, l) and (b, e, f) belong to $\mathbb{C} \times \mathbb{C}^m \times \mathbb{C}^m$ and (x, y) to $\mathbb{C}^m \times \mathbb{C}^m$. Working with the properties of symplectic matrices given in formulas (2.1), one can prove that:

- $e = f = 0$;
- $b = \alpha^{-1}$;
- $\begin{pmatrix} A & C \\ B & D \end{pmatrix}$ belongs to $\mathrm{Sp}(m, \mathbb{C})$.

Then, again using the conditions that a symplectic matrix must satisfy, one shows that:

- (i) $k\alpha^{-1} + {}^tAy = {}^tBx$;
- (ii) $l\alpha^{-1} + {}^tCy = {}^tDx$.

Comparing with the Q-form matrix obtained in the first part of the proof, we can identify s, u, v:

- $s = \alpha^{-1}t$;
- $u = \alpha^{-1}k$;
- $v = \alpha^{-1}l$.

So that (i) and (ii) become:

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} {}^tB & -{}^tA \\ {}^tD & -{}^tC \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Inverting this gives:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -C & A \\ -D & B \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}.$$

So we see that $q = \begin{pmatrix} \alpha & \alpha {}^t u & \alpha s & \alpha {}^t v \\ 0 & A & Av - Cu & C \\ 0 & 0 & \alpha^{-1} & 0 \\ 0 & B & Bv - Du & D \end{pmatrix}$ is indeed a Q-form matrix, that we can write

$$q = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & A & 0 & C \\ 0 & 0 & \alpha^{-1} & 0 \\ 0 & B & 0 & D \end{pmatrix} \begin{pmatrix} 1 & {}^t u & s & {}^t v \\ 0 & I & v & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -u & I \end{pmatrix}$$

and that finally does belong to $Q = MAN$.

End of proof.

3.2 Specific induced representations

Throughout this section we consider any $\lambda \in \mathbb{R}$ and $\delta \in \mathbb{Z}$.

3.2.1 Usual definitions

Induced picture

Consider $(\lambda, \delta) \in \mathbb{R} \times \mathbb{Z}$ and the character $\chi_{i\lambda,\delta}$ of the parabolic subgroup $Q = MAN$ (introduced in section 3.1) defined by

$$\chi_{i\lambda,\delta}(man) = e^{i\delta\theta(m)} \left(\alpha(a)\right)^{i\lambda+N}.$$

Consider the complex vector space $V^0_{i\lambda,\delta}$ of all functions $f \in C^0(G)$ such that for all $(g, m, a, n) \in G \times M \times A \times N$:

$$f(gman) = \chi^{-1}_{i\lambda,\delta}(man)\, f(g).$$

The induced representation $\pi_{i\lambda,\delta} = \mathrm{Ind}_Q^G \chi_{i\lambda,\delta}$ is obtained by considering the left action of G on $V^0_{i\lambda,\delta}$ and completing $V^0_{i\lambda,\delta}$ (extending the action of G accordingly) with respect to the norm $\|\cdot\|$ defined by

$$\|f\|^2 = \int_K |f(k)|^2 \, dk,$$

where dk denotes the Haar measure (unique up to a constant) of K. Functions of $V^0_{i\lambda,\delta}$ are said to be (λ, δ)-*covariant* (or just *covariant*, for short).

The representations $\pi_{i\lambda,\delta}$ obtained by varying the parameters λ and δ are continuous, unitary and form a family called the *degenerate principal series* of G (the word degenerate refers to the fact that the parabolic subgroup we have chosen is not minimal). They were studied in [18], where it is proved that $\pi_{i\lambda,\delta}$ is irreducible if and only if $(\lambda, \delta) \neq (0, 0)$.

One can change the carrying space and define representations that are equivalent to $\pi_{i\lambda,\delta}$. To specify which one of them is considered, one uses the word *picture*, by which we mean the description of the action and the carrying space. The above description is called the *induced picture*. The equivalences between different pictures are due to structure theory, which, as mentioned previously, we have chosen not to discuss.

Compact picture

The compact picture is obtained by restricting functions of $V^0_{i\lambda,\delta}$ to K. One thus considers the complex vector space

$$\left\{ f \in C^0(K) \;/\; \forall k \in K,\ \forall m \in M \cap K:\ f(km) = e^{-i\delta\theta(m)} f(k) \right\},$$

denoting it again by $V^0_{i\lambda,\delta}$ and completing it with respect to the norm $\|\cdot\|$ defined by

$$\|f\|^2 = \int_K |f(k)|^2 \, dk.$$

Identification of the carrying spaces of the induced and compact pictures is based on a specific decomposition of elements of G: structure theory shows that $G = KMAN$ and restriction to K then defines a one-to-one correspondance between functions of $V^0_{i\lambda,\delta}$ seen in the induced picture and functions of $V^0_{i\lambda,\delta}$ seen in the compact picture. The action of G in the compact picture is not as simple as one might think, because multiplying elements of K by elements of G can lead to elements that do not belong to K anymore. However, the restriction of the action to K is just the left action of K.

Non-compact picture

One can show that restricting functions of $V^0_{i\lambda,\delta}$ (in the induced picture) to $\overline{N}$ gives continuous elements of $L^2(\overline{N}, d\overline{n})$, where $d\overline{n}$ denotes the Haar measure (unique up to a constant) of $\overline{N}$. So, in the non-compact picture, one chooses as carrying space the whole of $L^2(\overline{N})$. Here as well, the action of G is not as simple as one might think, because multiplying elements of $\overline{N}$ by elements of G can lead to elements that do not belong to $\overline{N}$ anymore; but the restriction of the action to $\overline{N}$ is just the left action of $\overline{N}$. Identification of the carrying spaces of the induced and non-compact pictures is slightly more subtle than for the compact picture, because based on a decomposition of G that only works for almost all $g \in G$: $\overline{N}MAN$ is dense in G.

3.2.2 Changing the carrying spaces

Another way to change the carrying spaces is to make use of the natural action of the group G on $\mathbb{C}^N$. Each of the three pictures described previously then leads to another picture; in the process, for simplicity, we do not rename the pictures and we keep on writing $\pi_{i\lambda,\delta}$ for the induced representations.

Another induced picture

Definition 3.2 (Induced picture of $\pi_{i\lambda,\delta}$)**.** *It is obtained by considering the left action of G on the complex vector space*

$$V^0_{i\lambda,\delta} = \left\{ f \in C^0\left(\mathbb{C}^N \backslash \{0\}\right) \big/ \forall c \in \mathbb{C}^\times : \ f(c\cdot) = \left(\frac{c}{|c|}\right)^{-\delta} |c|^{-i\lambda - N} f(\cdot) \right\}$$

and completing this space (extending the action of G accordingly) with respect to the norm $\|\cdot\|$ defined by

$$\|f\|^2 = \int_{S^{2N-1}} |f(x)|^2 \, d\sigma(x).$$

The completion of $V^0_{i\lambda,\delta}$ is denoted by $V_{i\lambda,\delta}$. Functions of $V^0_{i\lambda,\delta}$ are also said to be (λ,δ)-*covariant* (or just *covariant*, for short). Let us point out that covariance makes $\|\cdot\|$ a proper norm, in the sense that the norm of $f \in V^0_{i\lambda,\delta}$ is equal to 0 if and only if f vanishes everywhere.

Naturally, we have the same properties as in the initial induced picture:

- $\pi_{i\lambda,\delta}$ is continuous and unitary;
- $\pi_{i\lambda,\delta}$ is irreducible if and only if $(\lambda,\delta) \neq (0,0)$.

It is explained in [8], amongst many other things, how $\pi_{0,0}$ decomposes into the direct sum of two irreducible invariant subspaces.

It is both interesting and important to understand how the correspondence works between both versions of the induced picture.

Though we said we would not change notation, it will be helpful for our explanations (till the end of the present subsection) to write $\tilde{V}^0_{i\lambda,\delta}$ (resp. $\tilde{\pi}_{i\lambda,\delta}$ and $\lceil\cdot\rceil$) instead of $V^0_{i\lambda,\delta}$ (resp. $\pi_{i\lambda,\delta}$ and $\|\cdot\|$) when considering the initial induced picture.

Consider the vector $e_1 = (1,0,\ldots,0)$ of $\mathbb{C}^N$. Consider the natural action of G on vectors of $\mathbb{C}^N \backslash \{0\}$ (matrices times column vectors). This action is transitive and the stabiliser S of e_1 is the subgroup of matrices of G such that the coefficients of the first column are all zero except the one at the top that has to be 1. Looking back at the proof of Proposition 3.1, we see that this stabiliser is a subgroup of the parabolic subgroup $Q = MAN$; more accurately, elements of S can be written $s = man$ with:

- $m = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & A & 0 & C \\ \hline 0 & 0 & 1 & 0 \\ 0 & B & 0 & D \end{array}\right)$ with $\left(\begin{array}{c|c} A & C \\ \hline B & D \end{array}\right) \in \mathrm{Sp}(m,\mathbb{C})$;

- $a = I_N$;

- $n = \left(\begin{array}{cc|cc} 1 & {}^t u & 2s & {}^t v \\ 0 & I_m & v & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & -u & I_m \end{array}\right)$ with $(s, u, v) \in \mathbb{C} \times \mathbb{C}^m \times \mathbb{C}^m$.

Consider a function $\tilde{f}$ that belongs to the space $\tilde{V}^0_{i\lambda,\delta}$.

Elements man of S satisfy $e^{i\theta(m)} = \alpha(a) = 1$, so covariance of $\tilde{f}$ implies that $\tilde{f}$ is constant when restricted to representatives of a same coset of G/S and thus defines a function f_S on G/S by:

$$f_S(gS) = \tilde{f}(g).$$

Since G/S is homeomorphic to $\mathbb{C}^N \setminus \{0\}$, one can then define a (complex-valued) function f on $\mathbb{C}^N \setminus \{0\}$ by

$$f(x) \;=\; f_S(gS) \;=\; \tilde{f}(g),$$

where one chooses any element g that satisfies $g(e_1) = x$.

Take $c \in \mathbb{C} \setminus \{0\}$ and set

$$C = \left(\begin{array}{cc|cc} c & 0 & 0 & 0 \\ 0 & I_m & 0 & 0 \\ \hline 0 & 0 & c^{-1} & 0 \\ 0 & 0 & 0 & I_m \end{array}\right).$$

This matrix C belongs to Q because it can be written $C = man$ with

$$m = \left(\begin{array}{cc|cc} \frac{c}{|c|} & 0 & 0 & 0 \\ 0 & I_m & 0 & 0 \\ \hline 0 & 0 & \left(\frac{c}{|c|}\right)^{-1} & 0 \\ 0 & 0 & 0 & I_m \end{array}\right), \; a = \left(\begin{array}{cc|cc} |c| & 0 & 0 & 0 \\ 0 & I_m & 0 & 0 \\ \hline 0 & 0 & |c|^{-1} & 0 \\ 0 & 0 & 0 & I_m \end{array}\right)$$

and $n = I_N$. Now if $g \in G$ is such that $g(e_1) = x$, one can write

$$\begin{array}{rcl} f(cx) & = & f(cg(e_1)) \\ & = & f(g(ce_1)) \\ & = & f(gC(e_1)) \\ & = & \tilde{f}(gC) \\ & = & \left(\frac{c}{|c|}\right)^{-\delta} |c|^{-i\lambda-N}\, \tilde{f}(g) \end{array}$$

and consequently

$$f(cx) = \left(\frac{c}{|c|}\right)^{-\delta} |c|^{-i\lambda-N} f(x). \tag{3.1}$$

Maps f_S and f are continuous, so we have, so far, associated in a one-to-one fashion covariant functions $\tilde{f}$ on G to continuous functions f on $\mathbb{C}^N \setminus \{0\}$ that satisfy (3.1) for all non-zero complex numbers c. This correspondence makes representations and identifications compatible, in the sense that we have a commutative diagram for each element g of G:

$$\begin{array}{ccc} \tilde{f} & \longrightarrow & \tilde{\pi}_{i\lambda,\delta}(g)\tilde{f} \\ \downarrow & & \downarrow \\ f & \longrightarrow & \pi_{i\lambda,\delta}(g)f. \end{array}$$

Now, what is a relevant norm for $V^0_{i\lambda,\delta}$?

Because K is a subgroup of $\mathrm{U}(N)$, the sphere S^{2N-1} and its Euclidean measure σ are invariant under the natural action of K. The restriction of the natural action of K to S^{2N-1} is transitive and the stabiliser of e_1 is $S_K = S \cap K$. Thus the map (taking, as above, any k such that $ke_1 = x$)

$$\begin{array}{rcccc} \Psi & : & S^{2N-1} & \longrightarrow & K/S_K \\ & & x & \longmapsto & kS_K \end{array}$$

is a homeomorphism between S^{2N-1} and the quotient K/S_K; let us denote equivalence classes by $[k]$ (k refering to any representative of the coset kS_K). Consider the function $F = f \circ \psi^{-1}$ on K/S_K. Then:

$$f(x) = F([k]) = \tilde{f}(k),$$

where k is any element of K such that $x = ke_1$.

The natural action of K on S^{2N-1} induces a natural action (simply denoted here by a dot) on K/S_K defined on equivalence classes by (taking k_1 and k_2 in K)

$$k_1 \cdot [k_2] = [k_1 k_2].$$

The Euclidean measure σ can be transported to K/S_K so as to define a K-invariant measure μ (the image measure of σ):

$$\mu(B) = \sigma(\Psi^{-1}(B)),$$

for all measurable subsets B of K/S_K. One has:

$$\int_{S^{2N-1}} |f(x)|^2 \, d\sigma(x) = \int_{K/S_K} |F([k])|^2 \, d\mu([k]). \tag{3.2}$$

Because S_K is a closed subgroup of K and because K is compact, the homogeneous space K/S_K has a unique left invariant Borel measure $\tilde{\mu}$ (up to a constant). This follows from a standard integration theorem that one can find for instance in [28] (Theorem 8.36 – this theorem is in fact stated for real-valued functions, but it passes on to complex-valued ones). So measures μ and $\tilde{\mu}$ are proportional. Moreover, the same theorem says that $\tilde{\mu}$ can be normalised so that for all continuous functions ϕ on K one has:

$$\int_K \phi(k) \, dk = \int_{K/S_K} \left(\int_{S_K} \phi(ks) ds \right) \, d\tilde{\mu}([k]),$$

where ds denotes the Haar measure of S_K (induced by the normalised left Haar measure of K). Applying this to $\phi = \left| \tilde{f}|_K \right|^2$ and denoting by $\mathrm{Vol}(S_K)$ the volume of S_K with respect to the Haar measure dk, we get:

$$\int_K |\tilde{f}(k)|^2 \, dk = \mathrm{Vol}(S_K) \int_{K/S_K} |F([k])|^2 \, d\tilde{\mu}([k]). \tag{3.3}$$

Because μ and $\tilde{\mu}$ are proportionnal, (3.2) and (3.3) imply that the norms $\lceil \cdot \rceil$ and $\| \cdot \|$ are proportionnal. These norms therefore define the same topology and unitarity in one picture corresponds to unitarity in the other picture. These considerations justify our choice of norm $\| \cdot \|$.

Another compact picture

Definition 3.3 (Compact picture of $\pi_{i\lambda,\delta}$). *Here, the carrying space is the Hilbert space*

$$V_{i\lambda,\delta} \;=\; \left\{ f \in L^2(S^{2N-1}) \;/\; \forall \theta \in \mathbb{R} \;:\; f(e^{i\theta}\cdot) = e^{-i\delta\theta} f(\cdot) \right\}$$

with respect to the norm $\|\cdot\|$ defined by:

$$\|f\|^2 = \int_K |f(k)|^2 \; dk.$$

We say that elements of $V_{i\lambda,\delta}$ are δ-covariant (or just covariant, for short). Again, the action of G is not as simple as one might think, but its restriction to K is just the left action of K.

The parameter λ does not explicitly appear in the compact picture, but is hidden in the following observation. Restriction of functions F of $V^0_{i\lambda,\delta}$ to S^{2N-1} (in the new induced picture) establishes a one-to-one correspondence with continuous elements f of the space $V_{i\lambda,\delta}$ in the new compact picture. This correspondence works as follows:

$$F(x) = \|x\|^{-i\lambda - N} f\left(\frac{x}{\|x\|}\right),$$

for $x \in \mathbb{C}^N \setminus \{0\}$.

Another non-compact picture

This picture is based on two observations:

- The image of $e_1 = (1, 0, \ldots, 0) \in C^N$ under the natural action of the subgroup $\overline{N}$ is the (affine) hyperplane

  $$\mathcal{P} = \{1\} \times \mathbb{C}^m \times \mathbb{C} \times \mathbb{C}^m .$$

  Indeed, given an element $n \in N$ written as in Section 3.1, the element ${}^t n \in \overline{N}$ assigns to e_1 the point $(1, u, 2s, v) \in \mathcal{P}$.

- The Haar measure of $\overline{N}$ is the Lebesgue measure $ds\, du\, dv$.

Restricting functions of $V^0_{i\lambda,\delta}$ to $\overline{N}$, in the initial induced picture, corresponds to restricting functions of $V^0_{i\lambda,\delta}$ to the hyperplane $\mathcal{P}$, in the new induced picture, thereby obtaining continuous functions of $L^2(\mathcal{P})$, with respect to the Lebesgue measure $ds\,du\,dv$. These considerations naturally lead us to the following choice:

Definition 3.4 (Non-compact picture of $\pi_{i\lambda,\delta}$)**.** *The carrying space is $L^2(\mathcal{P})$, with respect to the Lebesgue measure $ds\,du\,dv$, where $(1, u, 2s, v)$ denote the coordinates on $\mathcal{P}$. Again, the action of G is not as simple as one might think, but the action restricted to $\overline{N}$ is just the left action of $\overline{N}$.*

One has to be careful with the way the induced picture and the non-compact picture correspond to one another. As much as one can always restrict functions of $V^0_{i\lambda,\delta}$, in the induced picture, to obtain continuous elements of $L^2(\mathcal{P})$, the reverse procedure via covariance only gives functions defined on $\mathbb{C}^N \cap (z_1 \neq 0)$. In more detail, a continuous function f on $\mathcal{P}$ defines a continous function F on $S^{2N-1} \cap (z_1 \neq 0)$:

$$F(z, w) = \left(\frac{z_1}{|z_1|}\right)^{-\delta} |z_1|^{-i\lambda-N} f\left(1, \frac{z_2}{z_1}, \ldots, \frac{z_n}{z_1}, \frac{w_1}{z_1}, \ldots, \frac{w_n}{z_1}\right).$$

The measure of $S^{2N-1} \cap (z_1 = 0)$ is 0 in S^{2N-1}, so this procedure enables one to obtain the whole of $V_{i\lambda,\delta}$ in the new compact and induced pictures.

Via the bijection

$$\begin{array}{ccc} \mathbb{C} \times \mathbb{C}^m \times \mathbb{C}^m & \longrightarrow & \{1\} \times \mathbb{C}^m \times \mathbb{C} \times \mathbb{C}^m \\ (s, u, v) & \longmapsto & (1, u, 2s, v) \end{array}$$

we will identify the spaces $L^2(\mathcal{P})$ and $L^2(\mathbb{C}^{2m+1})$ (again, with respect to the Lebesgue measures $ds\,du\,dv$).

<u>Remark</u>: the parameters λ and δ do not actually appear in this picture; they are hidden in the restriction/extension process.

Chapter 4

Actions of $\mathrm{Sp}(n)$ and $\mathrm{Sp}(1)$

In this section, we work with the compact picture of the induced representations $\pi_{i\lambda,\delta}$. Because the restrictions $\pi_{i\lambda,\delta}\big|_K$ coincide with the left action of K on $V_{i\lambda,\delta}$, we now simply write L instead of $\pi_{i\lambda,\delta}\big|_K$, whatever the value of δ. By changing the values of δ, one can reconstruct the whole of $V = L^2\left(S^{2N-1}\right)$, as we will see. We intend to study how this Hilbert space decomposes into irreducible invariant subspaces under two actions: the left action L of $K = \mathrm{Sp}(n)$ and the right action of $\mathrm{Sp}(1)$ (which we shall define later on).

4.1 Left action of $\mathrm{Sp}(n)$

4.1.1 Preliminaries

For the time being, denote the coordinates of $\mathbb{R}^{2N}$ by $(x_1, \dots x_{2N})$. Consider the Laplace operator $\Delta_{\mathbb{R}}$ defined by:

$$\Delta_{\mathbb{R}} = \sum_{i=1}^{2N} \frac{\partial^2}{\partial x_i^2}.$$

For $k \in \mathbb{N}$, denote by H^k the complex vector space of polynomial functions f defined on $\mathbb{R}^{2N}$, with complex coefficients and such that:

1. f is homogeneous of degree k;

2. f is harmonic, that is, $\Delta_{\mathbb{R}}(f) = 0$.

Denote by $\mathcal{Y}^k$ the complex vector space whose elements are the restrictions to S^{2N-1} of elements of H^k; these restrictions are called *spherical harmonics*. It is well known that (see for example [12], Chapter 9):

$$L^2\left(S^{2N-1}\right) = \widehat{\bigoplus_{k\in\mathbb{N}}} \mathcal{Y}^k, \tag{4.1}$$

where the various spaces $\mathcal{Y}^k$ are orthogonal to one another in $L^2(S^{2N-1})$. It is also well known that the subspaces $\mathcal{Y}^k$ are stable under the left action of $\mathrm{SO}(2N)$ and that they define irreducible pairwise inequivalent representations of $\mathrm{SO}(2N)$ (see Chapter 9 of [52]).

Because K can be seen as a subgroup of $\mathrm{SU}(N)$ which can itself be seen as a subgroup of $\mathrm{SO}(2N)$, the sphere S^{2N-1} is stable under the natural actions of K and $\mathrm{SU}(N)$ and one can therefore consider the left actions of K and $\mathrm{SU}(N)$ on $\mathcal{Y}^k$. It so happens that the right action of $\mathrm{Sp}(1)$ that we will define later also preserves $\mathcal{Y}^k$. So, to understand how $L^2(S^{2N-1})$ decomposes under the left action of K and the right action of $\mathrm{Sp}(1)$, one just needs to concentrate on each $\mathcal{Y}^k$.

Let us switch to complex coordinates. Put $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$, $y = (x_{N+1}, \ldots, x_{2N}) \in \mathbb{R}^N$ and $z = x + iy = (z_1, \ldots, z_N) \in \mathbb{C}^N$. Because $x = \frac{z+\bar{z}}{2}$ and $y = \frac{z-\bar{z}}{2i}$, a function f of x and y can be written as a function F of the complex variable z and its conjugate $\bar{z}$:

$$f(x, y) = F(z, \bar{z}).$$

An element u of $\mathrm{SU}(N)$ can be viewed as a matrix $\tilde{u}$ of $\mathrm{SO}(2N)$, through the following embedding of the set of complex $N \times N$ matrices into the set of real $2N \times 2N$ matrices (the way we have identified $\mathbb{R}^{2N}$ and $\mathbb{C}^N$ is precisely designed to match this embedding): one decomposes u into two real matrices A and B, writing $u = A + iB$, then defines

$$\tilde{u} = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}$$

and checks that $\tilde{u}$ belongs to $\mathrm{SO}(2N)$ and that the mapping $u \longrightarrow \tilde{u}$ is an injective group morphism. The left action of $\tilde{u}$ on f then transfers to F in the obvious way:

$$L(u)P(z,\bar{z}) = P(u^{-1}z,\, \overline{u^{-1}}\bar{z}) = P(u^{-1}z,\, {}^t u\bar{z}). \tag{4.2}$$

The last equality holds because u is unitary and therefore $\overline{u^{-1}} = {}^t u$. In the coordinates $(z, \overline{z})$, the Laplace operator becomes:

$$\Delta_{\mathbb{C}} = 4\sum_{i=1}^{N} \frac{\partial^2}{\partial z_i \partial \bar{z}_i}.$$

For $\alpha, \beta \in \mathbb{N}$, consider the space $\mathrm{H}^{\alpha,\beta}$ of polynomials $P(z, \bar{z})$ ($z \in \mathbb{C}^N$) such that:

1. P is homogeneous of degree α in z and of degree β in $\bar{z}$;
2. $\Delta_{\mathbb{C}}(P) = 0$.

Each $\mathrm{H}^{\alpha,\beta}$ is invariant under the left action of unitary matrices; the resulting representations of the unitary group $\mathrm{SU}(N)$ on $\mathrm{H}^{\alpha,\beta}$ are irreducible and pairwise inequivalent (again, see [52], chapter 11).

Proposition 4.1. *The dimension of* $\mathrm{H}^{\alpha,\beta}$ *is given by the following formula:*

$$\dim \mathrm{H}^{\alpha,\beta} = \frac{(\alpha+\beta+N-1)(\alpha+N-2)!(\beta+N-2)!}{(N-1)!(N-2)!\alpha!\beta!}.$$

Via coordinate identifications, one can consider spaces $\mathrm{H}^{\alpha,\beta}$ as subspaces of $\mathrm{H}^{\alpha+\beta}$. Given $k \in \mathbb{N}$, this leads to the natural isomorphism (see [52], Chapter 11):

$$\mathrm{H}^k \simeq \bigoplus_{\substack{(\alpha,\beta)\in\mathbb{N}^2 \\ \alpha+\beta=k}} \mathrm{H}^{\alpha,\beta}\,. \tag{4.3}$$

We denote by $\mathcal{Y}^{\alpha,\beta}$ the space of restrictions of elements of $\mathrm{H}^{\alpha,\beta}$ to the unit sphere S^{4N-1}. From (4.1) and (4.3) we deduce:

$$L^2\left(S^{2N-1}\right) = \widehat{\bigoplus_{(\alpha,\beta)\in\mathbb{N}^2}} \mathcal{Y}^{\alpha,\beta}\,.$$

Consequently, in the compact picture:

$$V_{i\lambda,\delta} = \widehat{\bigoplus_{\substack{(\alpha,\beta)\in\mathbb{N}^2 \\ \delta=\beta-\alpha}}} \mathcal{Y}^{\alpha,\beta} . \tag{4.4}$$

We now discuss a basic yet important point: we can work with homogeneous harmonic polynomials rather than spherical harmonics. In order to explain why, let us fix any non-negative integers α and β and denote by $\mathcal{R}$ the map that restricts polynomials of $\mathrm{H}^{\alpha,\beta}$ to the unit sphere S^{2N-1}; the target space of $\mathcal{R}$ is precisely $\mathcal{Y}^{\alpha,\beta}$. Because of homogeneity, the map $\mathcal{R}$ is a bijection; it is in fact a linear isomorphism. Let us:

- denote by π the representation of K that corresponds to the left action on $\mathcal{Y}^{\alpha,\beta}$.
- denote by π' the representation of K that corresponds to the left action on $\mathrm{H}^{\alpha,\beta}$.

The restriction map $\mathcal{R}$ commutes with both left actions of K. Said otherwise, the following diagram commutes (with $k \in K$):

$$\begin{tikzcd} \mathrm{H}^{\alpha,\beta} \arrow[r, "\pi'(k)"] \arrow[d, "\mathcal{R}"] & \mathrm{H}^{\alpha,\beta} \arrow[d, "\mathcal{R}"] \\ \mathcal{Y}^{\alpha,\beta} \arrow[r, "\pi(k)"'] & \mathcal{Y}^{\alpha,\beta} . \end{tikzcd}$$

Because the spaces $\mathrm{H}^{\alpha,\beta}$ and $\mathcal{Y}^{\alpha,\beta}$ are finite-dimensional, π and π' are differentiable. The differentials (at the identity) $d\pi$ and $d\pi'$ are also related by a commuting diagram (with $X \in \mathfrak{k}$):

$$\begin{tikzcd} \mathrm{H}^{\alpha,\beta} \arrow[r, "d\pi'(X)"] \arrow[d, "\mathcal{R}"] & \mathrm{H}^{\alpha,\beta} \arrow[d, "\mathcal{R}"] \\ \mathcal{Y}^{\alpha,\beta} \arrow[r, "d\pi(X)"'] & \mathcal{Y}^{\alpha,\beta} . \end{tikzcd}$$

In other words, the linear isomorphism $\mathcal{R}$ intertwines $d\pi$ and $d\pi'$. This implies that weights of π coincide with weights of π', weight vectors of

π correspond to weight vectors of π' and highest weight vectors of π correspond to highest weight vectors of π'.

This is why we now identify π and π', going back to our notation L and denoting by dL the differential of L (at the identity).

Remark: we will often identify, without mentioning it, polynomials of $\mathrm{H}^{\alpha,\beta}$ and the corresponding spherical harmonics of $\mathcal{Y}^{\alpha,\beta}$.

We now work with polynomials, looking for highest weight vectors and the corresponding highest weights.

Given $(\alpha,\beta)\in\mathbb{N}^2$, the space $\mathrm{H}^{\alpha,\beta}$ is stable under the action of K, but not necessarily irreducible: it can break up into (obviously) finite-dimensional irreducible invariant subspaces. Each of these subspaces is generated by some highest weight vector under the action of K. Also, because these subspaces are finite-dimensional, the restriction of L to any one of them can be differentiated. Remembering (4.2), if X belongs to the Lie algebra $\mathfrak{k}$ of K, if P belongs to the carrying space of L and if z belongs to $\mathbb{C}^N$, then by definition:

$$dL(X)(P)(z,\bar z)=\frac{d}{dt}\Big|_{t=0}\Big(P\left(\exp(-tX)z,{}^t\big(\exp(tX)\big)\bar z\right)\Big).$$

The chain rule of differentiation applied to functions of real variables and then rewritten in terms of complex variables gives

$$dL(X)(P)(z,\bar z)=\left(\begin{array}{c|c}\frac{\partial P}{\partial z}(z,\bar z) & \frac{\partial P}{\partial \bar z}(z,\bar z)\end{array}\right)\left(\begin{array}{c}\frac{d}{dt}\Big|_{t=0}\Big(\exp(-tX)z\Big)\\ \frac{d}{dt}\Big|_{t=0}\Big({}^t\big(\exp(tX)\big)\bar z\Big)\end{array}\right)$$

and so we have

$$dL(X)(P)(z,\bar z)=\left(\begin{array}{c|c}\frac{\partial P}{\partial z}(z,\bar z) & \frac{\partial P}{\partial \bar z}(z,\bar z)\end{array}\right)\left(\begin{array}{c}-Xz\\ {}^tX\bar z\end{array}\right). \qquad (4.5)$$

The infinitesimal action dL can be complexified, in other words extended in the natural way to the complexification $\mathfrak{g}=\mathfrak{sp}(n,\mathbb{C})$ of $\mathfrak{k}=\mathfrak{sp}(n)$. This complexification of dL is also defined by Formula (4.5), but this time with $X\in\mathfrak{g}$.

4.1.2 Highest weight vectors

We fix $k \in \mathbb{N}$ and $(\alpha, \beta) \in \mathbb{N}^2$ such that $\alpha + \beta = k$.

To make polynomial calculations clearer, we change our system of notation for complex variables: we write $(z, w) = (z_1, ..., z_n, w_1, ..., w_n)$ instead of our initial $z = (z_1, ..., z_N)$, with of course $z = (z_1, ..., z_n) \in \mathbb{C}^n$ and $w = (w_1, ..., w_n) \in \mathbb{C}^n$. We now state and prove a theorem which is the heart of this chapter:

Theorem 4.2. *Denote by $I^{\alpha,\beta}$ the set of integers γ such that*

$$0 \leq \gamma \leq \min(\alpha, \beta).$$

For $\gamma \in I^{\alpha,\beta}$, consider the polynomial $P_\gamma^{\alpha,\beta}$ defined by:

$$P_\gamma^{\alpha,\beta}(z, w, \bar{z}, \bar{w}) = w_1^{\alpha-\gamma} \bar{z_1}^{\beta-\gamma} (w_2 \bar{z_1} - w_1 \bar{z_2})^\gamma.$$

1. *$P_\gamma^{\alpha,\beta}$ belongs to $\mathrm{H}^{\alpha,\beta}$.*

2. *For any element H of the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$:*

   $$dL(H)P_\gamma^{\alpha,\beta} = \Big((\alpha + \beta - \gamma)L_1 + \gamma L_2\Big)(H)P_\gamma^{\alpha,\beta}.$$

   This means that $P_\gamma^{\alpha,\beta}$ is a weight vector associated to the weight

   $$\sigma_\gamma^{\alpha,\beta} = (\alpha + \beta - \gamma, \gamma, 0, \dots, 0) = (k - \gamma, \gamma, 0, \dots, 0).$$

3. *If i and j denote positive integers, if X denotes $E_{i,j} - E_{n+j,n+i}$ or $E_{i,n+j} + E_{j,n+i}$ when $1 \leq i < j \leq n$, or $E_{i,n+i}$ when $1 \leq i \leq n$, then:*

   $$dL(X)P_\gamma^{\alpha,\beta} = 0.$$

   This means that the highest weight condition is satisfied.

In other words, $P_\gamma^{\alpha,\beta}$ is a highest weight vector of L. Under the action of L, $P_\gamma^{\alpha,\beta}$ generates an irreducible invariant subspace $V_\gamma^{\alpha,\beta}$ of $\mathrm{H}^{\alpha,\beta}$. We will call $V_\gamma^{\alpha,\beta}$ a <u>component</u> of L.

Proof:

For simplicity, let us just write P instead of $P_\gamma^{\alpha,\beta}$.

1. Multiplying out the brackets of P gives a list of monomials that are all homogeneous of degree γ in both w and $\bar{z}$. Combining this with the powers of the terms w_1 and $\bar{z_1}$ that sit before the brackets and the fact that no variable appears at the same time as its conjugate, we see that P belongs to $\mathrm{H}^{\alpha,\beta}(\mathbb{C}^N)$.

2. and 3. We use Formula (4.5) for the specific elements of $\mathfrak{g}$ we need to consider:

- if H belongs to $\mathfrak{h}$ and if $h_1, ..., h_n, -h_1, ..., -h_n$ denote the complex diagonal terms of H:

$$dL(H)P(z,w,\bar{z},\bar{w}) = \sum_{i=1}^{n} h_i \left(-z_i \frac{\partial P}{\partial z_i} + w_i \frac{\partial P}{\partial w_i} + \bar{z_i} \frac{\partial P}{\partial \bar{z_i}} - \bar{w_i} \frac{\partial P}{\partial \bar{w_i}} \right).$$

So:

$$dL(H)P(z,w,\bar{z},\bar{w}) = h_1 w_1 \frac{\partial P}{\partial w_1} + h_2 w_2 \frac{\partial P}{\partial w_2} + h_1 \bar{z_1} \frac{\partial P}{\partial \bar{z_1}} + h_2 \bar{z_2} \frac{\partial P}{\partial \bar{z_2}}. \quad (4.6)$$

- if $X = E_{i,j} - E_{n+j,n+i}$ $(1 \leq i < j \leq n)$:

$$dL(X)P(z,w,\bar{z},\bar{w}) = -z_j \frac{\partial P}{\partial z_i} + w_i \frac{\partial P}{\partial w_j} + \bar{z_i} \frac{\partial P}{\partial \bar{z_j}} - \bar{w_j} \frac{\partial P}{\partial \bar{w_i}}.$$

So:

  - for $j \geq 3$:

$$dL(X)P = 0.$$

  - for $i = 1$ and $j = 2$ (only remaining case):

$$dL(X)P(z,w,\bar{z},\bar{w}) = w_1 \frac{\partial P}{\partial w_2} + \bar{z}_1 \frac{\partial P}{\partial \bar{z}_2}. \quad (4.7)$$

- if $X = E_{i,n+j} + E_{j,n+i}$ $(1 \leq i < j \leq n)$:

$$dL(X)P(z,w,\bar{z},\bar{w}) = -w_j \frac{\partial P}{\partial z_i} - w_i \frac{\partial P}{\partial z_j} + \bar{z}_j \frac{\partial P}{\partial \bar{w}_i} + \bar{z}_i \frac{\partial P}{\partial \bar{w}_j}.$$

  So:

$$dL(X)P = 0.$$

- If $X = E_{i,n+i}$ $(1 \leq i \leq n)$:

$$dL(X)P(z,w,\bar{z},\bar{w}) = -w_i \frac{\partial P}{\partial z_i} + \bar{z}_i \frac{\partial P}{\partial \bar{w}_i}.$$

  So:

$$dL(X)P = 0.$$

All that remains to be done is compute the partial derivatives in formulas (4.6) and (4.7) to find:

- (i) $\forall\ H \in \mathfrak{h}$: $dL(H)P = \big((\alpha + \beta - \gamma)h_1 + \gamma h_2\big)P$.
- (ii) $dL(E_{1,2} - E_{n+2,n+1})P = 0$.

To make computations easier we set $B = w_2 \bar{z_1} - w_1 \bar{z_2}$.

To establish (i), let us compute $dL(H)P(z,w,\bar{z},\bar{w})$:

$$\begin{aligned} h_1 w_1 \left((\alpha - \gamma) w_1^{\alpha-\gamma-1} \bar{z_1}^{\beta-\gamma} B^{\gamma} - \gamma \bar{z_2} w_1^{\alpha-\gamma} \bar{z_1}^{\beta-\gamma} B^{\gamma-1}\right) + \\ h_2 w_2 \left(\gamma \bar{z_1} w_1^{\alpha-\gamma} \bar{z_1}^{\beta-\gamma} B^{\gamma-1}\right) + \\ h_1 \bar{z_1} \left((\beta - \gamma) w_1^{\alpha-\gamma} \bar{z_1}^{\beta-\gamma-1} B^{\gamma} + \gamma w_2 w_1^{\alpha-\gamma} \bar{z_1}^{\beta-\gamma} B^{\gamma-1}\right) + \\ h_2 \bar{z_2} \left(-\gamma w_1 w_1^{\alpha-\gamma} \bar{z_1}^{\beta-\gamma} B^{\gamma-1}\right). \end{aligned} \tag{4.8}$$

Then we can organise the terms of (4.8) to get

$$\begin{aligned} B^{\gamma} \left(h_1(\alpha - \gamma) w_1^{\alpha-\gamma} \bar{z_1}^{\beta-\gamma} + h_1(\beta - \gamma) w_1^{\alpha-\gamma} \bar{z_1}^{\beta-\gamma}\right) + \\ B^{\gamma-1} h_1 \gamma w_1^{\alpha-\gamma} \bar{z_1}^{\beta-\gamma} \left(w_2 \bar{z_1} - w_1 \bar{z_2}\right) + \\ B^{\gamma-1} h_2 \gamma w_1^{\alpha-\gamma} \bar{z_1}^{\beta-\gamma} \left(w_2 \bar{z_1} - w_1 \bar{z_2}\right), \end{aligned} \tag{4.9}$$

which can be rewritten

$$((\alpha+\beta-\gamma)h_1+\gamma h_2)w_1^{\alpha-\gamma}\bar{z_1}^{\beta-\gamma}B^{\gamma}. \tag{4.10}$$

We finally recognise the desired expression: $((\alpha+\beta-\gamma)h_1+\gamma h_2)P$.

To establish (ii), let us compute $dL(E_{1,2}-E_{n+2,n+1})P(z,w,\bar{z},\bar{w})$:

$$\left(w_1\frac{\partial P}{\partial w_2}+\bar{z_1}\frac{\partial P}{\partial \bar{z_2}}\right)(z,w,\bar{z},\bar{w}) = \\ w_1\gamma\bar{z_1}w_1^{\alpha-\gamma}\bar{z_1}^{\beta-\gamma}B^{\gamma-1}-\bar{z_1}\gamma w_1 w_1^{\alpha-\gamma}\bar{z_1}^{\beta-\gamma}B^{\gamma-1},$$

which obviously equals 0.

End of proof.

Each component corresponds to a specific eigenvalue of the Casimir operator Ω_L of L. Proposition 2.6 and Theorem 4.2 imply:

Corollary 4.3. *Consider the irreducible representation $(L,V_\gamma^{\alpha,\beta})$. We remind the reader that its highest weight is $(\alpha+\beta-\gamma)L_1+\gamma L_2$. Then (denoting by Id the identity map):*

$$\Omega_L=-\Big((\alpha+\beta-\gamma)^2+2n(\alpha+\beta)+\gamma^2-2\gamma\Big)\,Id.$$

4.1.3 Isotypic decomposition with respect to $\mathrm{Sp}(n)$

Again we fix $k\in\mathbb{N}$ and $(\alpha,\beta)\in\mathbb{N}^2$ such that $\alpha+\beta=k$. We now want to establish that $\mathrm{H}^{\alpha,\beta}$ is the direct sum of all subspaces $V_\gamma^{\alpha,\beta}$. We do this by computing dimensions and showing that the dimensions of the various $V_\gamma^{\alpha,\beta}$ add up to the dimension of $\mathrm{H}^{\alpha,\beta}$. We point out that in the formulas to come, we use the standard convention $0!=1$.

Proposition 4.4. *The dimension of $V_\gamma^{\alpha,\beta}$ is given by the following formula:*

$$\mathrm{dim}V_\gamma^{\alpha,\beta}=\frac{(k-\gamma+N-2)!\,(\gamma+N-3)!\,(k-2\gamma+1)\,(k+N-1)}{(k-\gamma+1)!\,\gamma!\,(N-1)!\,(N-3)!}.$$

We denote this dimension d_γ^k (omitting the values of α and β).

Proof:

We apply Weyl's dimension formula (2.4), choosing the highest weight $\sigma = \sigma_\gamma^{\alpha,\beta}$ (given by Theorem 4.2), $\rho_K = (n, n-1, ..., 1)$ and using the three types of positive roots $L_i - L_j$, $L_i + L_j$ and $2L_i$ introduced in Section 2.3.1 of Chapter 2. With these choices, $\dim V_\gamma^{\alpha,\beta}$ is equal to:

$$\prod_{i<j} \frac{<\sigma + \rho_K, L_i - L_j>}{<\rho_K, L_i - L_j>} \prod_{i<j} \frac{<\sigma + \rho_K, L_i + L_j>}{<\rho_K, L_i + L_j>} \prod_{i=1}^{n} \frac{<\sigma + \rho_K, 2L_i>}{<\rho_K, 2L_i>}. \quad (4.11)$$

Looking into each product, we see that the terms for $i \geq 3$ all cancel out so that (4.11) can be rewritten as:

$$\prod_{j=2}^{n} \frac{<\sigma + \rho_K, L_1 - L_j>}{<\rho_K, L_1 - L_j>} \prod_{j=3}^{n} \frac{<\sigma + \rho_K, L_2 - L_j>}{<\rho_K, L_2 - L_j>} \prod_{j=2}^{n} \frac{<\sigma + \rho_K, L_1 + L_j>}{<\rho_K, L_1 + L_j>} \prod_{j=3}^{n} \frac{<\sigma + \rho_K, L_2 + L_j>}{<\rho_K, L_2 + L_j>} \frac{<\sigma + \rho_K, 2L_1>}{<\rho_K, 2L_1>} \frac{<\sigma + \rho_K, 2L_2>}{<\rho_K, 2L_2>}. \quad (4.12)$$

With $\sigma + \rho_K = (k - \gamma + n, \gamma + n - 1, n - 2, ..., 1)$, (4.12) becomes:

$$\frac{(k-\gamma+n)-(\gamma+n-1)}{n-(n-1)} \prod_{j=3}^{n} \frac{(k-\gamma+n)-(n-j+1)}{n-(n-j+1)} \frac{(k-\gamma+n)+(\gamma+n-1)}{n+(n-1)} \prod_{j=3}^{n} \frac{(\gamma+n-1)-(n-j+1)}{(n-1)-(n-j+1)} \prod_{j=3}^{n} \frac{(k-\gamma+n)+(n-j+1)}{n+(n-j+1)} \prod_{j=3}^{n} \frac{(\gamma+n-1)+(n-j+1)}{(n-1)+(n-j+1)} \frac{2(k-\gamma+n)}{2n} \frac{2(\gamma+n-1)}{2(n-1)}. \quad (4.13)$$

Formula (4.13) can be rewritten as:

$$\frac{k-2\gamma+1}{1}\prod_{j=3}^{n}\frac{k-\gamma+j-1}{j-1}\prod_{j=3}^{n}\frac{\gamma+j-2}{j-2}$$
$$\frac{k+N-1}{N-1}\prod_{j=3}^{n}\frac{k-\gamma+N-j+1}{N-j+1}\prod_{j=3}^{n}\frac{\gamma+N-j}{N-j}$$
$$\frac{k-\gamma+n}{n}\frac{\gamma+n-1}{n-1}. \quad (4.14)$$

Formula (4.14) can be reorganised as:

$$\prod_{j=2}^{n}\frac{k-\gamma+j}{j}\prod_{j=1}^{n-1}\frac{\gamma+j}{j}$$
$$\prod_{j=3}^{n}\frac{k-\gamma+N+1-j}{N+1-j}\prod_{j=3}^{n}\frac{\gamma+N-j}{N-j}$$
$$\frac{(k-2\gamma+1)(k+N-1)}{(N-1)}. \quad (4.15)$$

Writing products in terms of factorials we get the following expression:

$$\frac{(k-\gamma+n)!}{(k-\gamma+1)!n!}\frac{(\gamma+n-1)!}{\gamma!(n-1)!}$$
$$\frac{(k-\gamma+N+1-3)!(N+1-n-1)!}{(k-\gamma+N+1-n-1)!(N+1-3)!}\frac{(\gamma+N-3)!(N-n-1)!}{(\gamma+N-n-1)!(N-3)!}$$
$$\frac{(k-2\gamma+1)(k+N-1)}{(N-1)}. \quad (4.16)$$

Formula (4.16) becomes:

$$\frac{(k-\gamma+n)!}{(k-\gamma+1)!n!}\frac{(\gamma+n-1)!}{\gamma!(n-1)!}$$
$$\frac{(k-\gamma+N-2)!n!}{(k-\gamma+n)!(N-2)!}\frac{(\gamma+N-3)!(n-1)!}{(\gamma+n-1)!(N-3)!}$$
$$\frac{(k-2\gamma+1)(k+N-1)}{(N-1)}.$$

One finishes the proof by noticing various cancellations.

End of proof.

Proposition 4.5. *The dimension of* $\mathrm{H}^{\alpha,\beta}$ *can be written as the following sum:*

$$\dim \mathrm{H}^{\alpha,\beta} = \sum_{\gamma \in I_\gamma^{\alpha,\beta}} \dim V_\gamma^{\alpha,\beta}.$$

Proof:

We start by supposing $\alpha \leq \beta$, so that $\gamma \leq \alpha$. Using Propositions 4.1 and 4.4 and several cancelations, we just need to prove this equality:

$$\sum_{\gamma=0}^{\alpha} \frac{(k-\gamma+N-2)!(\gamma+N-3)!(k-2\gamma+1)}{(k-\gamma+1)!\gamma!} = \frac{(\alpha+N-2)!(\beta+N-2)!}{(N-2)\alpha!\beta!}. \quad (4.17)$$

If $\beta \leq \alpha$, so that $\gamma \leq \beta$, we need to prove:

$$\sum_{\gamma=0}^{\beta} \frac{(k-\gamma+N-2)!(\gamma+N-3)!(k-2\gamma+1)}{(k-\gamma+1)!\gamma!} = \frac{(\alpha+N-2)!(\beta+N-2)!}{(N-2)\alpha!\beta!}.$$

These two formulas are in fact equivalent, because of the symmetrical roles of α and β. So one just has to prove Formula (4.17); in this formula, let us set $\mu = k+1$ and $A = N-3$, which changes Formula (4.17) into

$$\sum_{\gamma=0}^{\alpha} \frac{(A+\mu-\gamma)!(A+\gamma)!(\mu-2\gamma)}{(\mu-\gamma)!\gamma!} = \frac{(\alpha+A+1)!(\mu-\alpha+A)!}{(A+1)\alpha!(\mu-\alpha-1)!}. \quad (4.18)$$

Forgetting about the exact expression of μ, we shall prove Formula (4.18) by induction on α, proving the following statement:

$$\forall \alpha \in \mathbb{N},\ \forall \mu \geq 2\alpha+1 : \text{ Formula (4.18) is true.} \quad (4.19)$$

One easily sees that Statement (4.19) is true for $\alpha = 0$.

Let us now suppose that it is true for given $\alpha \in \mathbb{N}$ and consider $\alpha + 1$ and some $\mu \geq 2(\alpha + 1) + 1$. Then

$$\sum_{\gamma=0}^{\alpha+1} \frac{(A + \mu - \gamma)!(A + \gamma)!(\mu - 2\gamma)}{(\mu - \gamma)!\gamma!}$$

separates into

$$\left(\sum_{\gamma=0}^{\alpha} \frac{(A + \mu - \gamma)!(A + \gamma)!(\mu - 2\gamma)}{(\mu - \gamma)!\gamma!} \right) + \frac{(A + \mu - \alpha - 1)!(A + \alpha + 1)!(\mu - 2\alpha - 2)}{(\mu - \alpha - 1)!(\alpha + 1)!}.$$

Because $\mu \geq 2(\alpha + 1) + 1 \geq 2\alpha + 1$, we can apply the induction hypothesis to the sum and obtain:

$$\frac{(\alpha + A + 1)!(\mu - \alpha + A)!}{(A + 1)\alpha!(\mu - \alpha - 1)!} + \frac{(A + \mu - \alpha - 1)!(A + \alpha + 1)!(\mu - 2\alpha - 2)}{(\mu - \alpha - 1)!(\alpha + 1)!} = \frac{(\alpha + A + 1)!(\mu - \alpha + A - 1)!}{(\mu - \alpha - 1)!\alpha!} \left(\frac{\mu - \alpha + A}{A + 1} + \frac{\mu - 2\alpha - 2}{\alpha + 1} \right). \quad (4.20)$$

One can check that the brackets can be written as

$$\frac{(\mu - \alpha - 1)(\alpha + A + 2)}{(A + 1)(\alpha + 1)}$$

so that (4.20) becomes:

$$\frac{(\alpha + A + 1)!(\mu - \alpha + A - 1)!}{(\mu - \alpha - 1)!\alpha!} \frac{(\mu - \alpha - 1)(\alpha + A + 2)}{(A + 1)(\alpha + 1)} = \frac{(\alpha + A + 2)!(A + \mu - \alpha - 1)!(\mu - \alpha - 1)}{(A + 1)(\mu - \alpha - 1)!(\alpha + 1)!} = \frac{((\alpha + 1) + A + 1)!(\mu - (\alpha + 1) + A)!}{(A + 1)(\alpha + 1)!(\mu - (\alpha + 1) - 1)!}.$$

This finishes the induction step and thus the proof.

End of proof.

The isotypic decomposition of the restriction of L to $\mathrm{H}^{\alpha,\beta}$ follows from:

Theorem 4.6. $\mathrm{H}^{\alpha,\beta} = \bigoplus_{\gamma \in I^{\alpha,\beta}} V_\gamma^{\alpha,\beta}$.

Proof:

The Peter-Weyl theorem 2.2 implies that two inequivalent subrepresentations of a unitary representation of a compact Lie group must act on orthogonal subspaces. Therefore the spaces $V_\gamma^{\alpha,\beta}$ are pairwise orthogonal (the corresponding subrepresentations each have a different highest weight, so they are pairwise inequivalent). Proposition 4.5 says that the dimensions of all the $V_\gamma^{\alpha,\beta}$ add up to the dimension of $\mathrm{H}^{\alpha,\beta}$; Theorem 4.6 follows.

End of proof.

To summarise the whole of Section 4.1, putting it back into context with regards to our induced representations $\pi_{i\lambda,\delta}$, we have proved:

Theorem 4.7 (Isotypic decomposition of $\pi_{i\lambda,\delta}$)**.** *Consider any* $\lambda \in \mathbb{R}$ *and* $\delta \in \mathbb{Z}$. *Then:*

$$\pi_{i\lambda,\delta}\big|_K \;\cong\; \sum_{\substack{(\alpha,\beta)\in\mathbb{N}^2 \\ \delta=\beta-\alpha \\ \gamma\in I^{\alpha,\beta}}}^{\oplus} L\big|_{V_\gamma^{\alpha,\beta}}.$$

In this isotypic decomposition, the multiplicity of each K*-type is* 1 *(one says that the* K*-types are multiplicity free).*

4.2 Right action of Sp(1)

4.2.1 Isotypic decomposition with respect to Sp(1)

Denote by

$$(z, w) = (z_1, \ldots, z_n, w_1, \ldots, w_n)$$

the coordinates of $\mathbb{C}^N$, by $h = (h_1, \dots, h_n)$ the coordinates of $\mathbb{H}^n$ and use the identification $h = (z + jw) \in \mathbb{H}^n \longleftrightarrow (z, w) \in \mathbb{C}^N$.

Right multiplication of a quaternionic vector h by a quaternion q consists simply in multiplying all quaternionic coordinates by q on the right, obtaining the quaternionic vector hq.

Consider any subset S of $\mathbb{H}^n$ and assume that it is stable under right multiplication by unit quaternions. Let $\mathcal{F}$ be any subset of the complex vector space $\{f : S \longrightarrow \mathbb{C}\}$. Then the *right action* R of $\mathrm{U}_{\mathbb{H}}$ on $\mathcal{F}$ is defined by:

$$R(q)f(x) = f(xq),$$

for all $(q, f, x) \in \mathrm{U}_{\mathbb{H}} \times \mathcal{F} \times S$; we write $R(q)f$ instead of $\big(R(q)\big)(f)$.

Consider a unit quaternion $q = a + jb \in \mathrm{U}_{\mathbb{H}}$. The rules of quaternionic multiplication imply (as explained in Section 2.4.3):

$$\forall h = z + jw \in \mathbb{H}^n,\ hq = (az - b\overline{w}) + j(aw + b\overline{z}). \tag{4.21}$$

The subset S of $\mathbb{H}^n$ identifies with a subset of $\mathbb{C}^N$, that we also denote by S. As explained in Chapter 2 (section 2.4.3) the quaternion q (seen as a 1×1 matrix) identifies with the matrix

$$\begin{pmatrix} a & -\overline{b} \\ b & \overline{a} \end{pmatrix} \in \mathrm{Sp}(1)$$

that we also denote by q. These considerations explain our choice of action of $\mathrm{Sp}(1)$ on functions of the complex variables (z, w):

Definition 4.8. *The right action of* $\mathrm{Sp}(1)$ *on* $\mathcal{F}$*, again denoted by* R*, is defined for all* $q \in \mathrm{Sp}(1)$ *and* $(z, w) \in S$ *by:*

$$R(q)f(z, w) = f\left(az - b\overline{w}, aw + b\overline{z}\right).$$

An element f of $\mathcal{F}$ is *right-invariant* if it is invariant under the right action of $\mathrm{Sp}(1)$, meaning that for all $q \in \mathrm{Sp}(1)$: $R(q)f = f$.

Let us take $S = S^{2N-1}$ and come back to $L^2\left(S^{2N-1}\right)$. One can show:

Proposition 4.9. *The right action R of* $\mathrm{Sp}(1)$ *on* $L^2\left(S^{2N-1}\right)$ *is a continuous (it satisfies the continuity property stated in Section 2.1) and unitary representation.*

Let us fix $k \in \mathbb{N}$. Using the usual identification $\mathbb{C}^N \simeq \mathbb{R}^{2N}$, denote by $\mathcal{F}^k$ the subspace of $L^2\left(S^{2N-1}\right)$ that consists of functions on S^{2N-1} (one should really speak of their equivalence classes) that are restrictions of homogeneous polynomial functions on $\mathbb{R}^{2N}$ of degree k; $\mathcal{F}^k$ is stable under the right action R of $\mathrm{Sp}(1)$. Because $\mathcal{F}^k$ is finite-dimensional, all its elements are smooth vectors of R. As we have done previously, homogeneity enables us to identify elements of $\mathcal{F}^k$ with the homogeneous polynomials they come from. The right action of $\mathrm{Sp}(1)$ is compatible with these identifications. We also denote by R the right action of $\mathrm{Sp}(1)$ on homogeneous polynomials of degree k; they are all smooth vectors of R.

Ultimately, our aim is to focus on the subspace $\mathcal{Y}^k$ of $\mathcal{F}^k$ and the corresponding space of harmonic polynomials H^k; these spaces are invariant under R, as we shall see.

As we have already mentioned, to study irreducible invariant subspaces under the action of a compact group, one complexifies its Lie algebra. The complexification of $\mathfrak{sp}(1) = \mathfrak{su}(2)$ is:

$$\mathfrak{su}(2) \oplus i\,\mathfrak{su}(2) = \mathfrak{sl}(2,\mathbb{C}).$$

A basis over the field $\mathbb{R}$ of $\mathfrak{su}(2)$ is given by the three matrices :

$$A = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad ; \quad B = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad ; \quad C = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

A basis over the field $\mathbb{C}$ of $\mathfrak{sl}(2,\mathbb{C})$ is given by the three matrices :

- $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 0 + i(-A) \in \mathfrak{su}(2) \oplus i\mathfrak{su}(2);$
- $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \frac{-C}{2} + i\left(\frac{-B}{2}\right) \in \mathfrak{su}(2) \oplus i\mathfrak{su}(2);$

- $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \frac{C}{2} + i\left(\frac{-B}{2}\right) \in \mathfrak{su}(2) \oplus i\mathfrak{su}(2).$

We have for $t \in \mathbb{R}$ the following exponentials:

- $exp(-tA) = \begin{pmatrix} e^{-it} & 0 \\ 0 & e^{it} \end{pmatrix}$;

- $exp\left(\frac{-tB}{2}\right) = \begin{pmatrix} \cos\left(\frac{t}{2}\right) & -i\sin\left(\frac{t}{2}\right) \\ -i\sin\left(\frac{t}{2}\right) & \cos\left(\frac{t}{2}\right) \end{pmatrix}$;

- $exp\left(\frac{-tC}{2}\right) = \begin{pmatrix} \cos\left(\frac{t}{2}\right) & \sin\left(\frac{t}{2}\right) \\ -\sin\left(\frac{t}{2}\right) & \cos\left(\frac{t}{2}\right) \end{pmatrix}$.

These exponentials belong to Sp(1) and therefore correspond to specific unit quaternions $q_t = a_t + jb_t$, with $(a_t, b_t) \in \mathbb{C} \times \mathbb{C}$ (see Lemma 2.11 to recall how one identifies quaternionic matrices with complex matrices):

- $\exp(-tA)$ corresponds to $q_t = e^{-it} + 0j$;
- $\exp\left(\frac{-tB}{2}\right)$ corresponds to $q_t = \cos\left(\frac{t}{2}\right) - j\,i\,\sin\left(\frac{t}{2}\right)$;
- $\exp\left(\frac{-tC}{2}\right)$ corresponds to $q_t = \cos\left(\frac{t}{2}\right) - j\sin\left(\frac{t}{2}\right)$.

Consider any homogeneous polynomial P of degree k. We know that P is a smooth vector of R, so, by definition, we have for all $X \in \mathfrak{sp}(1)$ and $(z, w) \in \mathbb{C}^n \times \mathbb{C}^n$:

$$dR(X)P(z,w) = \left.\frac{d}{dt}\right|_{t=0} \Big(R\left(\exp(tX)\right) P(z,w)\Big).$$

Extend dR to a complex representation $d_{\mathbb{C}}R$ of $\mathfrak{sl}(2,\mathbb{C})$ by defining for all $Z = X + iY \in \mathfrak{sl}(2,\mathbb{C})$ (taking $(X,Y) \in \mathfrak{sp}(1) \times \mathfrak{sp}(1)$):

$$d_{\mathbb{C}}R(Z) = dR(X) + idR(Y).$$

We will write Z instead of $d_{\mathbb{C}}R(Z)$ and $Z \cdot P$ instead of $d_{\mathbb{C}}R(Z)P$ (in other words, we see Z as a differential operator).

Using the formulas that link e, f, h to A, B, C, the exponentials of $-tA$, $\frac{-tB}{2}$, $\frac{-tC}{2}$ and the definitions of dR and $d_{\mathbb{C}}R$, one can prove:

Proposition 4.10. *Simply denote by e, f, g the respective differential operators $d_{\mathbb{C}}R(e)$, $d_{\mathbb{C}}R(f)$, $d_{\mathbb{C}}R(h)$. Then:*

- $e = \sum_{r=1}^{n} \left(w_r \frac{\partial}{\partial \overline{z_r}} - z_r \frac{\partial}{\partial \overline{w_r}} \right)$;
- $f = \sum_{r=1}^{n} \left(\overline{z_r} \frac{\partial}{\partial w_r} - \overline{w_r} \frac{\partial}{\partial z_r} \right)$;
- $h = \sum_{r=1}^{n} \left(z_r \frac{\partial}{\partial z_r} - \overline{z_r} \frac{\partial}{\partial \overline{z_r}} + w_r \frac{\partial}{\partial w_r} - \overline{w_r} \frac{\partial}{\partial \overline{w_r}} \right)$.

Let us now look into the direct sum $\mathrm{H}^k = \bigoplus_{s=0}^{k} \mathrm{H}^{k-s,s}$. We know that each $H^{k-s,s}$ breaks into $\bigoplus_{\gamma=0}^{\min(k-s,s)} V_\gamma^{k-s,s}$ and that each $V_\gamma^{k-s,s}$ contains a highest weight vector denoted by $P_\gamma^{k-s,s}$ (see Theorem 4.2). Using formulas of Proposition 4.10, one shows:

Theorem 4.11. *Consider integers k, α, β, γ such that $0 \leq \gamma \leq \min(\alpha, \beta)$ and $\alpha + \beta = k$. Then:*

- $h \cdot P_\gamma^{\alpha,\beta} = (\alpha - \beta) P_\gamma^{\alpha,\beta}$;
- *When* $\gamma \leq \beta \,:\, e \cdot P_\gamma^{\alpha,\beta} = (\beta - \gamma) P_\gamma^{\alpha+1,\beta-1}$;
- *When* $\gamma \leq \alpha \,:\, f \cdot P_\gamma^{\alpha,\beta} = (\alpha - \gamma) P_\gamma^{\alpha-1,\beta+1}$.

4.2.2 K-type diagram

Figure 4.1 captures the contents of Theorems 4.2 and 4.11. In this diagram, the thick black dots represent the highest weight vectors $P_\gamma^{\alpha,\beta}$ $(\alpha + \beta = k)$.

Let us take a closer look at Figure 4.1. For any fixed integer γ such that $0 \leq \gamma \leq \mathbb{E}\left(\frac{k}{2}\right)$ (the function $\mathbb{E}$ assigns to a real number its integer part):

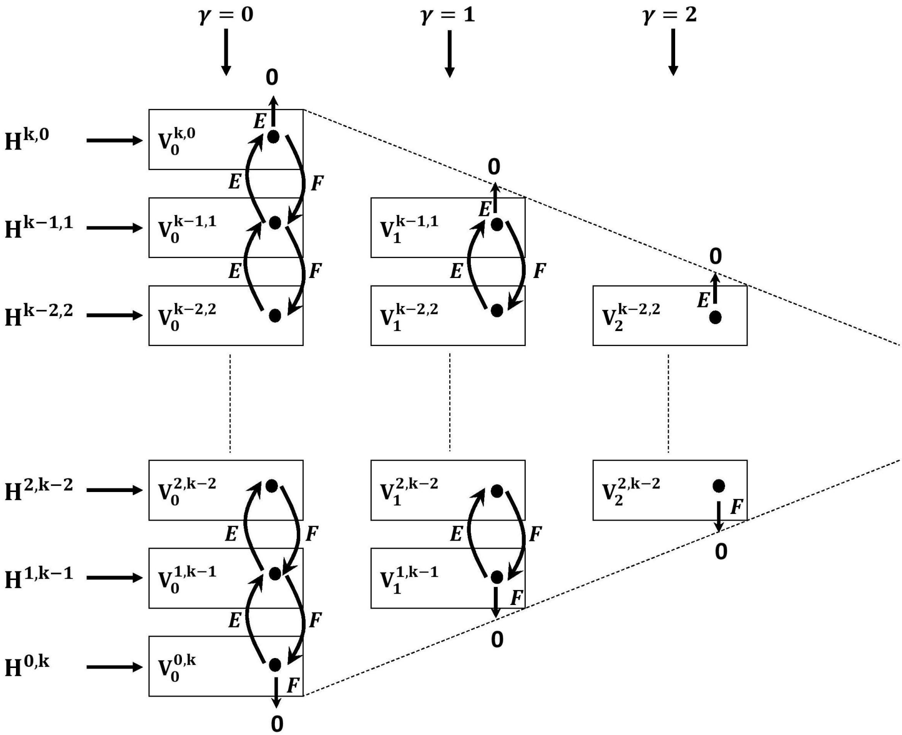

Figure 4.1: combining the left action of $\mathrm{Sp}(n)$ and the right action of $\mathrm{Sp}(1)$ within H^k, assuming here $k \geq 5$.

- The arrow beneath γ points to a **vertical set of components** whose direct sum defines an invariant subspace V_γ^k (obviously not irreducible) under the left action of $\mathrm{Sp}(n)$:

$$V_\gamma^k = \bigoplus_{\alpha=\gamma}^{k-\gamma} V_\gamma^{\alpha,k-\alpha}.$$

- The **components in** V_γ^k all correspond to the **highest weight** $(k-\gamma,\gamma,0,\cdots,0)$ (given by Theorem 4.2) and are therefore equivalent; in particular they all have the **same dimension** d_γ^k.

- A **vertical set of thick black dots** defines a basis of an irreducible invariant subspace W_γ^k under the right action of $\mathrm{Sp}(1)$ (of dimension $k-2\gamma+1$):

$$W_\gamma^k = \mathrm{Vect}_{\mathbb{C}}\{P_\gamma^{\alpha,k-\alpha}\}_{\alpha=\gamma,\ldots,k-\gamma} \subset V_\gamma.$$

  The polynomial $P_\gamma^{k-\gamma,\gamma}$ is a highest weight vector of W_γ^k and corresponds to the highest weight $k-2\gamma$.

- Because the left action of $\mathrm{Sp}(n)$ and the right action of $\mathrm{Sp}(1)$ commute, applying L to W_γ^k gives irreducible invariant subspaces of R contained in V_γ^k; these subspaces define subrepresentations of R which are equivalent to the restriction of R to W_γ^k; therefore, they all have the same dimension $k-2\gamma+1$.

Because the left action L of $\mathrm{Sp}(n)$ is irreducible in each component, we have proved:

Theorem 4.12. *The isotypic of the right action R of* $\mathrm{Sp}(1)$ *on* H^k *is given by:*

$$\mathrm{H}^k = \bigoplus_{\gamma=0}^{\mathbb{E}\left(\frac{k}{2}\right)} d_\gamma^k\, W_\gamma^k.$$

In particular, the multiplicity of $R|_{W_\gamma^k}$ is equal to d_γ^k.

Remark 4.13. *The structure of* H^k*, which we have studied with respect to the left action of* $\mathrm{Sp}(n)$ *and the right action of* $\mathrm{Sp}(1)$*, appears in [22]* *(see Proposition 5.1), where it is expressed as follows in terms of tensor products (using our system of notation):*

$$\mathrm{H}^k\big|_{\mathrm{Sp}(n)\times\mathrm{Sp}(1)} = \sum_{\gamma=0}^{\mathbb{E}\left(\frac{k}{2}\right)} V_\gamma^{k-\gamma,\gamma}\otimes W_\gamma^k.$$

The credit one can give to our work is to explain this structure in a personal and self-contained way.

4.2.3 Bi-invariant polynomials

This result follows from Theorem 4.12:

Corollary 4.14.

1. *Consider $k\in\mathbb{N}$ and $P\in\mathrm{H}^k$. If the subspace $\mathrm{Vect}_{\mathbb{C}}\{P\}$ is stable under the right action R of $\mathrm{Sp}(1)$, then k is even and P belongs to $V_\alpha^{\alpha,\alpha}$, where we set $\alpha=\frac{k}{2}$.*
2. *Consider $k\in\mathbb{N}$, suppose k is even and set $\alpha=\frac{k}{2}$.*
   - *The subspace $\mathrm{Vect}_{\mathbb{C}}\{P_\alpha^{\alpha,\alpha}\}$ is stable under the right action R of $\mathrm{Sp}(1)$; in fact, $P_\alpha^{\alpha,\alpha}$ is right-invariant.*
   - *The component $V_\alpha^{\alpha,\alpha}$ decomposes into d_α^k one-dimensional irreducible subspaces, along which R is just the identity representation. This implies that all elements of $V_\alpha^{\alpha,\alpha}$ are right-invariant.*

Another consequence of the previous section is:

Corollary 4.15. *Consider an element f of $L^2(S^{2N-1})$. Then f is right-invariant if and only if f belongs to the following Hilbert sum:*

$$\widehat{\bigoplus_{\alpha\in\mathbb{N}}} V_\alpha^{\alpha,\alpha}.$$

Let us now consider the subgroup $1\times\mathrm{Sp}(n-1)\subset\mathrm{Sp}(n)$: it consists of block diagonal quaternionic matrices $\begin{pmatrix}1&0\\0&A\end{pmatrix}$, where $A\in\mathrm{Sp}(n-1)$.

Definition 4.16. *Consider $k \in \mathbb{N}$. A polynomial of* H^k *and its corresponding spherical harmonic in* $\mathcal{Y}^k$ *are said to be bi-invariant if:*

- *they are invariant under the left action of* $1 \times \mathrm{Sp}(n-1)$*;*
- *they are also invariant under the right action of* $\mathrm{Sp}(1)$*.*

Theorem 4.17.

1. *Consider any $(\alpha, \beta) \in \mathbb{N}^2$ and any $\gamma \in \mathbb{N}$ such that $\gamma \leq \min(\alpha, \beta)$. Denote by* $\mathrm{Inv}_\gamma^{\alpha,\beta}$ *the complex vector space that consists of all polynomials of $V_\gamma^{\alpha,\beta}$ which are invariant under the left action of* $1 \times \mathrm{Sp}(n-1)$*. Then:*

$$\dim_{\mathbb{C}}(\mathrm{Inv}_\gamma^{\alpha,\beta}) = \alpha + \beta - 2\gamma + 1.$$

   We point out that this dimension is higher than or equal to 1 *and that it equals* 1 *if and only if $\alpha = \beta = \gamma$.*

2. *Consider $k \in \mathbb{N}$.*
   - *If a polynomial $P \in \mathrm{H}^k$ is bi-invariant, then k is even and P belongs to $V_\alpha^{\alpha,\alpha}$, where we set $\alpha = \frac{k}{2}$.*
   - *Assume that k is even and set $\alpha = \frac{k}{2}$. Then* H^k *contains a unique, up to a constant, bi-invariant polynomial (and this polynomial belongs to $V_\alpha^{\alpha,\alpha}$).*

Proof:

Item 1: our proof relies on Želobenko's branching theorem with respect to the pair $(\mathrm{Sp}(n), \mathrm{Sp}(n-1))$, namely Theorem 4 of Chapter XVIII in [54]. This theorem implies that the number of times an irreducible representation of $1 \times \mathrm{Sp}(n-1) \simeq \mathrm{Sp}(n-1)$ with highest weight $c = (c_1, \cdots, c_{n-1})$ occurs in an irreducible representation of $\mathrm{Sp}(n)$ with highest weight $a = (a_1, \cdots, a_n)$ is equal to the number of non-negative integral n-tuples $b = (b_1, \cdots, b_n)$ such that the following two rows of inequalities are satisfied:

$$\begin{array}{cccccccccc} a_1 \geq & b_1 \geq & a_2 \geq & b_2 \geq & a_3 \geq & b_3 \geq & \cdots \geq & a_n \geq & b_n \geq & 0; \\ & b_1 \geq & c_1 \geq & b_2 \geq & c_2 \geq & b_3 \geq & \cdots \geq & c_{n-1} \geq & b_n. & \end{array}$$

Remember that the highest weight of the left action of $\mathrm{Sp}(n)$ on $V_\gamma^{\alpha,\beta}$ is the n-tuple $a = (\alpha+\beta-\gamma, \gamma, 0, \cdots, 0)$.

Also, the highest weight theorem (see for instance [27], Section 7 of Chapter IV) implies that a polynomial $P \in V_\gamma^{\alpha,\beta}$ is invariant under the left action of $1 \times \mathrm{Sp}(n-1)$ if and only if the restriction of the left action of $1 \times \mathrm{Sp}(n-1)$ to the 1-dimensional space $\mathrm{Vect}_{\mathbb{C}}(P)$ defines an irreducible representation of $\mathrm{Sp}(n-1)$ whose highest weight is $c = (0, \cdots, 0)$.

Taking $a = (\alpha+\beta-\gamma, \gamma, 0, \cdots, 0)$ and $c = (0, \cdots, 0)$, the two rows of inequalities become:

$$\begin{array}{ccccccccccccc}
\alpha+\beta-\gamma \geq & b_1 \geq & \gamma \geq & b_2 \geq & 0 = & b_3 = & \cdots = & 0 = & b_n = & 0; \\
& b_1 \geq & 0 = & b_2 = & 0 = & b_3 = & \cdots = & 0 = & b_n. &
\end{array}$$

Thus the only non-zero coordinate of the n-tuple b is b_1 and we are left with counting the number of integers b_1 between $\alpha+\beta-\gamma$ and γ. This number is $\alpha+\beta-2\gamma+1$. Because $\gamma \in [0, \min(\alpha,\beta)]$, this number is higher than or equal to 1 and it equals 1 if and only if $\alpha = \beta = \gamma$.

<u>Item 2</u>: it just follows from Item 1 and Corollary 4.14.

End of proof.

Let us fix any $\alpha \in \mathbb{N}$ and write $k = 2\alpha$. We now set to determine, explicitly, the unique (up to a constant) bi-invariant polynomial of H^k (given by Theorem 4.17).

Define the following set:

$$\mathcal{A}_\alpha = \{(a,b) \in \mathbb{N}^2 \ / \ a+b = \alpha\}.$$

Let us consider the polynomials U, V, P defined by:

- $U = z_1\overline{z_1} + w_1\overline{w_1}$;
- $V = \sum_{r=2}^n (z_r\overline{z_r} + w_r\overline{w_r})$;

- $P = \sum_{A\in\mathcal{A}_\alpha} \nu_A\, U^a\, V^b$, where the various ν_A denote complex scalars (undetermined at this stage).

Polynomial P is obviously:

- homogeneous, of homogeneous degree (α, α);
- bi-invariant.

We want P to be harmonic (in which case P actually belongs to the subspace $V_\alpha^{\alpha,\alpha}$ of $\mathrm{H}^{\alpha,\alpha}$). To ensure that this be indeed the case, we need to apply the complex Laplace operator and make P belong to its kernel. Remember that this operator is:

$$\Delta_{\mathbb{C}} = 4\sum_{r=1}^{n}\left(\frac{\partial^2}{\partial z_r \partial \overline{z_r}} + \frac{\partial^2}{\partial w_r \partial \overline{w_r}}\right).$$

One easily checks that:

$$\Delta_{\mathbb{C}} P(z,w) \;=\; \nu_A(a^2+a)\, U^{a-1}\, V^b \;+\; \nu_A(b^2+(2n-3)b)\, U^a\, V^{b-1}.$$

Given $A \in \mathcal{A}$, let us define:

- $C_1(A) = \nu_A(a^2+a)$ and $T_1(A) \;=\; U^{a-1}\, V^b$;
- $C_2(A) = \nu_A(b^2+(2n-3)b)$ and $T_2(A) \;=\; U^a\, V^{b-1}$.

We have:

$$\Delta_{\mathbb{C}} P(z;w) = 4\sum_{A\in\mathcal{A}} \left(C_1(A)\cdot T_1(A) + C_2(A)\cdot T_2(A)\right). \tag{4.22}$$

By multiplying the brackets out in powers of U and V, we can write down the list of monomials that come from $T_i(A)$, given $A = (a,b) \in \mathcal{A}$ and $i \in \{1,2\}$; let us denote by $\mathcal{M}_i(A)$ the set of such monomials.

Proposition 4.18. *Consider two pairs $A = (a,b)$ and $A' = (a',b')$ such that $A \neq A'$. Then:*

1. $\mathcal{M}_1(A) \cap \mathcal{M}_1(A') = \emptyset$;
2. $\mathcal{M}_2(A) \cap \mathcal{M}_2(A') = \emptyset$;

3. $(\mathcal{M}_1(A) \cap \mathcal{M}_2(A') \neq \emptyset) \Leftrightarrow (A' = (a-1, b+1))$;

4. $(\mathcal{M}_2(A) \cap \mathcal{M}_1(A') \neq \emptyset) \Leftrightarrow (A' = (a+1, b-1))$.

Proof:

The monomials of $\mathcal{M}_i(A)$ (resp. $\mathcal{M}_i(A')$), again with $i \in \{1, 2\}$, split into a first part which is a homegeneous polynomial of the variables $z_1, w_1, \overline{z}_1, \overline{w}_1$ and of homogeneous degree (a, a) (resp. (a', a')) and a second part which is a homegeneous polynomial of the variables $z_2, \ldots, z_n, w_2, \ldots, w_n, \overline{z}_2, \ldots \overline{z}_n, \overline{w}_2, \ldots \overline{w}_n$ and of homogeneous degree (b, b) (resp. (b', b')). For monomials to coincide, their homogeneous degrees must of course coincide, which is enough to establish (1), (2), (3) and (4).

End of proof.

Consider two pairs $A = (a, b) \in \mathcal{A}$ and $A' = (a', b') \in \mathcal{A}$ such that $A \neq A'$.

- If a monomial appears in both $\mathcal{M}_1(A)$ and $\mathcal{M}_2(A')$ (implying $A' = (a-1, b+1)$), then the coefficients it appears with are $C_1(A)$ times some combinatorial coefficient and $C_2(A')$ times the same combinatorial coefficient. Thus we must have:

$$C_1(A) + C_2(A') = 0.$$

- If a monomial appears in both $\mathcal{M}_2(A)$ and $\mathcal{M}_1(A')$ (implying $A' = (a+1, b-1)$), then the coefficients it appears with are $C_2(A)$ times a combinatorial coefficient and $C_1(A')$ times the same combinatorial coefficient. Thus we must have:

$$C_2(A) + C_1(A') = 0. \tag{4.23}$$

Above considerations finally show us how to choose coefficients ν_A in the definition of P so as to ensure that P be harmonic. Figure 4.2 helps understand the following steps (assuming that $\alpha \geq 1$):

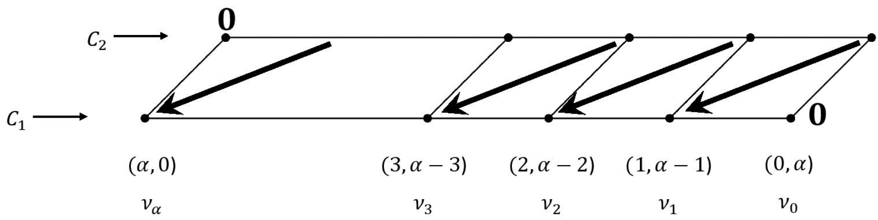

Figure 4.2: induction method to find bi-invariant polynomials.

- List all pairs of $\mathcal{A}_\alpha$ from left to right:

$$(\alpha, 0)\,,\ (\alpha-1, 1)\,,\ \ldots\,,\ (1, \alpha-1)\,,\ (0, \alpha).$$

- Start for instance with the pair $A_0 = (0, \alpha)$ on the far right and assign to it any coefficient $\nu_{A_0} = \nu_0$; it determines $C_1(A_0)$ (which is in fact 0) and $C_2(A_0)$.

- Move on to the pair $A_1 = (1, \alpha-1)$ on the left and compute the only suitable coefficient $\nu_{A_1} = \nu_1$ by using Equation (4.23).

- And so on, until the pair $A_\alpha = (\alpha, 0)$ has been reached and its coefficient $\nu_{A_\alpha} = \nu_\alpha$ computed.

Let us refer to this as the *induction method.*

Theorem 4.19. *Given any even integer $k \in \mathbb{N}$ and setting $\alpha = \frac{k}{2}$, the induction method computes the unique (up to a constant) bi-invariant polynomial of H^k (and this polynomial belongs to $V_\alpha^{\alpha,\alpha}$).*

Examples 4.2.1.

- *For $\alpha = 1$, one finds $P = (1-n)\,U\, +\, V$.*

- *For $\alpha = 2$, one finds $P = \frac{(2n-1)(n-1)}{3}\,U^2\, +\, (1-2n)\,UV\, +\, V^2$.*

Chapter 5

Compact picture and hypergeometric equation

In this chapter, we work with the compact picture of our representations $\pi_{i\lambda,\delta}$, using the setting and results of Chapter 4.

5.1 Quaternionic projective space

In this work, straight lines are defined with respect to right multiplication: given $x \in \mathbb{H}^n$, the *quaternionic line* through x is the vector space

$$x\,\mathbb{H} = \{xh \ / \ h \in \mathbb{H}\}.$$

The *quaternionic projectice space* $\mathrm{P}^{n-1}(\mathbb{H})$ is naturally the set of quaternionic lines of $\mathbb{H}^n$. Quaternionic matrices act naturally on quaternionic lines: given $x \in \mathbb{H}^n$, an invertible matrix $M \in \mathrm{GL}(n, \mathbb{H})$ assigns to $x\,\mathbb{H}$ the quaternionic line through Mx. We call this action the *natural action* of $\mathrm{GL}(n, \mathbb{H})$ on $\mathrm{P}^{n-1}(\mathbb{H})$. We now study the orbits under the restriction of this action to the subgroup $1 \times \mathrm{Sp}(n-1)$. Given $x\,\mathbb{H} \in \mathrm{P}^{n-1}(\mathbb{H})$, we denote by $\mathcal{O}(x\,\mathbb{H})$ the orbit of $x\,\mathbb{H}$. We show in the next proposition that the orbits can be parametrised by a single real variable.

Proposition 5.1. *Consider any $x = (x_1, ..., x_n) \in \mathbb{H}^n$. There exists a unique $\theta \in \left[0, \frac{\pi}{2}\right]$ such that, denoting $x(\theta) = (\cos\theta, 0, ..., 0, \sin\theta) \in \mathbb{H}^n$,* *the quaternionic line $x(\theta)\,\mathbb{H}$ belongs to $\mathcal{O}(x\,\mathbb{H})$. Moreover, θ is explicitly given by the following formulas:*

- *if $x_1 \neq 0$, then $\theta = \arctan(\|x'\|)$, taking $x' = \left(\frac{x_2}{x_1}, ..., \frac{x_n}{x_1}\right)$;*
- *if $x_1 = 0$, then $\theta = \frac{\pi}{2}$.*

Proof:

Existence:

Suppose first that $x_1 \neq 0$. Then, writing $x' = \left(x_2 \frac{1}{x_1}, ..., x_n \frac{1}{x_1}\right)$, we have:

$$x\,\mathbb{H} = (x_1, ..., x_n)\,\mathbb{H} = \left(x_1 \frac{1}{x_1}, x_2 \frac{1}{x_1}, ..., x_n \frac{1}{x_1}\right)\mathbb{H} = (1, x')\mathbb{H}.$$

Because $1 \times \mathrm{Sp}(n-1)$ acts transitively on spheres of $\mathbb{H}^{n-1}$, there exists a matrix $\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$, with $k \in \mathrm{Sp}(n-1)$, that takes $(1, x')\mathbb{H}$ to the quaternionic line $(1, 0, ..., 0, \|x'\|)\,\mathbb{H}$ (remember that $\|\cdot\|$ denotes the norm of $\mathbb{H}^{n-1}$).

This quaternionic line can be rewritten

$$(1, 0, ..., 0, \|x'\|)\,\mathbb{H} = \left(\frac{1}{\sqrt{1+\|x'\|^2}}, 0, ..., 0, \frac{\|x'\|}{\sqrt{1+\|x'\|^2}}\right)\mathbb{H}$$

and then simply

$$(1, 0, ..., 0, \|x'\|)\,\mathbb{H} = (\cos\theta, 0, ..., 0, \sin\theta)\,\mathbb{H}$$

for some $\theta \in \left[0, \frac{\pi}{2}\right[$ (see Figure 5.1).

If now $x_1 = 0$, thus $\|x'\| = 1$, then one can choose k so that $\begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$ carries $x\,\mathbb{H}$ onto $(0, ...0, 1)\,\mathbb{H}$, which gives $\theta = \frac{\pi}{2}$.

This establishes existence and formulas of θ.

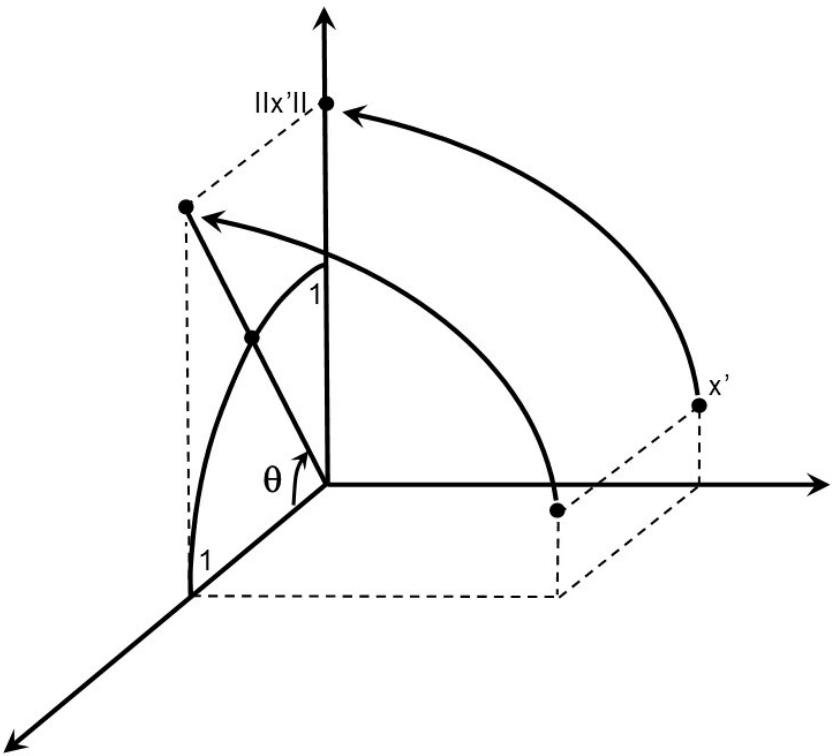

Figure 5.1: a single variable θ to parametrise orbits.

Uniqueness:

Suppose θ and θ' belong to $\left[0, \frac{\pi}{2}\right]$ and that the quaternionic lines $x(\theta)\,\mathbb{H}$ and $x(\theta')\,\mathbb{H}$ belong to $\mathcal{O}(x\,\mathbb{H})$. Then there exists $k \in 1 \times \mathrm{Sp}(n-1)$ such that $kx(\theta')\,\mathbb{H} = x(\theta)\,\mathbb{H}$, which implies that there exists $q \in \mathrm{U}_{\mathbb{H}}$ such that:

$$kx(\theta') = x(\theta)q. \tag{5.1}$$

Suppose that $n \geq 3$ and write

$$k = \begin{pmatrix} 1 & 0 & 0 \\ 0 & A & B \\ 0 & L & b \end{pmatrix}$$

with $b \in \mathbb{H}$, $A \in \mathrm{M}_{n-2,n-2}(\mathbb{H})$, $L \in \mathrm{M}_{1,n-2}(\mathbb{H})$ and $B \in \mathrm{M}_{n-2,1}(\mathbb{H})$. Equation (5.1) becomes:

$$\begin{pmatrix} \cos\theta' \\ B\sin\theta' \\ b\sin\theta' \end{pmatrix} = \begin{pmatrix} q\cos\theta \\ 0 \\ q\sin\theta \end{pmatrix}.$$

We now split the rest of the proof into three cases.

- Case 1: $\theta, \theta' \in \left[0, \frac{\pi}{2}\right[$. Then $q = \frac{\cos\theta'}{\cos\theta}$, $B\sin\theta' = 0$ and

  $$b\sin\theta' = \frac{\sin\theta\cos\theta'}{\cos\theta}. \tag{5.2}$$

  Subcase 1: $\theta' \neq 0$. Then B must be 0 and Proposition 2.14 (see Chapter 2) implies $L = 0$ but more importantly for us $|b| = 1$. Thus from (5.2) it follows that $|\tan\theta| = |\tan\theta'|$, which then implies $\tan\theta = \tan\theta'$ and $\theta = \theta'$.
  Subcase 2: $\theta' = 0$. Then $\sin\theta' = 0$ and $\cos\theta' = 1$, so (5.2) becomes $\tan\theta = 0$ and finally $\theta = 0 = \theta'$.

- Case 2: $\theta = \frac{\pi}{2}$. Then $\cos\theta = 0$ implies $\cos\theta' = 0$ and finally $\theta' = \frac{\pi}{2} = \theta$.

- Case 3: $\theta' = \frac{\pi}{2}$. Then $\cos\theta' = 0$ implies $q\cos\theta = 0$, thus $\cos\theta = 0$ and finally $\theta = \frac{\pi}{2} = \theta'$ (because $q \neq 0$).

There remains the case $n = 2$, for which we choose k as $\begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}$ with $b \in \mathrm{U}_{\mathbb{H}}$. Similarly to what is detailed above, we also obtain $\theta = \theta'$.

End of proof.

5.2 Reducing the number of variables

From now on, throughout the rest of this chapter, the setting is the following:

- We fix an even integer $k \in \mathbb{N}$ and write $\alpha = \frac{k}{2}$.
- We consider the space H^k and its unique, up to a constant, bi-invariant polynomial P, which in fact belongs to $V_\alpha^{\alpha,\alpha} \subset \mathrm{H}^{\alpha,\alpha}$ (see Theorem 4.19).
- We denote by f_P the spherical harmonic that corresponds to P, that is:

$$f_P = P|_{S^{2N-1}} \in \mathcal{Y}^{\alpha,\alpha}.$$

We remind the reader that, by definition, the right action of $\mathrm{Sp}(1)$ corresponds, in the quaternionic setting, to right scalar multiplication by unit quaternions.
Invariance under the right action of $\mathrm{Sp}(1)$ enables f_P to descend to a function f on $\mathrm{P}^{n-1}(\mathbb{H})$:

$$\begin{array}{cccll} f & : & \mathrm{P}^{n-1}(\mathbb{H}) & \longrightarrow & \mathbb{C} \\ & & x\,\mathbb{H} & \mapsto & f_P(x), \text{where } x \text{ is assumed to belong to } S^{2N-1}. \end{array}$$

Transferring, in the way one expects, the left action of K on spherical harmonics to a left action, also denoted by L, of K on complex-valued functions defined on $\mathrm{P}^{n-1}(\mathbb{H})$, we write

$$L(k)f(x\mathbb{H}) = f\big((k^{-1}x)\mathbb{H}\big)$$

for all $(k, x) \in K \times S^{2N-1}$.
Invariance of f_P under the left action of $1 \times \mathrm{Sp}(n-1)$ implies that f is invariant along the orbits of the left action of $1 \times \mathrm{Sp}(n-1)$ on $\mathrm{P}^{n-1}(\mathbb{H})$.

Therefore, f descends to a function on the classifying set $\mathcal{O}$ of orbits, leading in turn to the following new function (via Proposition 5.1):

$$\begin{array}{rccl} F & : & [0, \frac{\pi}{2}] & \longrightarrow & \mathbb{C} \\ & & \theta & \mapsto & F(\theta) = f\left(x(\theta)\mathbb{H}\right) = f_P\left(x(\theta)\right). \end{array}$$

Now, apply Formula (2.6) to write down the Casimir operator Ω_L of L. This is how this operator acts on f_P, given an element $x \in S^{4n-1}$:

$$\Omega_L(f_P)(x) = \textstyle\sum_{X\in\mathcal{B}_{\mathfrak{k}}} \frac{\partial^2}{\partial t^2}\Big|_{t=0}\Big(\pi(\exp(-tX))(f_P)(x)\Big) = \\ \sum_{X\in\mathcal{B}_{\mathfrak{k}}} \frac{\partial^2}{\partial t^2}\Big|_{t=0}\Big(f_P(\exp(-tX)x)\Big),$$

as long as each expression $\frac{\partial^2}{\partial t^2}\Big|_{t=0}\Big(f_P\left(\exp(-tX)x\right)\Big)$ above is well defined, which is the case because f_P is smooth.

Corollary 4.3 says that the component $V_\alpha^{\alpha,\alpha}$ is associated to the eigenvalue $\Lambda = -(2\alpha^2 + (4n-2)\alpha)$ of Ω_L:

$$\Omega_L(f_P)(x) = \Lambda f_P(x).$$

Given $\theta \in \left[0, \frac{\pi}{2}\right]$, one can apply this equation to $x = x(\theta)$ (see Section 5.1 for notation):

$$\sum_{X\in\mathcal{B}_{\mathfrak{k}}} \frac{\partial^2}{\partial t^2}\Big|_{t=0}\Big(f_P\left(\exp(-tX)x(\theta)\right)\Big) = \Lambda f_P\left(x(\theta)\right). \tag{5.3}$$

One can convert this into a differential equation satisfied by the function F of the real variable θ. But this requires to know for each $X \in \mathcal{B}_{\mathfrak{k}}$ and each $t \in \mathbb{R}$, which parameter $\xi_{X,\theta}(t) \in \left[0, \frac{\pi}{2}\right]$ labels the orbit of $\exp(-tX)x(\theta)\mathbb{H}$ under the left action of $1 \times \mathrm{Sp}(n-1)$. Proposition 5.1 gives us the value of $\xi_{X,\theta}(t)$. Indeed, denoting by $(y_1, ..., y_n)$ the coordinates of $\exp(-tX)x(\theta)$:

- when $y_1 \neq 0$, we have $\xi_{X,\theta}(t) = \arctan \|y'\|$, where $y' =$ stands for $\left(\frac{y_2}{y_1}, ..., \frac{y_n}{y_1}\right)$;

- when $y_1 = 0$, we have $\xi_{X,\theta}(t) = \frac{\pi}{2}$.

Then (5.3) can be written:

$$\sum_{X \in \mathcal{B}_{\mathfrak{k}}} \frac{\partial^2}{\partial t^2}\bigg|_{t=0} \Big(F(\xi_{X,\theta}(t)) \Big) = \Lambda F(\theta). \tag{5.4}$$

We point out that in this equation, the expressions that appear in the sum are automatically well defined. Finally, all we have to do now is determine the coordinates of the various $\exp(-tX)x(\theta)$ and use above formulas to compute $\xi_{X,\theta}(t)$; let us also point out that we obviously have:

$$\xi_{X,\theta}(0) = \theta.$$

5.3 Hypergeometric equation

We continue to work with the setting of Section 5.2.

In Equation (5.3), all terms corresponding to elements of the Lie algebra of $1 \times \mathrm{Sp}(1)$ vanish, precisely because f_P is invariant under the left action of $1 \times \mathrm{Sp}(1)$. So the relevant elements of $\mathcal{B}_{\mathfrak{k}}$ are :

$$Z_i = A_1 \ , \ Z_j = D_1 \ , \ Z_{ji} = E_1 \ , \ X_r = \frac{B_{1,r}}{\sqrt{2}} \ ,$$
$$Y_{r,i} = \frac{C_{1,r}}{\sqrt{2}} \ , \ Y_{r,j} = \frac{F_{1,r}}{\sqrt{2}} \ , \ Y_{r,ji} = \frac{G_{1,r}}{\sqrt{2}},$$

where r denotes an integer such that $2 \leq r \leq n$; let us denote by $\mathcal{I}$ the set of these integers.

Exponential formulas of Lemma 2.13 give, taking $r \in \mathcal{I}$, $t \in \mathbb{R}$ and $\eta \in \{i, j, ji\}$:

$$\exp(-tZ_\eta) = \begin{pmatrix} \cos t - \eta \sin t & 0 \\ 0 & I_m \end{pmatrix},$$

$$\exp(-tX_r) = \begin{pmatrix} \cos\tau & \cdots & -\sin\tau & \cdots \\ \vdots & \ddots & \vdots & \\ \sin\tau & \cdots & \cos\tau & \cdots \\ \vdots & & \vdots & \ddots \end{pmatrix},$$

$$\exp(-tY_{r,\eta}) = \begin{pmatrix} \cos\tau & \cdots & -\eta\sin\tau & \cdots \\ \vdots & \ddots & \vdots & \\ -\eta\sin\tau & \cdots & \cos\tau & \cdots \\ \vdots & & \vdots & \ddots \end{pmatrix},$$

where:

- the non-explicited entries are 1 on the diagonal and 0 elsewhere;
- the dots show the diagonal, Rows 1 and r and Columns 1 and r;
- the variable τ denotes $\frac{t}{\sqrt{2}}$.

<u>Dealing first with matrices Z_η:</u>

The following proposition is straightforward:

Proposition 5.2.

1. $\exp(-tZ_\eta)x(\theta) = ((\cos t - \eta\sin t)\cos\theta, 0, ..., 0, \sin\theta)$.
2. *Suppose* $\theta \in \left[0, \frac{\pi}{2}\right[$. *Then:*

$$\xi'_{Z_\eta,\theta}(0) = 0 \quad ; \quad \xi''_{Z_\eta,\theta}(0) = 0.$$

<u>Dealing now with matrices X_r and $Y_{r,\eta}$:</u>

<u>Case 1: $r = n$</u>

This proposition is also straightforward:

Proposition 5.3.

1. $\exp(-tX_n)x(\theta) =$

$$(\cos\tau\cos\theta - \sin\tau\sin\theta, 0, ..., 0, \sin\tau\cos\theta + \cos\tau\sin\theta).$$

2. Suppose $\theta \in \left]0, \frac{\pi}{2}\right[$. *Then for* t *small enough we have*

$$cos\tau\cos\theta - \sin\tau\sin\theta \neq 0$$

and

$$\xi_{X_n,\theta}(t) = \tau + \theta \in \left]0, \frac{\pi}{2}\right[.$$

Also:

$$\xi'_{X_n,\theta}(0) = \frac{1}{\sqrt{2}} \quad ; \quad \xi''_{X_n,\theta}(0) = 0.$$

Proposition 5.4. *Consider* $\eta \in \{i, j, ji\}$.

1. $\exp(-tY_{n,\eta})x(\theta) =$

$$(\cos\tau\cos\theta - \eta\sin\tau\sin\theta, 0, ..., 0, -\eta\sin\tau\cos\theta + \cos\tau\sin\theta).$$

2. Suppose $\theta \in \left]0, \frac{\pi}{2}\right[$. *Then* $\cos\tau\cos\theta - \eta\sin\tau\sin\theta \neq 0$ *and*

$$\xi_{Y_{n,\eta},\theta}(t) = \arctan\sqrt{\frac{\cos^2\tau\sin^2\theta + \sin^2\tau\cos^2\theta}{\cos^2\tau\cos^2\theta + \sin^2\tau\sin^2\theta}} \in \left]0, \frac{\pi}{2}\right[.$$

Also:

$$\xi'_{Y_{n,\eta},\theta}(0) = 0 \quad ; \quad \xi''_{Y_{n,\eta},\theta}(0) = \frac{1}{\tan(2\theta)}.$$

Proof:

A simple matrix multiplication proves Item 1. Let us now look at Item 2. Because $\cos\theta$ and $\sin\theta$ are both non-zero and $\cos\tau$ and $\sin\tau$ cannot be simultaneously equal to 0, one necessarily has

$$cos\tau\cos\theta - \eta\sin\tau\sin\theta \neq 0.$$

Then the square root formula comes from Proposition 5.1 and the fact that, given any $(a, b, \eta) \in \mathbb{R}\times\mathbb{R}\times\{i, j, ji\}$:

$$|a + \eta b| = \sqrt{a^2 + b^2}.$$

The reason why $\xi_{Y_\eta,\theta}(t)$ belongs to the open interval $\left]0, \frac{\pi}{2}\right[$ is due to the following facts:

- arctan maps $[0, +\infty[$ onto $[0, \frac{\pi}{2}[$;
- the square root expression cannot take the value 0 because, again, neither $\cos\theta$ nor $\sin\theta$ are 0 and $\cos\tau$ and $\sin\tau$ cannot simultaneously be equal to 0.

Now let us now compute $\xi'_{Y_\eta,\theta}(0)$ and $\xi'_{Y_\eta,\theta}(0)$. For this purpose, define functions T, U, V, W, R, S, ϕ by (t, x, y are real variables):

- $T(t) = \frac{t}{\sqrt{2}}$;
- $U(x) = \cos^2 x \sin^2\theta + \sin^2 x \cos^2\theta$;
- $V(x) = \cos^2 x \cos^2\theta + \sin^2 x \sin^2\theta$;
- $W = \frac{U}{V}$;
- $R(y) = \sqrt{y}$;
- $S = R \circ W$;
- $\phi = \arctan \circ S$.

One obviously has

$$U(0) = \sin^2\theta\,,\; V(0) = \cos^2\theta\,,\; W(0) = \tan^2(\theta)\,,\; S(0) = \tan\theta\,,\; \phi(0) = \theta$$

and easily checks

$$U'(0) = 0\;,\; V'(0) = 0\;,\; U''(0) = 2\cos(2\theta) \text{ and } V''(0) = -2\cos(2\theta).$$

By writing $W' = \frac{U'V - UV'}{V^2}$ and $S' = \frac{W'}{2\sqrt{W}}$ and by differentiating W' and S' one obtains:

$$W'(0) = 0\;,\; W''(0) = \frac{2\cos(2\theta)}{\cos^4\theta}\;,\; S'(0) = 0 \text{ and } S''(0) = \frac{\cos(2\theta)}{\cos^4\theta\tan\theta}\;.$$

By writing $\phi' = \frac{S'}{1+S^2}$ and by differentiating ϕ' one gets:

$$\phi'(0) = 0 \text{ and } \phi''(0) = \frac{2}{\tan(2\theta)}\;.$$

Finally, writing $\xi = \phi \circ T$, we have:

- $\xi'_{Y_{\eta},\theta}(t) = (\phi \circ T)'(t) = \phi'(T(t)) \cdot T'(t) = \frac{1}{\sqrt{2}} \cdot \phi'(T(t))$;
- $\xi''_{Y_{\eta},\theta}(t) = \frac{1}{\sqrt{2}} \cdot (\phi' \circ T)'(t) = \frac{1}{\sqrt{2}} \cdot \phi''(T(t)) \cdot T'(t) = \frac{1}{2} \cdot \phi''(T(t))$;
- $\xi'_{Y_{\eta},\theta}(0) = 0$;
- $\xi''_{Y_{\eta},\theta}(0) = \frac{1}{\tan(2\theta)}$.

Throughout the calculations, one can check that all expressions are well defined.

End of proof.

Case 2: $2 \leq r < n$ (when $n \geq 3$ only)

Proposition 5.5.

1. $\exp(-tX_r)x(\theta) = (\cos\tau\cos\theta, 0, ..., 0, \sin\tau\cos\theta, 0, ..., 0, \sin\theta)$, *where the term* $\sin\tau\cos\theta$ *is the* r^{th} *coordinate.*
2. *Take* $\theta \in \left]0, \frac{\pi}{2}\right[$. *If* $\tau \in \mathbb{R}$ *is such that* $|\tau|$ *is small enough, then*

$$\xi_{X_r,\theta}(t) = \arctan\left(\frac{1}{\cos\tau}\sqrt{\sin^2\tau + \tan^2\theta}\right).$$

Also:

$$\xi'_{X_r,\theta}(0) = 0 \quad ; \quad \xi''_{X_r,\theta}(0) = \frac{1}{2\tan\theta}.$$

Proof:

The proof is straightforward; here are the steps:

- Define $U(\tau) = \sin^2\tau + \tan^2\theta$ and check:

$$U(0) = \tan^2(\theta)\ ,\ U'(0) = 0 \text{ and } U''(0) = 2.$$

- Define $V = \sqrt{U}$ and check:

$$V(0) = \tan\theta\ ,\ V'(0) = 0 \text{ and } V''(0) = \frac{1}{\tan\theta}.$$

- Define $W(\tau) = \cos\tau$ and $Z = \frac{V}{W}$; then check:

$$Z(0) = \tan\theta\ ,\ Z'(0) = 0 \text{ and } Z''(0) = \frac{2}{\sin(2\theta)}.$$

- Define $A = \arctan \circ Z$ and check:

$$A(0) = \theta\ ,\ A'(0) = 0 \text{ and } A''(0) = \frac{1}{\tan\theta}.$$

These are the calculations with respect to the variable τ; remembering that $\tau = \frac{t}{\sqrt{2}}$ establishes Item 2.

End of proof.

Above calculations also establish:

Proposition 5.6. *Consider $\eta \in \{i, j, ji\}$.*

1. $\exp(-tY_{r,\eta})x(\theta) = (\cos\tau\cos\theta, 0, ..., 0, -\eta\sin\tau\cos\theta, 0, ..., 0, \sin\theta)$, *where the term $-\eta\sin\tau\cos\theta$ is the r^{th} coordinate.*

2. *Take $\theta \in \left]0, \frac{\pi}{2}\right[$. If $\tau \in \mathbb{R}$ is such that $|\tau|$ is small enough, then*

$$\xi_{Y_r,\theta}(t) = \arctan\left(\frac{1}{\cos\tau}\sqrt{\sin^2\tau + \tan^2\theta}\right).$$

Also:

$$\xi'_{Y_r,\theta}(0) = 0 \quad ; \quad \xi''_{Y_r,\theta}(0) = \frac{1}{2\tan\theta}.$$

Assembling all terms in (5.4)

Equation (5.4) can be written:

$$\sum_{\substack{\eta \in \{i,j,ji\} \\ 2 \le r \le n \\ M \in \{X_r, Y_{r,\eta}, Z_\eta\}}} F''\Big(\xi_{M,\theta}(0)\Big)\left(\xi'_{M,\theta}(0)\right)^2 + F'\Big(\xi_{M,\theta}(0)\Big)\xi''_{M,\theta}(0) = \Lambda F(\theta).$$

Combining this to Propositions 5.2, 5.3, 5.4, 5.5 and 5.6 finally gives for $\theta \in \left]0, \frac{\pi}{2}\right[$:

$$F''(\theta) + \left(\frac{6}{\tan(2\theta)} + \frac{4n-8}{\tan\theta}\right) F'(\theta) - 2\Lambda F(\theta) = 0. \tag{5.5}$$

If one considers the smooth diffeomorphism (onto)

$$\begin{array}{rcccc} \psi & : & \left]0, \frac{\pi}{2}\right[& \longrightarrow &]0,1[\\ & & \theta & \longmapsto & \cos^2\theta \end{array}$$

and applies the change of variables $u = \psi(\theta)$, then one obtains the standard hypergeometric equation given in the theorem below, which is one of our main results and which summarises this chapter. Before we state it:

Definition 5.7. *The reduced version of f_P is the function*

$$\varphi_P = F \circ \psi^{-1}.$$

Theorem 5.8 (Compact picture and hypergeometric equation)**.** *Consider $\alpha \in \mathbb{N}$ and $k = 2\alpha$. Consider the unique (up to a constant) bi-invariant spherical harmonic of $\mathcal{Y}^k$. Then the restriction φ of its reduced version to the open interval $]0,1[$ satisfies the following hypergeometric equation:*

$$u(1-u)\varphi''(u) + (2 - Nu)\varphi'(u) - \frac{\Lambda}{2}\varphi(u) = 0,$$

where $\Lambda = -\Big(2\alpha^2 + (4n-2)\alpha\Big)$.

Chapter 6

Non-standard picture and Bessel functions

We use the definition of the non-standard picture given in [8] (Section 5.1, where it is actually called the non-standard model). This picture was initiated by the authors of [35], who studied degenerate principal series of the real symplectic group $\mathrm{Sp}(n, \mathbb{R})$.

We now fix $(\lambda, \delta) \in \mathbb{R} \times \mathbb{Z}$.

6.1 Non-standard picture

Let us denote by (s, u, v) the coordinates on $\mathbb{C} \times \mathbb{C}^m \times \mathbb{C}^m \simeq \mathbb{C}^{2m+1}$ and define some partial Fourier transforms along specific variables of $\mathbb{C}^{2m+1}$:

- Denote by L^1_τ the set of equivalence classes of functions

  $$g : \mathbb{C}^{2m+1} \longrightarrow \mathbb{C}$$

  such that for almost all $(u, v) \in \mathbb{C}^m \times \mathbb{C}^m$:

  $$\int_{\mathbb{C}} |g(\tau, u, v)| \, d\tau < +\infty.$$

For such functions g and such pairs (u, v) we set:

$$\mathcal{F}_\tau(g)(s, u, v) = \int_{\mathbb{C}} g(\tau, u, v)\, e^{-2i\pi \mathrm{Re}(s\tau)}\, d\tau.$$

- Denote by L^1_ξ the set of equivalence classes of functions

$$g : \mathbb{C}^{2m+1} \longrightarrow \mathbb{C}$$

such that for almost all $(s, u) \in \mathbb{C} \times \mathbb{C}^m$:

$$\int_{\mathbb{C}^m} |g(s, u, \xi)|\, d\xi < +\infty.$$

For such functions g and such pairs (s, u) we set:

$$\mathcal{F}_\xi(g)(s, u, v) = \int_{\mathbb{C}^m} g(s, u, \xi)\, e^{-2i\pi \mathrm{Re} < v, \xi >}\, d\xi.$$

Density of $L^1_\tau \cap L^2(\mathbb{C}^{2m+1})$ and $L^1_\xi \cap L^2(\mathbb{C}^{2m+1})$ in $L^2(\mathbb{C}^{2m+1})$ implies that above formulas completely define two partial Fourier transforms, written respectively:

$$\mathcal{F}_\tau : L^2(\mathbb{C}^{2m+1}) \longrightarrow L^2(\mathbb{C}^{2m+1});$$
$$\mathcal{F}_\xi : L^2(\mathbb{C}^{2m+1}) \longrightarrow L^2(\mathbb{C}^{2m+1}).$$

We now define the partial Fourier transform $\mathcal{F}$ on which is based the non-standard picture:

$$\mathcal{F} = \mathcal{F}_\tau \circ \mathcal{F}_\xi. \tag{6.1}$$

For functions $f \in L^1(\mathbb{C}^{2m+1}) \cap L^2(\mathbb{C}^{2m+1})$, one can write:

$$\mathcal{F}(f)(s, u, v) = \int_{\mathbb{C} \times \mathbb{C}^m} f(\tau, u, \xi)\, e^{-2i\pi \mathrm{Re}(s\tau + \langle v, \xi \rangle)}\, d\tau d\xi. \tag{6.2}$$

Finally:

Definition 6.1. *The non-standard picture of $\pi_{i\lambda,\delta}$ has $L^2(\mathbb{C}^{2m+1})$ as carrying space. The action of G is then the conjugate under $\mathcal{F}$ of the action of G in the non-compact picture; in other words, $\mathcal{F}$ intertwines the action of G in the non-compact picture and the action of G in the non-standard picture.*

6.2 Aim of this chapter: specific highest weight vectors

Decomposition (4.4) and Theorem 4.6 tell us what the K-types of $\pi_{i\lambda,\delta}$ are: they are the isotypic components of the left action of K which sit in the spaces $\mathrm{H}^{\alpha,\beta}$ such that $\delta = \beta - \alpha$. This point of view comes from the compact picture. Though this picture has helped us describe all K-types, it is interesting to see what they look like in other pictures, in particular the non-standard one. This is the point of the rest of this chapter. Recalling Theorem 4.2, for technical issues we restrict to K-types labelled by $\gamma = 0$, namely the components $V_0^{\alpha,\beta}$ whose highest weight vectors are

$$P_0^{\alpha,\beta}(z, w, \bar{z}, \bar{w}) = w_1^{\alpha} \bar{z_1}^{\beta}.$$

Because we intend to use the right action of $\mathrm{Sp}(1)$, we consider an entire space H^k of harmonic polynomials.

So we now fix (for the rest of this chapter) any $k \in \mathbb{N}$ and consider values of α and β such that:

$$\alpha + \beta = k.$$

We denote by $g_{\alpha,\beta}$ the restriction of $P_0^{\alpha,\beta}$ to the unit sphere S^{2N-1}.

Let us call g the function in the induced picture that corresponds to $g_{\alpha,\beta}$. It extends $g_{\alpha,\beta}$, meaning: $g_{|_{S^{2N-1}}} = g_{\alpha,\beta}$. Remember that g must satisfy the covariance relation for all non-zero complex numbers c:

$$g(c\,\cdot) = \left(\frac{c}{|c|}\right)^{-\delta} |c|^{-i\lambda - N}\, g.$$

We define

$$a(s, u) = \sqrt{1 + 4|s|^2 + \|u\|^2}$$

and

$$r(s, u, v) = \sqrt{a^2(s, u) + \|v\|^2}.$$

By restricting g to the complex hyperplane $\{1\} \times \mathbb{C} \times \mathbb{C}^m \times \mathbb{C}^m$, we obtain the function $G_{\alpha,\beta}$ defined on $\mathbb{C}^{2m+1}$ by:

$$\begin{aligned} G_{\alpha,\beta}(s,u,v) &= g(1,u,2s,v) \\ &= (r(s,u,v))^{-i\lambda-N}\, g\left(\frac{1}{r(s,u,v)}, \frac{u}{r(s,u,v)}, \frac{2s}{r(s,u,v)}, \frac{v}{r(s,u,v)}\right) \\ = \left(\frac{1}{r(s,u,v)}\right)^{i\lambda+N} & g_{\alpha,\beta}\left(\frac{1}{r(s,u,v)}, \frac{u}{r(s,u,v)}, \frac{2s}{r(s,u,v)}, \frac{v}{r(s,u,v)}\right). \end{aligned}$$

Finally, due to the total homogeneity degree k of $P_0^{\alpha,\beta}$:

$$G_{\alpha,\beta}(s,u,v) = \frac{(2s)^\alpha}{(a^2(s,u) + \|v\|^2)^{\frac{i\lambda+k}{2}+n}}. \tag{6.3}$$

The function $G_{\alpha,\beta}$ is the non-compact form of $g_{\alpha,\beta}$. We point out that we automatically know that $G_{\alpha,\beta}$ belongs to the Hilbert space $L^2(\mathbb{C}^{2m+1})$: this follows from the identifications between the carrying spaces in the various pictures.

The aim of the rest of this chapter is to determine the non-standard form $\mathcal{F}(G_{\alpha,\beta})$ of $G_{\alpha,\beta}$ (using the composition formula $\mathcal{F} = \mathcal{F}_\tau \circ \mathcal{F}_\xi$). The explicit non-standard form we end up with is given in Theorem 6.10 and also in Theorem 6.11, which is a recap that puts all these calculations back into context. Calculations will involve Bessel functions and various technical results which, for convenience, we put together in the next section.

6.3 Bessel functions and useful formulas

We shall need to compute various integrals. Some will be expressed in terms of Bessel functions. Let us recall definitions of these functions. In these definitions, following for instance [37] (Sections 5.3 and 5.7), we take ν to be any complex number (called the *order*) and the variable $z \in \mathbb{C}\setminus\{0\}$ to be such that $-\pi < \mathrm{Arg}(z) < \pi$:

1. The *Bessel function of the first kind* is the function J_ν defined

by:
$$J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(k+\nu+1)} \left(\frac{z}{2}\right)^{\nu+2k}.$$
We point out that for $\nu \in \mathbb{Z}$, $J_{-\nu}(z) = (-1)^\nu J_\nu(z)$.

2. The *Bessel function of the second kind* is the function Y_ν defined by:
$$Y_\nu(z) = \frac{J_\nu(z)\cos(\nu\pi) - J_{-\nu}(z)}{\sin(\nu\pi)}$$
when $\nu \notin \mathbb{Z}$ and, when $\nu \in \mathbb{Z}$, by
$$Y_\nu(z) = \lim_{\substack{\epsilon\to\nu \\ 0<|\epsilon-\nu|<1}} Y_\epsilon(z).$$
We point out that for $\nu \in \mathbb{Z}$, $Y_{-\nu}(z) = (-1)^\nu Y_\nu(z)$.

3. The *modified Bessel function of the first kind* is the function I_ν defined by:
$$I_\nu(z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+1)\Gamma(k+\nu+1)} \left(\frac{z}{2}\right)^{\nu+2k}.$$
We point out that for $\nu \in \mathbb{Z}$, $I_{-\nu}(z) = I_\nu(z)$.

4. The *modified Bessel function of the third kind* is the function K_ν defined by
$$K_\nu(z) = \frac{\pi}{2}\frac{I_{-\nu}(z) - I_\nu(z)}{\sin(\nu\pi)}$$
when $\nu \notin \mathbb{Z}$ and, when $\nu \in \mathbb{Z}$, by
$$K_\nu(z) = \lim_{\substack{\epsilon\to\nu \\ 0<|\epsilon-\nu|<1}} K_\epsilon(z).$$
We point out that $K_{-\nu}(z) = K_\nu(z)$.

Remarks:

- The Gamma function is a meromorphic function that has no zeros and whose poles are $0, -1, -2\cdots$ (they are all simple poles). In the series above, some coefficients may involve terms $\frac{1}{\Gamma(x)}$ for certain poles x (in which case there are only finitely many coefficients of this sort); but this is not a problem, since such coefficients are simply 0 (because $|\Gamma(x)| = +\infty$).

- At the end of this work, we recall the differential equations that are satisfied by the various Bessel functions.

When the order is a nonnegative integer, one can define Bessel functions as integrals (see [17], Chapter 8, Section 8.411, Formula 1.[11], or [10], Chapter VII, Section 7.3.1, Formula (2)):

Proposition 6.2 (Bessel integral representation)**.** *When $\nu \in \mathbb{N}$, one can define J_ν as follows:*

$$J_\nu(z) = \frac{1}{2\pi} \int_0^{2\pi} e^{iz\sin\theta}\, e^{-i\nu\theta} d\theta.$$

As a consequence, by writing

$$\cos a\, \cos\theta\, +\, \sin a\, \sin\theta\, =\, \cos(a-\theta)\, =\, \sin\left(\frac{\pi}{2} - a + \theta\right)$$

and using a change of variables one can prove:

Corollary 6.3 (Another Bessel integral representation)**.** *Given any $\nu \in \mathbb{N}$, $\rho > 0$ and $a \in \mathbb{R}$:*

$$\int_0^{2\pi} e^{i\nu\theta}\, e^{-i\rho(\cos a\, \cos\theta\, +\, \sin a\, \sin\theta)}\, d\theta\, =\, 2\pi e^{i\nu\left(a-\frac{\pi}{2}\right)}\, J_\nu(\rho).$$

Formulas in this proposition are stated in [11] (Chapter VIII: Formula (20) of Section 8.5 and Formula (35) of Section 8.14); one can also find them in [17] (Section 6.565, Formula 4., page 678 and Section 6.596, Formula 7.[8], page 693):

Proposition 6.4 (Two integral formulas involving Bessel functions)**.**

- *For any real number $y > 0$ and any complex numbers a, ν, μ such that $\mathrm{Re}(a) > 0$ and $-1 < \mathrm{Re}(\nu) < 2\mathrm{Re}(\mu) + \frac{3}{2}$, one has:*

$$\int_0^\infty x^{\nu+\frac{1}{2}} \left(x^2 + a^2\right)^{-\mu-1} J_\nu(xy)\sqrt{xy}\ dx =$$

$$\frac{a^{\nu-\mu} y^{\mu+\frac{1}{2}} K_{\nu-\mu}(ay)}{2^\mu \Gamma(\mu+1)}.$$

- *For any real number $y > 0$ and any complex numbers a, β, ν, μ such that $\mathrm{Re}(a) > 0$, $\mathrm{Re}(\beta) > 0$ and $\mathrm{Re}(\nu) > -1$, one has:*

$$\int_0^\infty x^{\nu+\frac{1}{2}} \left(x^2 + \beta^2\right)^{-\frac{\mu}{2}} K_\mu \left(a(x^2+\beta^2)^{\frac{1}{2}}\right) J_\nu(xy)\sqrt{xy}\ dx =$$

$$a^{-\mu} \beta^{\nu+1-\mu} y^{\nu+\frac{1}{2}} (a^2+y^2)^{\frac{\mu}{2}-\frac{\nu}{2}-\frac{1}{2}} K_{\mu-\nu-1}\left(\beta(a^2+y^2)^{\frac{1}{2}}\right).$$

This next formula is stated in [10] (Section 7.4.1, Item (4), page 24):

Proposition 6.5 (Asymptotic expansion for Modified Bessel functions)**.** *For any fixed $P \in \mathbb{N}\backslash\{0\}$ and $\nu \in \mathbb{C}$:*

$$K_\nu(z) = \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} e^{-z} \left(\left[\sum_{p=0}^{P-1} \frac{\Gamma\left(\frac{1}{2}+\nu+p\right)}{p!\ \Gamma\left(\frac{1}{2}+\nu-p\right)}\ (2z)^{-p}\right] + \mathrm{O}\left(|z|^{-P}\right)\right).$$

The following proposition can be derived from Section 9.6 of [12]:

Proposition 6.6 (The Bochner formula)**.** *Consider any integer $p \geq 2$. For $\xi' \in S^{p-1}$ and $s > 0$:*

$$\int_{S^{p-1}} e^{-2i\pi s \xi\cdot\xi'} d\sigma(\xi) = 2\pi s^{1-\frac{p}{2}} J_{\frac{p}{2}-1}(2\pi s),$$

where $\xi \cdot \xi'$ denotes the Euclidean scalar product of $\mathbb{R}^p$ applied to the elements of the sphere ξ and ξ' seen as elements of $\mathbb{R}^p$.

We end this section by recalling two standard results: the first gives sufficient conditions to differentiate integrals with respect to parameters and the second relates the standard polar change of coordinates.

Proposition 6.7 (Integrals and parameters)**.** *Given an integer $p \geq 1$ and a pair $(a,b) \in \mathbb{R}^2$ such that $a < b$, consider a function*

$$f : \mathbb{R}^p \times]a,b[\longrightarrow \mathbb{C}\,.$$

Denote by (x,λ) the coordinates on $\mathbb{R}^p \times]a,b[$ (λ is the parameter). Assume that:

- *for all fixed $\lambda \in]a,b|$, the function $x \longmapsto f(x,\lambda)$ is integrable on $\mathbb{R}^p$;*
- *the function $\lambda \longmapsto f(x,\lambda)$ is differentiable for all fixed $x \in \mathbb{R}^p$;*
- *there exists an integrable function $g : \mathbb{R}^p \longrightarrow [0,\infty[$ such that for all $(x,\lambda) \in \mathbb{R}^p \times]a,b[$:*

$$\left|\frac{\partial f}{\partial \lambda}(x,\lambda)\right| \leq g(x).$$

Then, for all $\lambda \in]a,b[$, the function $x \longmapsto \frac{\partial f}{\partial \lambda}(x,\lambda)$ is integrable on $\mathbb{R}^p$, the function

$$\begin{array}{cccc} \phi & : &]a,b[& \longrightarrow & \mathbb{C} \\ & & \lambda & \longmapsto & \int_{\mathbb{R}^p} f(x,\lambda)\, dx \end{array}$$

is differentiable and

$$\phi'(\lambda) = \int_{\mathbb{R}^p} \frac{\partial f}{\partial \lambda}(x,\lambda)\, dx.$$

Proposition 6.8 (Integrals and polar coordinates)**.** *Consider any integer $p \geq 2$ and a measurable function $f : \mathbb{R}^p \longrightarrow \mathbb{C}$ (with respect to the Lebesgue measure of $\mathbb{R}^p$). Write elements $x \in \mathbb{R}^p \setminus \{0\}$ as $r\xi$ with $r > 0$ and $\xi \in S^{p-1}$.*
Then f is integrable on $\mathbb{R}^p$ if and only if the function

$$(r,\xi) \in]0,+\infty[\times S^{p-1} \longmapsto f(r\xi)$$

is integrable on $]0,+\infty[\times S^{p-1}$ with respect to the measure $r^{p-1}dr\, d\sigma(\xi)$. When integrability is satisfied, we have:

$$\int_{\mathbb{R}^p} f(x)dx = \int_0^\infty \left(\int_{S^{p-1}} f(r\xi)d\sigma(\xi)\right) r^{p-1}dr.$$

We now set to prove the main result of this chapter. The calculations are long and technical. We spread them over two sections: Section 6.4 (which applies $\mathcal{F}_\xi$) and Section 6.5 (which applies $\mathcal{F}_\tau$). In Section 6.6, we summarise in a single and self-contained statement what we have achieved; this is our main theorem, namely Theorem 6.11.

6.4 First transform

By definition:

$$\mathcal{F}_\xi(G_{\alpha,\beta})(s,u,v) = \int_{\mathbb{C}^m} \frac{(2s)^\alpha}{(a^2(s,u)+\|\xi\|^2)^{\frac{i\lambda+k}{2}+n}}\, e^{-2i\pi \mathrm{Re}<v,\xi>}\, d\xi. \quad (6.4)$$

In real coordinates, writing $\xi = x+iy$ and $v = a+ib$ (elements x, y, a, b each belong to $\mathbb{R}^m$) and identifying ξ and v with the elements (x, y) and (a, b) of $\mathbb{R}^m \times \mathbb{R}^m$, formula (6.4) reads:

$$\mathcal{F}_\xi(G_{\alpha,\beta})(s,u,v) = \int_{\mathbb{R}^m\times\mathbb{R}^m} \frac{(2s)^\alpha}{(a^2(s,u)+\|x\|^2+\|y\|^2)^{\frac{i\lambda+k}{2}+n}}\, e^{-2i\pi(a\cdot x-b\cdot y)}\, dxdy. \quad (6.5)$$

We remind the reader that the dots in the exponential term refer to the usual Euclidian scalar product of $\mathbb{R}^m$. Proposition 6.8 allows us to switch to polar coordinates in Formula (6.5): given non-zero pairs (x, y) and $(a, -b)$, we write $(x, y) = rM$ and $(a, -b) = r'M'$, where we take M and M' to belong to the sphere S^{2m-1} of $\mathbb{R}^{2m}$ and where we set:

$$r = \sqrt{\|x\|^2+\|y\|^2} = \|\xi\| > 0;$$

$$r' = \sqrt{a^2+(-b)^2} = \|v\| > 0.$$

This is possible because the function

$$(r, M) \in]0, +\infty[\times S^{2m-1} \longmapsto G(s,u,rM)\, e^{-2i\pi M\cdot M'}$$

is integrable with respect to the measure $r^{2m-1}dr\, d\sigma(\xi)$. This follows from the inequality

$$\left|\frac{r^{2m-1}}{(a^2(s,u)+r^2)^{\frac{i\lambda+k}{2}+n}}\right| \leq \frac{1}{r^{k+3}}$$

and Riemann's usual criterion for integrability.

Here, we denote by $M \cdot M'$ the Euclidean scalar product of $\mathbb{R}^{2m}$ applied to the points M and M' of the sphere S^{2m-1} seen as vectors of $\mathbb{R}^{2m}$. Polar coordinates change the integral of Formula (6.5) into

$$\int_0^\infty \left(\int_{S^{2m-1}} \frac{(2s)^\alpha}{(a^2(s,u)+r^2)^{\frac{i\lambda+k}{2}+n}} e^{-2i\pi r r' M\cdot M'} d\sigma(M) \right) r^{2m-1} dr. \tag{6.6}$$

Integral (6.6) can be written:

$$\int_0^\infty \frac{(2s)^\alpha}{(a^2(s,u)+r^2)^{\frac{i\lambda+k}{2}+n}} \left(\int_{S^{2m-1}} e^{-2i\pi r r' M\cdot M'} d\sigma(M) \right) r^{2m-1} dr. \tag{6.7}$$

Proposition 6.6 then changes (6.7) into

$$\int_0^\infty \frac{(2s)^\alpha}{(a^2(s,u)+r^2)^{\frac{i\lambda+k}{2}+n}} 2\pi (rr')^{1-m} J_{m-1}(2\pi r r') r^{2m-1} dr. \tag{6.8}$$

Because $r' = \|v\|$, (6.8) becomes

$$2^{\alpha+1}\pi s^\alpha \|v\|^{1-m} \int_0^\infty \frac{r^m}{(a^2(s,u)+r^2)^{\frac{i\lambda+k}{2}+n}} J_{m-1}(2\pi\|v\| r) dr. \tag{6.9}$$

We now want to apply Proposition 6.4. But it uses another notation system than ours. To understand how to switch from one to the other, let us define new variables x, y, μ by:

$$x = r \ ; \ y = 2\pi\|v\| \ ; \ \mu = \frac{i\lambda+k}{2} + n - 1 \ ; \ \nu = m-1.$$

Then (6.9) becomes:

$$2^{\alpha+1}\pi s^\alpha \|v\|^{1-m} y^{-\frac{1}{2}} \int_0^\infty \frac{x^{\nu+\frac{1}{2}}}{(a^2(s,u)+r^2)^{\mu+1}} J_{m-1}(xy)\sqrt{xy}\, dx. \tag{6.10}$$

Proposition 6.4 (first formula) now gives (as long as $\|v\| > 0$)

$$2^{\alpha+1}\pi s^\alpha \|v\|^{1-m} y^{-\frac{1}{2}} \frac{a^{\nu-\mu} y^{\mu+\frac{1}{2}} K_{\nu-\mu}(ay)}{2^\mu \Gamma(\mu+1)},$$

which, back to our own notation choices, is equal to

$$\frac{2^{\alpha+1}s^{\alpha}\pi^{\frac{i\lambda+k}{2}+n}}{\Gamma\left(\frac{i\lambda+k}{2}+n\right)}\left(\frac{\|v\|}{a(s,u)}\right)^{\frac{i\lambda+k}{2}+1} K_{-\left(\frac{i\lambda+k}{2}+1\right)}(2\pi a(s,u)\|v\|).$$

Because $K_\nu = K_{-\nu}$ whatever the value of ν, we have proved so far:

Proposition 6.9. *Given any $\lambda \in \mathbb{R}$ and any $(\alpha, \beta) \in \mathbb{N}^2$, consider the non-compact version $G_{\alpha,\beta}$ of the highest weight vector $g_{\alpha,\beta}$. Then, for all (s, u, v) in $\mathbb{C} \times \mathbb{C}^m \times \mathbb{C}^m$ such that $v \neq 0$:*

$$\mathcal{F}_\xi(G_{\alpha,\beta})(s,u,v) = 2^{\alpha+1}s^{\alpha}\frac{\pi^{\Lambda+n}}{\Gamma(\Lambda+n)}\left(\frac{\|v\|}{a(s,u)}\right)^{\Lambda+1} K_{\Lambda+1}(2\pi a(s,u)\|v\|),$$

where $k = \alpha + \beta$ and $\Lambda = \frac{i\lambda+k}{2}$.

6.5 Second transform

Given any $(s, u, v) \in \mathbb{C} \times \mathbb{C}^m \times \mathbb{C}^m$ (we assume $v \neq 0$), we want to apply $\mathcal{F}_\tau$ to $\mathcal{F}_\xi(G_{\alpha,\beta})$ by writing

$$\mathcal{F}_\tau(\mathcal{F}_\xi(G_{\alpha,\beta}))(s,u,v) = \int_{\mathbb{C}} \mathcal{F}_\xi(G_{\alpha,\beta})(\tau,u,v)\, e^{-2i\pi \mathrm{Re}(s\tau)}\, d\tau. \quad (6.11)$$

Let us first check that the function $\tau \longmapsto \mathcal{F}_\xi(G_{\alpha,\beta})(\tau,u,v)$ is indeed integrable. In other words, we wish to know whether the integral

$$\int_{\mathbb{C}} \left|\mathcal{F}_\xi(G_{\alpha,\beta})(\tau,u,v)\right|\, d\tau$$

or, more explicitly,

$$\int_{\mathbb{C}} \left|2^{\alpha+1}\tau^{\alpha}\frac{\pi^{\Lambda+n}}{\Gamma(\Lambda+n)}\left(\frac{\|v\|}{a(\tau,u)}\right)^{\Lambda+1} K_{\Lambda+1}(2\pi a(\tau,u)\|v\|)\right|\, d\tau \quad (6.12)$$

is finite or not. Let us split (6.12) into two integrals:

- one integral, denoted by I_1, over the unit ball B_1 of $\mathbb{C}$;
- one integral, denoted by I_2, over $\mathbb{C} \setminus B_1$.

Because $\|v\| \neq 0$ and $a(\tau, u) \geq 1$, the function that sits under Integral (6.12) is continuous on B_1 and therefore I_1 is finite. Proposition 6.5 (using the integer $P = 1$) and Proposition 6.9 allow us to write $|\mathcal{F}_\xi(G_{\alpha,\beta})(\tau, u, v)|$ in the following way (A, B, C denote positive constants and $B \geq 1$):

$$\frac{A\ |\tau|^\alpha}{(B + 4|\tau|^2)^{\frac{k+3}{4}}}\ e^{-C(B+4|\tau|^2)^{\frac{1}{2}}} \left(1 + \mathrm{O}\left(\frac{1}{C(B + 4|\tau|^2)^{\frac{1}{2}}}\right)\right). \tag{6.13}$$

Then for $|\tau|$ large enough and a suitable constant $D > 0$,

$$\frac{A}{|\tau|^{\frac{k+3}{2}-\alpha}}\ e^{-C|\tau|} \left(1 + \frac{D}{|\tau|}\right)$$

is an upper-bound of (6.13). The exponential term forces convergence of I_2 and this finally establishes the desired integrability assumption, which will allow us to switch to polar coordinates.

Let us again use the letters a, b, x, y, but this time taking them to simply be real numbers such that $s = a + ib$ and $\tau = x + iy$. Then $\mathrm{Re}(s\tau) = ax - by$ and (6.11) becomes

$$\int_{\mathbb{R}^2} \frac{2^{\alpha+1}\tau^\alpha \pi^{\Lambda+n}}{\Gamma(\Lambda + n)} \left(\frac{\|v\|}{a(\tau, u)}\right)^{\Lambda+1} K_{\Lambda+1}(2\pi a(\tau, u)\|v\|)\, e^{-2i\pi(ax-by)}\, dxdy,$$

which can be re-organised as

$$\frac{2^{\alpha+1}\pi^{\Lambda+n}\|v\|^{\Lambda+1}}{\Gamma(\Lambda + n)} \int_{\mathbb{R}^2} \frac{\tau^\alpha K_{\Lambda+1}(2\pi a(\tau, u)\|v\|)}{\left(a(\tau, u)\right)^{\Lambda+1}}\, e^{-2i\pi(ax-by)}\, dxdy. \tag{6.14}$$

Let us again use polar coordinates (outside the origin):

- $(x, y) = rv_\theta$ with $r > 0$, $\theta \in \mathbb{R}$ and $v_\theta = (\cos\theta, \sin\theta)$; accordingly, $\tau = re^{i\theta}$.
- $(a, -b) = r'v_{\theta'}$ with $r' > 0$, $\theta' \in \mathbb{R}$ and $v_{\theta'} = (\cos\theta', \sin\theta')$.

Let us point out that we obviously have $r = |\tau|$ and $r' = |s|$. Let us also write $a(r, u)$ instead of $a(\tau, u)$:

$$a(r, u) = \sqrt{1 + 4r^2 + \|u\|^2}.$$

Integral 6.14 can now be written:

$$\frac{2^{\alpha+1}\pi^{\Lambda+n}\|v\|^{\Lambda+1}}{\Gamma(\Lambda+n)}\int_0^\infty \frac{r^\alpha K_{\Lambda+1}(2\pi a(r,u)\|v\|)}{\left(a(r,u)\right)^{\Lambda+1}}$$
$$\left(\int_0^{2\pi} e^{i\alpha\theta}\, e^{-2i\pi rr'(\cos\theta\cos\theta'+\sin\theta\sin\theta')}d\theta\right) rdr. \quad (6.15)$$

Following Corollary 6.3, the inner integral

$$\int_0^{2\pi} e^{i\alpha\theta}\, e^{-2i\pi rr'(\cos\theta\;\cos\theta'+\sin\theta\;\sin\theta')}\,d\theta$$

is equal to

$$2\pi e^{i\alpha\left(\theta'-\frac{\pi}{2}\right)}\,J_\alpha(2\pi rr'). \quad (6.16)$$

Remembering that $r'=|s|$ and $\theta'=\mathrm{Arg}(\overline{s})$, (6.16) is equal to:

$$2\pi e^{i\alpha\left(\mathrm{Arg}(\overline{s})-\frac{\pi}{2}\right)}\,J_\alpha(2\pi r|s|).$$

This turns (6.15) into:

$$\frac{2^{\alpha+2}\pi^{\Lambda+n+1}\|v\|^{\Lambda+1}e^{i\alpha\left(\mathrm{Arg}(\overline{s})-\frac{\pi}{2}\right)}}{\Gamma(\Lambda+n)}$$
$$\int_0^\infty \frac{r^{\alpha+1}\,K_{\Lambda+1}(2\pi a(r,u)\|v\|)}{\left(a(r,u)\right)^{\Lambda+1}}\,J_\alpha(2\pi r|s|)\,dr. \quad (6.17)$$

We can now apply the second formula of Proposition 6.4. To help follow notation choices made in this proposition, we set:

- $x=2r$ and $dx=2dr$;
- $\beta=\sqrt{1+\|u\|^2}>0$;
- $a=2\pi\|v\|>0$ (careful: this variable a is not what we have denoted $a(r,u)$);
- $y=\pi|s|>0$;
- $\nu=\alpha$;

- $\mu = \Lambda + 1$.

Plugging these expressions in (6.17) and using Proposition 6.4, we finally achieve what we announced at the end of Section 6.2: compute the non-standard form $\mathcal{F}_\tau(\mathcal{F}_\xi(G_{\alpha,\beta}))$ of $G_{\alpha,\beta}$ (therefore of $g_{\alpha,\beta}$). The formula we obtain is:

Theorem 6.10.

$$\mathcal{F}(G_{\alpha,\beta})(s,u,v) = \int_{\mathbb{C}\times\mathbb{C}^m} \frac{(2\tau)^\alpha}{(1+4|\tau|^2+\|u\|^2+\|\xi\|^2)^{\frac{i\lambda+k}{2}+n}} e^{-2i\pi \mathrm{Re}(s\tau+\langle v,\xi\rangle)}\, d\tau d\xi = R(s,u,v)\, K_{\frac{i\lambda+\delta}{2}}\left(\pi\sqrt{1+\|u\|^2}\sqrt{|s|^2+4\|v\|^2}\right),$$

where

$$R(s,u,v) = \frac{(-i\,\overline{s})^\alpha\, \pi^{i\lambda+\beta+n}}{2^{\frac{i\lambda+k}{2}+1}\,\Gamma(\frac{i\lambda+k}{2}+n)} \left(\frac{\sqrt{|s|^2+4\|v\|^2}}{\pi\sqrt{1+\|u\|^2}}\right)^{\frac{i\lambda+\delta}{2}}$$

and:

- $(k,\alpha,n)\in\mathbb{N}^3$, $0\le\alpha\le k$ *and* $n\ge 2$;
- $(s,u,v)\in\mathbb{C}\times\mathbb{C}^m\times\mathbb{C}^m$ *and* $(\tau,\xi)\in\mathbb{C}\times\mathbb{C}^m$;
- $s\neq 0$ *and* $v\neq 0$.

6.6 Statement of main result

The following result, which serves as a recap, binds together the formula of Theorem 6.10 and all the ingredients that are needed to understand the meaning of this formula.

In this theorem, we consider three groups M, A and N, respectively isomorphic to the groups $\mathrm{U}(1)\times\mathrm{Sp}(m,\mathbb{C})$, $\mathbb{R}_\times^{+,*}$ (multiplicative group of positive reals) and $\mathrm{H}_\mathbb{C}^{2m+1}$ (complex Heisenberg group), embedded in $\mathrm{Sp}(n,\mathbb{C})$ as follows (denoting elements of M, A, N respectively by m, a, n):

- $m = \left(\begin{array}{cc|cc} e^{i\theta(m)} & 0 & 0 & 0 \\ 0 & A & 0 & C \\ \hline 0 & 0 & e^{-i\theta(m)} & 0 \\ 0 & B & 0 & D \end{array}\right)$,
  with $\theta(m) \in \mathbb{R}$ and $\left(\begin{array}{c|c} A & C \\ \hline B & D \end{array}\right) \in \mathrm{Sp}(m, \mathbb{C})$;

- $a = \left(\begin{array}{cc|cc} \alpha(a) & 0 & 0 & 0 \\ 0 & I_m & 0 & 0 \\ \hline 0 & 0 & (\alpha(a))^{-1} & 0 \\ 0 & 0 & 0 & I_m \end{array}\right)$,
  with $\alpha(a) \in]0, \infty[$;

- $n = \left(\begin{array}{cc|cc} 1 & {}^t u & 2s & {}^t v \\ 0 & I_m & v & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & -u & I_m \end{array}\right)$,
  with $(s, u, v) \in \mathbb{C} \times \mathbb{C}^m \times \mathbb{C}^m$.

We consider the group $Q = MAN$.

Theorem 6.11 (Non-standard picture and Bessel functions)**.**
*Consider $n \in \mathbb{N}$ such that $n \geq 2$ and set $N = 2n$, $m = n - 1$.
Let G be the group $\mathrm{Sp}(n, \mathbb{C})$ and $Q = MAN$ its parabolic subgroup introduced above; let K be the subgroup $\mathrm{Sp}(n)$. Consider $(\lambda, \delta) \in \mathbb{R} \times \mathbb{Z}$ and the character $\chi_{i\lambda,\delta}$ defined on Q by*

$$\chi_{i\lambda,\delta}(man) = e^{i\delta\theta(m)} \left(\alpha(a)\right)^{i\lambda+N}.$$

Consider the degenerate principal series representations of G (see Section 3.2.1 of Chapter 3 for details):

$$\pi_{i\lambda,\delta} = \mathrm{Ind}_Q^G \, \chi_{i\lambda,\delta}.$$

Consider $k \in \mathbb{N}$ and any $(\alpha, \beta) \in \mathbb{N}^2$ such that $\alpha + \beta = k$; set $\delta = \beta - \alpha$. Consider the irreducible (finite-dimensional) subrepresentation of $\pi_{i\lambda,\delta}\big|_K$ whose highest weight is $(k, 0, \cdots, 0)$. Then the corresponding highest weight vector (up to a constant) is given:

- *in the compact picture by the function*

$$\begin{array}{rcccc} g_{\alpha,\beta} & : & S^{2N-1} \subset \mathbb{C}^n \times \mathbb{C}^n & \longrightarrow & \mathbb{C} \\ & & (z,w) & \longmapsto & w_1^{\alpha} \bar{z_1}^{\beta}; \end{array}$$

- *in the non-compact picture by the function*

$$\begin{array}{rcccc} G_{\alpha,\beta} & : & \mathbb{C} \times \mathbb{C}^m \times \mathbb{C}^m & \longrightarrow & \mathbb{C} \\ & & (s,u,v) & \longmapsto & \frac{(2s)^{\alpha}}{(1+4|s|^2+\|u\|^2+\|v\|^2)^{\frac{i\lambda+k}{2}+n}}; \end{array}$$

- *in the non-standard picture by the function* $\mathcal{F}(G_{\alpha,\beta})$*, which is defined for all* (s,u,v) *in* $\mathbb{C} \times \mathbb{C}^m \times \mathbb{C}^m$ *such that* $v \neq 0$ *and* $s \neq 0$ *by*

$$\mathcal{F}(G_{\alpha,\beta})(s,u,v) \;=\; R(s,u,v)\; K_{\frac{i\lambda+\delta}{2}} \left(\pi\sqrt{1+\|u\|^2}\sqrt{|s|^2+4\|v\|^2}\right),$$

with

$$R(s,u,v) = \frac{(-i\,\overline{s})^{\alpha}\,\pi^{i\lambda+\beta+n}}{2^{\frac{i\lambda+k}{2}+1}\,\Gamma(\frac{i\lambda+k}{2}+n)} \left(\frac{\sqrt{|s|^2+4\|v\|^2}}{\pi\sqrt{1+\|u\|^2}}\right)^{\frac{i\lambda+\delta}{2}}.$$

Remark: as we have pointed out before, the symbol N refers to a group, Symbols n, m to elements of groups, while the three symbols also refer to dimensions; context makes intended meanings of these symbols clear.

6.7 Two interesting properties

In this section, we make two observations that seem relevant to us: we feel they may prove useful in further investigation of the K-types of the various representations $\pi_{i\lambda,\delta}$ in the non-standard picture.

6.7.1 A differential operator that connects certain K-types

If we look back at Figure 4.1 of Chapter 4, we see that the highest weight vectors we have studied in this present chapter are those that correspond to the components of the column on the far left. We saw in Theorem 4.11 that the operator e applies a highest weight vector onto the one immediately above (up to a constant). When restricting to the highest weight vectors of the left column, the action of e can be interpreted, with respect to their non-standard versions given by Theorem 6.11, as a differential operator that acts on Fourier transforms; this present section makes this clear.

In our study of properties of $\mathcal{F}$ with respect to differentiablity and multiplication by coordinates:

- (τ, u, ξ) will denote the coordinates on $\mathbb{C} \times \mathbb{C}^m \times \mathbb{C}^m$ with respect to initial functions f;
- (s, u, v) will then denote the coordinates on $\mathbb{C} \times \mathbb{C}^m \times \mathbb{C}^m$ with respect to $\mathcal{F}(f)$.

Theorem 6.12 (Differential induction formula)**.** *Suppose $k \geq 2$ and $\beta \geq 2$. Then:*

$$\mathcal{F}(G_{\alpha+1,\beta-1})(s,u,v) = \frac{2}{-i\pi}\frac{\partial}{\partial s}(\mathcal{F}(G_{\alpha,\beta})(s,u,v)).$$

Proof:

For $0 \leq \alpha \leq k-1$ and $(\tau, u, \xi) \in \mathbb{C} \times \mathbb{C}^m \times \mathbb{C}^m$:

$$\begin{aligned} G_{\alpha+1,\beta-1}(\tau,u,\xi) &= \left(r(\tau,u,\xi)\right)^{-i\lambda-N-k} g_{\alpha+1,\beta-1}(1,u;2\tau,\xi) \\ &= \left(r(\tau,u,\xi)\right)^{-i\lambda-N-k}(2\tau)^{\alpha+1} \\ &= 2\tau\left(r(\tau,u,\xi)\right)^{-i\lambda-N-k}(2\tau)^{\alpha} \\ &= 2\tau\; G_{\alpha,\beta}(\tau,u,\xi). \end{aligned}$$

Theorem 6.12 then follows from Lemmas 6.13 and 6.14 given below.

End of proof.

Remark: as we see in the proof, one switches from $G_{\alpha,\beta}$ to $G_{\alpha+1,\beta-1}$ by multiplying by 2τ; this corresponds to the action of the operator e studied in section 4.2.1.

Lemma 6.13. *If $\beta \geq 2$, then, given any $u \in \mathbb{C}^m$, the maps*

$$(\tau, \xi) \in \mathbb{C} \times \mathbb{C}^m \longmapsto G_{\alpha,\beta}(\tau, u, \xi)$$

and

$$(\tau, \xi) \in \mathbb{C} \times \mathbb{C}^m \longmapsto \tau\, G_{\alpha,\beta}(\tau, u, \xi)$$

are both integrable.

Proof:

Let us call ψ the map $(\tau, \xi) \longmapsto \tau\, G_{\alpha,\beta}(\tau, u, \xi)$ (we point out that integrability of this map forces integrability of the other map).

There is no problem as to the integrability of ψ on the unit ball $B_1(0)$ whose center is the origin. So one just needs to check integrability on the set $\mathbb{R}^{2n} \setminus B_1(0)$.

Given a non-zero element (τ, ξ) of $\mathbb{C} \times \mathbb{C}^m \simeq \mathbb{C}^n$, there exists a unique $r \in]0, +\infty[$ and a unique $M \in S^{2n-1}$ such that if $(\tau_M, \xi_M) \in \mathbb{C} \times \mathbb{C}^m$ denote the coordinates of M seen as an element of $\mathbb{C} \times \mathbb{C}^m$, then:

$$(\tau, \xi) = r(\tau_M, \xi_M).$$

Identifying complex coordinates on $\mathbb{C} \times \mathbb{C}^m$ with real coordinates on $\mathbb{R}^2 \times \mathbb{R}^{2m} \simeq \mathbb{R}^{2n}$ and applying Proposition 6.8, we see that ψ is integrable if the function

$$(r, M) \in]0, +\infty[\times S^{2n-1} \longmapsto r\, G_{\alpha,\beta}(r\tau_M, u, r\xi_M)$$

is itself integrable on $]0, +\infty[\times S^{2n-1}$ with respect to the measure $r^{2n-1} dr\, d\sigma(M)$. We have:

$$\begin{aligned}
|r\, G_{\alpha,\beta}(r\tau_M, u, r\xi_M)|\ r^{2n-1} &= \left| \frac{2^\alpha (r\tau_M)^\alpha\, r^{2n}}{(1 + 4|r\tau_M|^2 + \|u\|^2 + \|r\xi_M\|^2)^{\frac{i\lambda+k}{2}+n}} \right| \\
&\leq \frac{2^\alpha r^{\alpha+2n}}{(r^2)^{\frac{k}{2}+n}} \\
&= \frac{2^\alpha}{r^\beta}.
\end{aligned}$$

This finishes the proof, because $\beta \geq 2$ and $r \geq 1$.

End of proof.

Lemma 6.14. *Consider a function* $f : \mathbb{C} \times \mathbb{C}^m \times \mathbb{C}^m \longrightarrow \mathbb{C}$*. Assume that the maps*

$$(\tau, \xi) \longmapsto f(\tau, u, \xi)$$

and

$$(\tau, \xi) \longmapsto \tau\, f(\tau, u, \xi)$$

are both integrable on $\mathbb{C} \times \mathbb{C}^m$ *for all* $u \in \mathbb{C}^m$*. Consider the map* $\psi : (\tau, u, \xi) \longmapsto \tau\, f(\tau, u, \xi)$*. Then:*

$$\mathcal{F}(\psi)(s, u, v) \;=\; \frac{1}{-i\pi}\frac{\partial}{\partial s}\Big(\mathcal{F}(f)(s, u, v)\Big).$$

Proof:

Switch to real variables:

- $s = x_s + iy_s$, with $(x_s, y_s) \in \mathbb{R}^2$;
- $u = x_u + iy_u$, with $(x_u, y_u) \in \mathbb{R}^m \times \mathbb{R}^m$;
- $v = x_v + iy_v$, with $(x_v, y_v) \in \mathbb{R}^m \times \mathbb{R}^m$.

Similarly for the integration variables τ and ξ that enter the definition of the non-standard Fourier transform:

- $\tau = x_\tau + iy_\tau$, with $(x_\tau, y_\tau) \in \mathbb{R}^2$;
- $\xi = x_\xi + ix_\xi$ with $(x_\xi, x_\xi) \in \mathbb{R}^m \times \mathbb{R}^m$.

To make expressions in the integral shorter, define:

- $w = (x_\tau, x_u, x_\xi, y_\tau, y_u, y_\xi)$;
- $dw = dx_\tau\, dx_u\, dx_\xi\, dy_\tau\, dy_u\, dy_\xi$.

We remind the reader that the dot simply refers to the standard eucliean product (of $\mathbb{R}^m$ here). With these notation choices, Definition 6.2 reads:

$$\mathcal{F}(s, u, v) = \int_{\mathbb{R}^{2n}} f_{\mathbb{R}}(w) \exp^{-2i\pi(x_s x_\tau - y_s y_\tau + x_v \cdot x_\xi - y_v \cdot y_\xi)}\, dw.$$

Coordinates x_s and y_s are just parameters that appear in the integral. Lemma 6.13 and Proposition 6.7 enable us to differentiate this integral with respect to both parameters and obtain:

$$\frac{\partial}{\partial x_s}(\mathcal{F}(s,u,v)) = \int_{\mathbb{R}^{2n}} -2i\pi x_\tau f_{\mathbb{R}}(w) \exp^{-2i\pi(x_s x_\tau - y_s y_\tau + x_v \cdot x_\xi - y_v \cdot y_\xi)} dw;$$

$$\frac{\partial}{\partial y_s}(\mathcal{F}(s,u,v)) = \int_{\mathbb{R}^{2n}} 2i\pi y_\tau f_{\mathbb{R}}(w) \exp^{-2i\pi(x_s x_\tau - y_s y_\tau + x_v \cdot x_\xi - y_v \cdot y_\xi)} dw.$$

This finishes the proof because, by definition, the operator $\frac{\partial}{\partial s}$ is equal to $\frac{1}{2}\left(\frac{\partial}{\partial x_s} - i\frac{\partial}{\partial y_s}\right)$.

End of proof.

Corollary 6.15. *Consider integers k, α, β such that $k \geq 2$, $\alpha + \beta = k$ and $\beta \geq 1$. Then:*

$$\mathcal{F}(G_{\alpha,\beta})(s,u,v) = \left(\frac{2}{-i\pi}\right)^{\alpha} \frac{\partial^\alpha}{\partial s^\alpha}\Big(\mathcal{F}(G_{0,k})(s,u,v)\Big).$$

The following diagram shows how the operators of this section relate to one another, connecting functions and their transforms (assuming that $\beta \geq 2$):

$$\begin{array}{ccccc}
g_{\alpha+1,\beta-1} & \longrightarrow & G_{\alpha+1,\beta-1} & \longrightarrow & \mathcal{F}(G_{\alpha+1,\beta-1}) \\
\Big\uparrow {\scriptstyle \frac{1}{\beta}e} & & \Big\uparrow {\scriptstyle 2\tau Id} & & \Big\uparrow {\scriptstyle \frac{2}{-i\pi}\frac{\partial}{\partial s}} \\
g_{\alpha,\beta} & \longrightarrow & G_{\alpha,\beta} & \longrightarrow & \mathcal{F}(G_{\alpha,\beta}).
\end{array}$$

6.7.2 Underlying Emden-Fowler equations

Theorem 6.11 gives the following expression for the highest weight vectors $g_{\alpha,\beta}$ in the non standard picture:

$$\frac{(-i\,\overline{s})^{\alpha}\,\pi^{i\lambda+\beta+n}}{2^{\frac{i\lambda+k}{2}+1}\,\Gamma(\frac{i\lambda+k}{2}+n)}\left(\frac{\sqrt{|s|^2+4\|v\|^2}}{\pi\sqrt{1+\|u\|^2}}\right)^{\frac{i\lambda+\delta}{2}}$$
$$K_{\frac{i\lambda+\delta}{2}}\left(\pi\sqrt{1+\|u\|^2}\sqrt{|s|^2+4\|v\|^2}\right).$$

One inevitably notices a sort of two variable structure in this expression. Indeed, the particular value $\alpha = 0$ and the square root terms lead one to study the functions

$$\begin{array}{rccc} \psi_\nu : &]0,+\infty[\times]0,+\infty[& \longrightarrow & \mathbb{C} \\ & (x,y) & \longmapsto & \left(\frac{x}{y}\right)^\nu K_\nu(xy), \end{array}$$

where the parameter ν is taken to be any complex number.

For any given $\nu \in \mathbb{C}$, $x_0 > 0$ and $y_0 > 0$, define the functions:

$$\begin{array}{rccc} \varphi_{x_0,\nu} : &]0,+\infty[& \longrightarrow & \mathbb{C} \\ & y & \longmapsto & \psi_\nu(x_0,y); \end{array}$$

$$\begin{array}{rccc} \phi_{y_0,\nu} : &]0,+\infty[& \longrightarrow & \mathbb{C} \\ & x & \longmapsto & \psi_\nu(x,y_0). \end{array}$$

Because the function K_ν is a modified Bessel function of the third kind, we have:

$$K_\nu''(\xi) + \frac{1}{\xi}K_\nu'(\xi) - \left(1 + \frac{\nu^2}{\xi^2}\right) K_\nu(\xi) = 0.$$

In this equation, ξ can be taken as a real or a complex variable; here, we choose $\xi \in \mathbb{R}$. Combining this equation with the first and second derivatives of the functions $\varphi_{x_0,\nu}$ and $\phi_{y_0,\nu}$, one can check the following proposition.

Proposition 6.16. *Given any $\nu \in \mathbb{C}$, $x_0 > 0$, $y_0 > 0$ and taking $y > 0$:*

$$\varphi_{x_0,\nu}''(y) + \frac{(1+2\nu)}{y}\,\varphi_{x_0,\nu}'(y) - x_0^2\,\varphi_{x_0,\nu}(y) = 0;$$

$$\phi_{y_0,\nu}''(x) + \frac{(1-2\nu)}{x}\,\phi_{y_0,\nu}'(x) - y_0^2\,\phi_{y_0,\nu}(x) = 0.$$

Both equations can be written as (taking $a \in \mathbb{C}$, $b > 0$ and $t > 0$):

$$u''(t) + \frac{a}{t}\,u'(t) - b\,u(t) = 0. \tag{6.18}$$

Such equations belong to the family of *Emden-Fowler equations* (or *Lane-Emden equations*), which appear in various forms and have been studied in many works (see for instance [6], [2], [3], [43] and [46]).

The general solutions of (6.18) can be written as the following combinations (with t-dependent coefficients) of Bessel functions of the first and second kind:

$$u(t) = C_1\, t^{\frac{1-a}{2}}\, J_{\frac{a-1}{2}}\left(-it\sqrt{b}\right) + C_2\, t^{\frac{1-a}{2}}\, Y_{\frac{a-1}{2}}\left(-it\sqrt{b}\right).$$

This implies that $\varphi_{x_0,\nu}$ and $\varphi_{x_0,\nu}$ are such combinations, which gives new formulas for the function ψ_ν, thus other formulas for the non-standard version of the highest weight vector of Theorem 6.11 (when $\alpha = 0$).

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