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think!
NEW SYLLABUS MATHEMATICS

8th Edition

Nautilus Shell

For
Cambridge O Level and Cambridge IGCSE Mathematics

4

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PREFACE

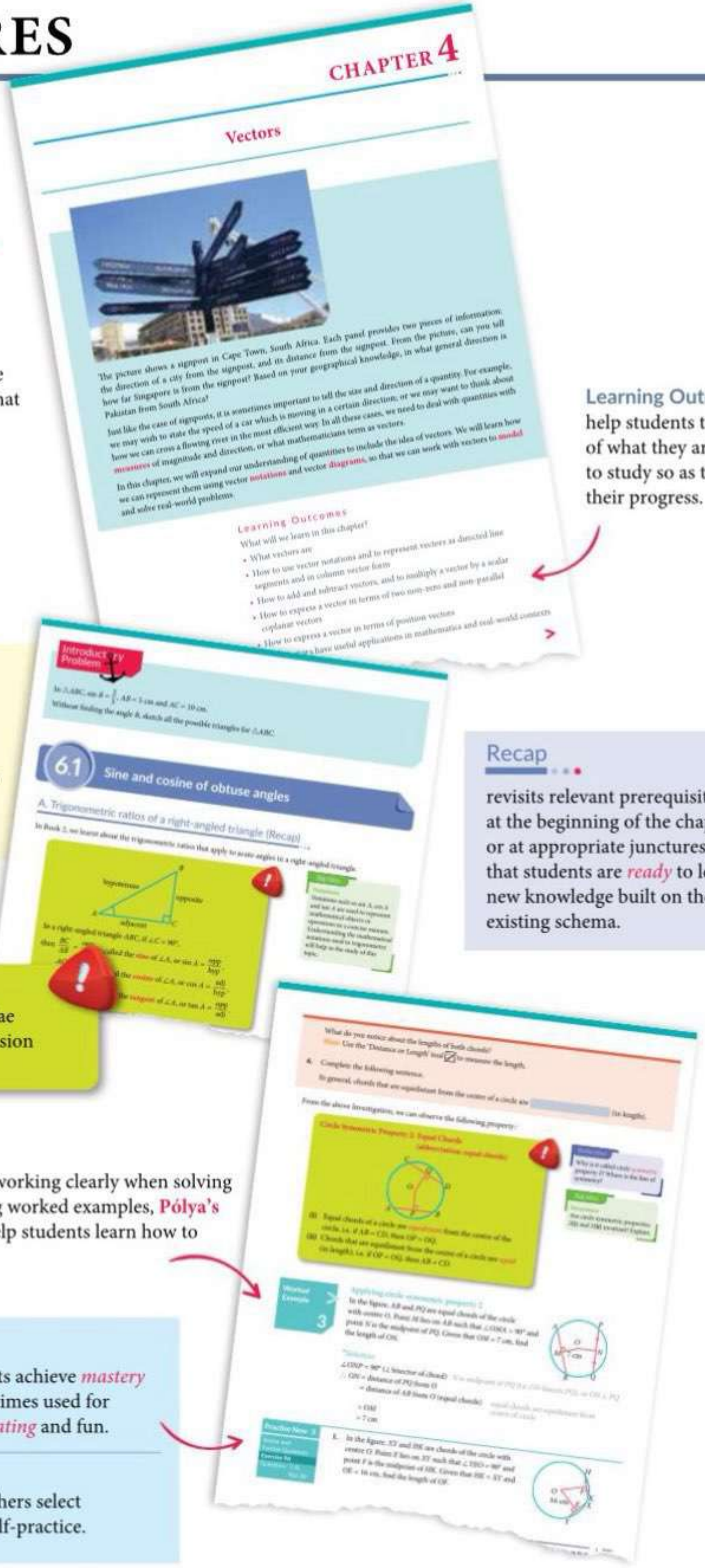
think! Mathematics is a series of textbooks specially designed to provide students valuable **learning experiences** by engaging their minds and hearts as they learn mathematics.

The features of this textbook series reflect the important shifts towards the development of 21st century competencies and a greater appreciation of mathematics, as articulated in the Singapore mathematics curriculum and other international curricula. Every chapter begins with a Chapter Opener and an Introductory Problem to motivate the development of the key concepts in the topic. The Chapter Opener gives a coherent overview of the **big ideas** that will frame the study of the topic, while the Introductory Problem positions problem solving at the heart of learning mathematics. Two key considerations guide the development of every chapter – seeing mathematics as a tool and as a discipline. Opportunities to engage in Investigation, Class Discussion, Thinking Time, Journal Writing and Performance Tasks are woven throughout the textbook to enhance students' learning experiences. Stories, songs, videos and puzzles serve to arouse interest and pique curiosity. Real-life examples serve to influence students to appreciate the beauty and usefulness of mathematics in their surroundings.

Underpinning the writing of this textbook series is the belief that all students can learn and appreciate mathematics. Worked Examples are carefully selected, questions in the Reflection section prompt students to reflect on their learning, and problems are of varying difficulty levels to ensure a high baseline of mastery, and to stretch students with special interest in mathematics. The use of ICT helps students to visualise and manipulate mathematical objects with ease, hence promoting interactivity.

We hope you will enjoy the subject as we embark on this exciting journey together to develop important mathematical dispositions that will certainly see you through beyond the examinations, to appreciate mathematics as an important tool in life, and as a discipline of the mind.

KEY FEATURES



CHAPTER 4

Vectors

The picture shows a signpost in Cape Town, South Africa. Each panel provides two pieces of information: the direction of a city from the signpost, and its distance from the signpost. From the picture, can you tell how far Singapore is from the signpost? Based on your geographical knowledge, in what general direction is Pakistan from South Africa?

Just like the case of signposts, it is sometimes important to tell the size and direction of a quantity. For example, we may wish to state the speed of a car which is moving in a certain direction; or we may want to think about how we can cross a flowing river in the most efficient way. In all these cases, we need to deal with quantities with measures of magnitude and direction, or what mathematicians term as vectors.

In this chapter, we will expand our understanding of quantities to include the idea of vectors. We will learn how we can represent them using vector notations and vector diagrams, so that we can work with vectors to model and solve real-world problems.

Learning Outcomes

What will we learn in this chapter?

- What vectors are
- How to use vector notations and to represent vectors as directed line segments and in column vector form
- How to add and subtract vectors, and to multiply a vector by a scalar
- How to express a vector in terms of two non-zero and non-parallel coplanar vectors
- How to express a vector in terms of position vectors
- have useful applications in mathematics and real-world contexts

Introductory Problem

Without finding the angle B, sketch all the possible triangles for △ABC.

6.1 Sine and cosine of obtuse angles

A. Trigonometric ratios of a right-angled triangle (Recap)

In Book 2, we learnt about the trigonometric ratios that apply to acute angles in a right-angled triangle.

What do you notice about the lengths of both chords?

Hint: Use the 'Distance or Length' tool to measure the length.

4. Complete the following sentence.

In general, chords that are equidistant from the center of a circle are ______ (in length).

From the above investigation, we can observe the following property:

Worked Example 3

Practise Now 3

Chapter Opener
gives students an overview of the topic. It includes *rationales* for learning the chapter to arouse students' *interest* and **big ideas** that *connect* the concepts within the chapter or with other chapters.

Learning Outcomes
help students to be aware of what they are about to study so as to *monitor* their progress.

Introductory Problem
provides students with a more specific *motivation* to learn the topic, using a problem that helps develop a concept, or an application problem that students will revisit after they have gained necessary knowledge from the chapter.

Recap
revisits relevant prerequisites at the beginning of the chapter or at appropriate junctures so that students are *ready* to learn new knowledge built on their existing schema.

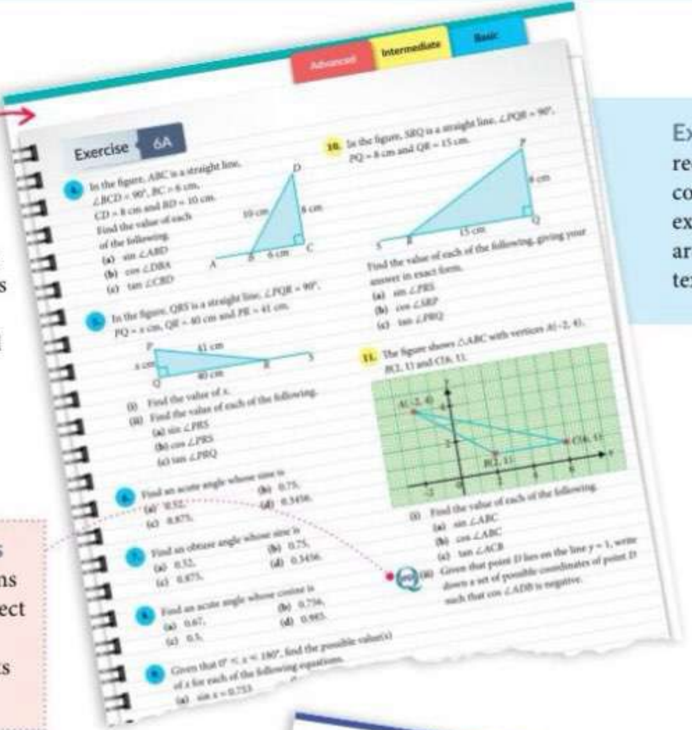
Important Results
summarise important concepts or formulae obtained from Investigation, Class Discussion or Thinking Time.

Worked Example
shows students how to present their working clearly when solving related *problems*. In more challenging worked examples, **Pólya's Problem Solving Model** is used to help students learn how to address a problem.

Practise Now
consists of questions that help students achieve *mastery* of procedural *skills*. Puzzles are sometimes used for consolidation to make practice *motivating* and fun.

Similar and Further Questions
follow after Practise Now to help teachers select appropriate questions for students' self-practice.

Exercise
questions are classified into three levels of difficulty – **Basic**, **Intermediate** and **Advanced**. Questions at the Basic level are usually short-answer items to test basic concepts and skills. The Intermediate level contains more structured questions, while the Advanced level involves applications and higher order thinking skills.

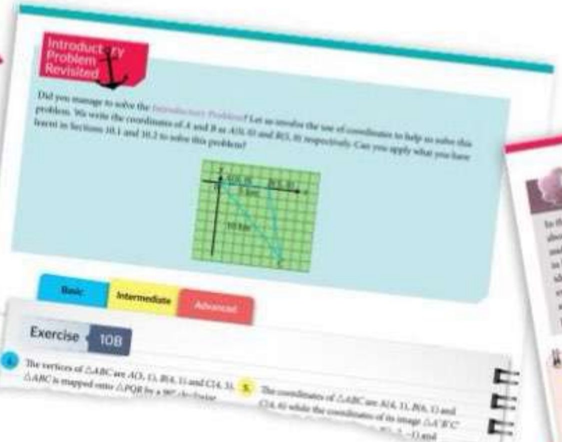


Explanation Questions
require students to communicate their explanations in writing and are spread throughout the textbook.

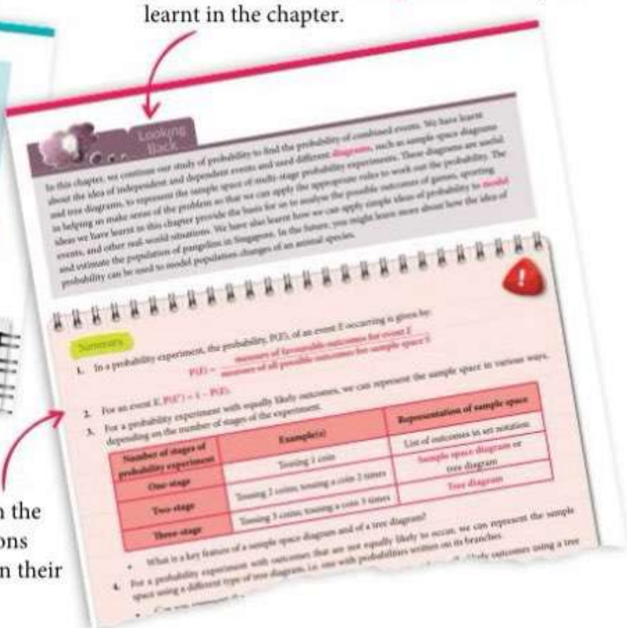
Open-ended Problems
are mathematics problems with more than one correct answer. Solving such problems expose students to real-world problems.

Performance Task
consists of mini-projects designed to develop research and presentation skills of students, through writing a report and/or giving an oral presentation.

Introductory Problem Revisited
revisits an application-based Introductory Problem later in the chapter. This is absent if the Introductory Problem leads directly to the development of a concept.



Looking Back
complements the Chapter Opener and helps students internalise the **big ideas** that they have learnt in the chapter.



Summary
compounds the key concepts taught in the chapter in a succinct manner. Questions are included to help students *reflect* on their learning.

Investigation

Guided investigation provides students the relevant *learning experiences* to explore and discover important mathematical *concepts*. It usually takes the **Concrete-Pictorial-Abstract (C-P-A)** approach to help students construct their knowledge meaningfully. The connections between concrete experiences (manipulative or examples), different pictorial representations and symbolic representations are explicitly made. Some investigations may also involve the use of **Information and Communication Technology (ICT)**.

Class Discussion

Questions are provided to **engage** students in discussion, with the teacher acting as the facilitator. Class discussions provide students the relevant *learning experiences* to think and *reason* mathematically, enhance their oral *communication* skills, and learn new *concepts* and *skills*.

Thinking Time

Key questions are included at appropriate junctures to provide students the relevant *learning experiences* to think critically on their own before sharing their thoughts with their classmates. Mathematical fallacies are sometimes included to check and test students' understanding.

Journal Writing

Journal writing provides opportunities for students to *reflect* on their learning and to *communicate* mathematically in writing. It can also be used as a formative assessment for the teacher to provide feedback for their students.

Reflection

Students are usually required to reflect on what they have learnt at the end of each section so as to *monitor* and *regulate* their own learning. The reflection questions provided can be generic prompts or specific to the topics in the section or chapter, to check if students have understood the key ideas.

MARGINAL NOTES

Big Idea

This provides additional details of the big idea mentioned in the main text.

Information

This includes information that may be of interest to students.

Internet Resources

This guides students to search the Internet for valuable information or interesting online games for their independent and self-directed learning.

Recall

Unlike the key feature 'Recap' in the main text, this contains just-in-time recall of prerequisite knowledge that students have already learnt.

Reflection

This guides students to think about different methods used to solve a problem.

Just For Fun

This contains puzzles, fascinating facts and interesting stories about mathematics as enrichment for students.

Attention

This contains important information that students should know.

Problem-solving Tip

This guides students on how to approach a problem in Worked Examples or Practise Now.

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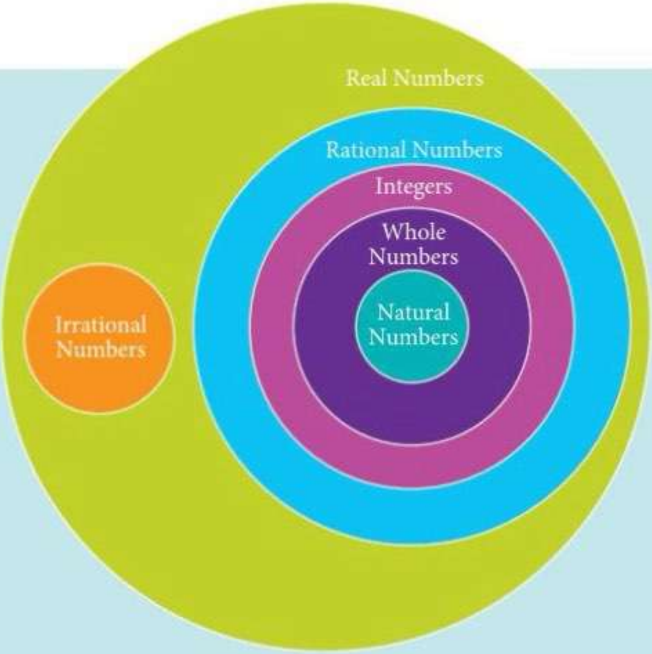
CHAPTER 1

Further Sets



Classification is an important strategy we use to make sense of the relationships between different types of objects. In biology, we can classify animals according to their habitats, i.e. land, water, or both. In music, we can classify the different musical pieces according to their genres. When we classify, we put things into different groups or collections so that we can see the properties of these things and examine the relationships between these different collections. In mathematics, we use the term 'set' to describe a collection of well-defined and distinct objects. For example, the set of natural numbers is the collection of numbers 1, 2, 3, and so on. The power of sets lies in how mathematicians represent these collections using precise set language, set **notations** and **diagrams** to clarify and communicate these relationships so that we can apply some of these ideas in real-world situations, such as more effective Internet searches.

Learning Outcomes

What will we learn in this chapter?

- How to use Venn diagrams to represent the intersection and union of two or three sets
- Why sets and Venn diagrams have useful applications in real life

Introductory Problem

In a class of 40 students, 15 students listen to electronic music, 24 students listen to rock music, and 12 students listen to neither. Find the number of students in the class who listen to both genres of music.

1.1 Intersection and union of two sets

A. Intersection of two sets

Consider the sets $A = \{1, 2, 3, 4, 5\}$ and $B = \{3, 5, 6, 7\}$.
How can we draw a Venn diagram to represent the sets A and B such that we do not repeat the common elements? Since all the elements in a set are distinct (i.e. we *cannot* write the same element more than once), we can draw the Venn diagram as shown in Fig. 1.1.

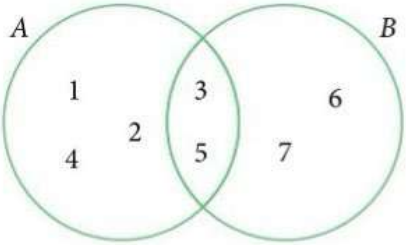


Fig. 1.1

We notice that the elements 3 and 5 are *common* to both sets A and B, and they lie in the **intersection** of A and B. We write $A \cap B = \{3, 5\}$, where $A \cap B$ is read as 'A intersect B'. Observe that $A \cap B$ is a *set*.

The **intersection** of sets A and B, denoted by $A \cap B$, is the set of all the elements which are common to both A and B.

Worked Example 1

Identifying elements belonging to the intersection of two sets

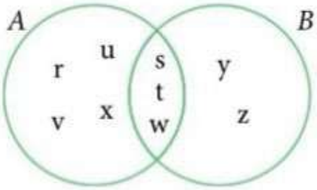
It is given that $A = \{r, s, t, u, v, w, x\}$ and
$B = \{s, t, w, y, z\}$.

(i) List all the elements of $A \cap B$ in set notation.
(ii) Draw a Venn diagram to represent the sets A and B.

Solution

(i) $A \cap B = \{s, t, w\}$

(ii)



Practise Now 1

Similar and Further Questions
Exercise 1A
Questions 1-3, 7-9, 13, 14

1. It is given that $C = \{a, e, f, g, i\}$ and
$D = \{b, e, g, h\}$.
(i) List all the elements of $C \cap D$ in set notation.
(ii) Draw a Venn diagram to represent the sets C and D.

2. It is given that $E = \{x : x \text{ is a multiple of 6 such that } 0 < x \leqslant 18\}$ and
$F = \{x : x \text{ is a multiple of 3 such that } 0 < x \leqslant 18\}$.
(i) List all the elements of E and of F in set notation.
(ii) Find $E \cap F$.
(iii) Draw a Venn diagram to represent the sets E and F.
(iv) Is $E \cap F = E$? Explain.

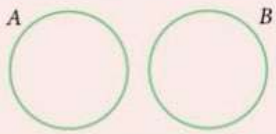
3. It is given that $G = \{x : x \text{ is a positive integer and a factor of } 12\}$ and
$H = \{x : x \text{ is a prime number between 5 and 13 inclusive}\}$.
(i) List all the elements of G and of H in set notation.
(ii) Find $G \cap H$. Explain.
(iii) Draw a Venn diagram to represent the sets G and H.

From Practise Now 1 Questions 2 and 3, we observe that:

If all the elements of A are also in B (i.e. A is a *subset* of B), then $A \cap B = A$.
If A and B do not share any common elements (i.e. A and B are *disjoint* sets), then $A \cap B = \varnothing$.

Attention

Two sets A and B are **disjoint sets** if they do not share any common elements.



B. Union of two sets

Consider the same sets $A = \{1, 2, 3, 4, 5\}$ and $B = \{3, 5, 6, 7\}$ as in Fig. 1.1, which is shown again below in Fig. 1.2.

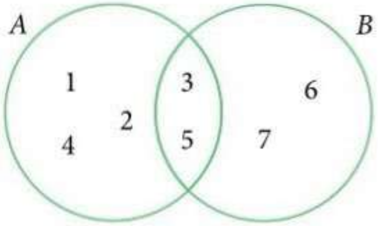


Fig. 1.2

If we list all the elements belonging to A or to B, we will get $\{1, 2, 3, 4, 5, 6, 7\}$.
This is called the **union** of A and B, and is denoted by $A \cup B$ (read as 'A union B'), i.e. $A \cup B = \{1, 2, 3, 4, 5, 6, 7\}$.

Observe that $A \cup B$ is also a *set*.

The **union** of sets A and B, denoted by $A \cup B$, is the set of all the elements belonging to A or to B.

Attention

In mathematics, all the elements belonging to A **or** to B *include* the elements belonging to both A **and** B.
We say 'A or B' for $A \cup B$. We do not say 'either A or B' because the word 'either' may imply 'not both'.
When writing the elements of $A \cup B$, we must not repeat the common elements because all elements in a set must be distinct.

Worked Example 2

Identifying elements belonging to the union of two sets

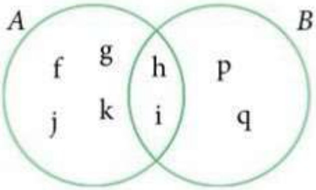
It is given that $A = \{f, g, h, i, j, k\}$ and
$B = \{h, i, p, q\}$.

(i) Draw a Venn diagram to represent the sets A and B.

(ii) From the Venn diagram, list all the elements of $A \cup B$ in set notation.

Solution

(i)



(ii) $A \cup B = \{f, g, h, i, j, k, p, q\}$

Problem-solving Tip

Identify $A \cap B$ before drawing the Venn diagram.

Practise Now 2

Similar and Further Questions
Exercise 1A
Questions 4–6, 10–12, 15

1. It is given that $C = \{\text{p, r, s, u, v}\}$ and
   $D = \{\text{q, r, t, u}\}$.
   (i) Draw a Venn diagram to represent the sets C and D.
   (ii) From the Venn diagram, list all the elements of $C \cup D$ in set notation.
2. It is given that $E = \{x : x \text{ is a positive integer and a factor of } 8\}$ and
   $F = \{x : x \text{ is a positive integer and a factor of } 16\}$.
   (i) List all the elements of E and of F in set notation.
   (ii) Draw a Venn diagram to represent the sets E and F.
   (iii) From the Venn diagram, find $E \cup F$.
   (iv) Is $E \cup F = F$? Explain.
3. It is given that $G = \{x : x \text{ is a multiple of 7 such that } 0 < x < 63\}$ and
   $H = \{x : x \text{ is a multiple of 9 such that } 0 < x < 63\}$.
   (i) List all the elements of G and of H in set notation.
   (ii) Draw a Venn diagram to represent the sets G and H.
   (iii) From the Venn diagram, find $G \cup H$.

From Practise Now 2 Question 2, we observe that:

If all the elements belonging to A also belong to B (i.e. A is a *subset* of B), then $A \cup B = B$.

Reflection

1. How do I explain the difference between the intersection and the union of two sets?
2. What have I learnt in this section that I am still unclear of?

Advanced Intermediate Basic

Exercise 1A

1. It is given that $A = \{1, 2, 3, 4, 7\}$ and
$B = \{2, 4, 8, 10\}$.
(i) List all the elements of $A \cap B$ in set notation.
(ii) Draw a Venn diagram to represent the sets A and B.

2. It is given that $C = \{\text{blue, green, yellow, orange, red, pink}\}$ and
$D = \{\text{yellow, pink, blue}\}$.
(i) List all the elements of $C \cap D$ in set notation.
(ii) Draw a Venn diagram to represent the sets C and D.

3. It is given that $E = \{p, q, r\}$ and
$F = \{s, t\}$.
(i) List all the elements of $E \cap F$ in set notation.
(ii) Draw a Venn diagram to represent the sets E and F.

4. It is given that $G = \{\text{apple, orange, banana, grape, durian, pear}\}$ and
$H = \{\text{apple, banana, grape, strawberry}\}$.
(i) Draw a Venn diagram to represent the sets G and H.
(ii) From the Venn diagram, list all the elements of $G \cup H$ in set notation.

Exercise 1A

5. It is given that $I = \{w, y\}$ and
$J = \{v, w, x, y, z\}$.

(i) Draw a Venn diagram to represent the sets I and J.

(ii) From the Venn diagram, list all the elements of $I \cup J$ in set notation.

6. It is given that $K = \{11, 13, 19, 21\}$ and
$L = \{12, 14, 15, 16, 17, 18, 20\}$.

(i) Draw a Venn diagram to represent the sets K and L.

(ii) From the Venn diagram, list all the elements of $K \cup L$ in set notation.

7. It is given that $M = \{x : x \text{ is a perfect square such that } 0 < x < 70\}$ and
$N = \{x : x \text{ is a perfect cube such that } 0 < x < 70\}$.

(i) List all the elements of M and of N in set notation.

(ii) Find $M \cap N$.

(iii) Draw a Venn diagram to represent the sets M and N.

8. It is given that $P = \{x : x \text{ is a multiple of 8 such that } 0 < x \leqslant 32\}$ and
$Q = \{x : x \text{ is a multiple of 4 such that } 0 < x \leqslant 32\}$.

(i) List all the elements of P and of Q in set notation.

(ii) Find $P \cap Q$.

(iii) Draw a Venn diagram to represent the sets P and Q.

(iv) Is $P \cap Q = P$? Explain.

9. It is given that $R = \{x : x \text{ is a positive integer and a factor of 18}\}$ and
$S = \{x : x \text{ is a composite number between 9 and 18}\}$.

(i) List all the elements of R and of S in set notation.

(ii) Find $R \cap S$. Explain.

(iii) Draw a Venn diagram to represent the sets R and S.

10. It is given that $T = \{x : x \text{ is a multiple of 3 such that } 0 < x \leqslant 18\}$ and
$V = \{x : x \text{ is a positive integer and a factor of 18}\}$.

(i) List all the elements of T and of V in set notation.

(ii) Draw a Venn diagram to represent the sets T and V.

(iii) Find $T \cup V$.

11. It is given that $W = \{x : x \text{ is a multiple of 4 such that } 1 \leqslant x < 16\}$ and
$X = \{x : x \text{ is a positive integer and a factor of 24}\}$.

(i) List all the elements of W and of X in set notation.

(ii) Draw a Venn diagram to represent the sets W and X.

(iii) From the Venn diagram, find $W \cup X$.

(iv) Is $W \cup X = X$? Explain.

12. It is given that $Y = \{x : x \text{ is a positive integer and a factor of 25}\}$ and
$Z = \{x : x \text{ is a multiple of 6 such that } 0 < x < 25\}$.

(i) List all the elements of Y and of Z in set notation.

(ii) Draw a Venn diagram to represent the sets Y and Z.

(iii) From the Venn diagram, find $Y \cup Z$.

13. It is given that $A = \{(x, y) : (x, y) \text{ are the coordinates of a point on the curve } y = x^2 - 3x + 2 \text{ such that } x \text{ and } y \text{ are integers}\}$ and
$B = \{(x, y) : (x, y) \text{ are the coordinates of a point on the line } y = 0 \text{ such that } x \text{ and } y \text{ are integers}\}$.
Describe $A \cap B$ in set notation.

14. It is given that $C = \{(x, y) : (x, y) \text{ lies on the curve } y = x^2 + x + 2\}$ and
$D = \{(x, y) : (x, y) \text{ lies on the line } y = 3x + 5\}$.
List all the elements of $C \cap D$ in set notation.

Exercise 1A

15. It is given that $E = \{y : y \text{ is the } y\text{-coordinate of a point on the curve } y = (x - 5)^2 + 3\}$ and $F = \{y : y \text{ is the } y\text{-coordinate of a point on the curve } y = (x + 2)^2 - 4\}$.
Describe $E \cup F$ in set notation.

1.2 Applications of sets in real-world contexts

In this section, we will learn how to apply the concept of sets and Venn diagrams to solve mathematical and real-life problems.

Worked Example 3

Solving problem involving universal set, and the intersection and union of two sets

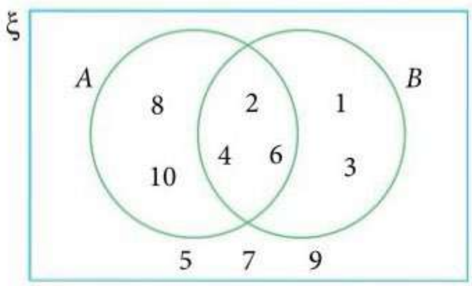
$\xi = \{x : x \text{ is a positive integer less than } 11\}$
$A = \{x : x \text{ is an even number}\}$
$B = \{x : x \text{ is a factor of } 12\}$

(i) List all the elements of ξ, A and B in set notation.
(ii) Draw a Venn diagram to represent the sets ξ, A and B.
(iii) Find
  (a) $(A \cup B)'$,
  (b) $A \cap B'$.

Solution

(i) $\xi = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
$A = \{2, 4, 6, 8, 10\}$
$B = \{1, 2, 3, 4, 6\}$

(ii)



(iii) **(a)** $(A \cup B)' = \{5, 7, 9\}$

(b) **Method 1:**
$A = \{2, 4, 6, 8, 10\}$
$B' = \{5, 7, 8, 9, 10\}$
$\therefore A \cap B' = \{8, 10\}$

Method 2:
From the Venn diagram,
$A \cap B' = \{8, 10\}$

Problem-solving Tip

(i) Since ξ is the set of positive integers less than 11, then A cannot contain all even numbers, but only those that are positive and less than 11.
Similarly, B does not contain 12 since 12 is not an element of ξ.
(ii) To draw a Venn diagram, always fill in the elements of $A \cap B$ first.
(iii) **(b)** For **Method 2**, identify the region $A \cap B'$ in the Venn diagram, i.e. the region in set A that does not belong to set B.

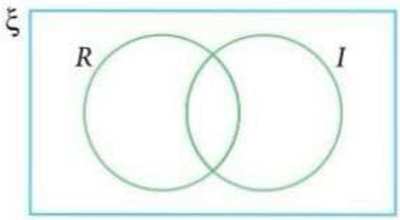
Practise Now 3

Similar and Further Questions
Exercise 1B
Questions 1-3, 8

1. $\xi = \{x : x \text{ is a positive integer not more than } 9\}$
$A = \{x : x \text{ is an odd number}\}$
$B = \{x : x \text{ is a multiple of } 3\}$

(i) List all the elements of ξ, A and B in set notation.
(ii) Draw a Venn diagram to represent the sets ξ, A and B.
(iii) Find
(a) $(A \cup B)'$, **(b)** $A \cap B'$.

2. $\xi = \{x : x \text{ is a triangle}\}$
$R = \{x : x \text{ is a right-angled triangle}\}$
$I = \{x : x \text{ is a triangle with exactly two equal sides}\}$
C is a triangle of sides 5 cm, 8 cm and 8 cm.
D is a right-angled isosceles triangle.
E is a triangle of sides 3 cm, 4 cm and 5 cm.
F is an equilateral triangle.
G is a triangle of sides 7 cm, 10 cm and 12 cm.
On the Venn diagram below, write C, D, E, F and G in the appropriate subsets.



Worked Example 4

Solving problem involving the intersection and union of three sets

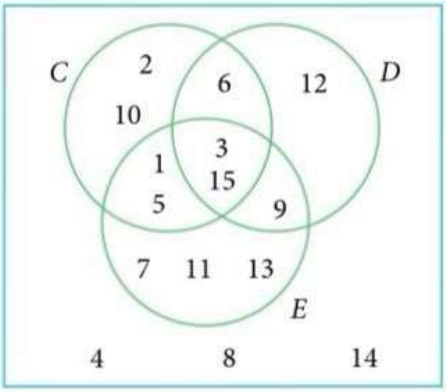
$\xi = \{x : x \text{ is an integer less than } 16\}$
$C = \{x : x \text{ is a factor of } 30\}$
$D = \{x : x \text{ is a multiple of } 3\}$
$E = \{x : x \text{ is an odd number}\}$

(i) List all the elements of ξ, C, D and E in set notation.
(ii) Draw a Venn diagram to represent the sets ξ, C, D and E.
(iii) Find
(a) $(C \cap E) \cup D$, **(b)** $(C \cup D)' \cup E$.

Solution

(i) $\xi = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}$
$C = \{1, 2, 3, 5, 6, 10, 15\}$
$D = \{3, 6, 9, 12, 15\}$
$E = \{1, 3, 5, 7, 9, 11, 13, 15\}$

(ii) ξ



Problem-solving Tip

Identify $C \cap D \cap E$, $C \cap D$, $D \cap E$ and $C \cap E$ before drawin the Venn diagram.

(iii) (a) $(C \cap E) \cup D = \{1, 3, 5, 6, 9, 12, 15\}$

(b) $(C \cup D)' \cup E = \{1, 3, 4, 5, 7, 8, 9, 11, 13, 14, 15\}$

Practise Now 4

Similar and Further Questions
Exercise 1B
Questions 9, 10(a), (b), 22, 23

$\xi = \{x : x \text{ is an integer between 1 and 20 inclusive}\}$
$F = \{x : x \text{ is a factor of 20}\}$
$G = \{x : x \text{ is a multiple of 2}\}$
$H = \{x : x \text{ is a factor of 144}\}$

(i) List all the elements of ξ, F, G and H in set notation.

(ii) Draw a Venn diagram to represent the sets ξ, F, G and H.

(iii) Find

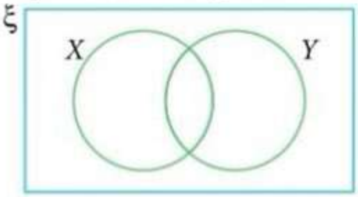
(a) $F \cap (G \cup H)'$, **(b)** $F \cup (G \cup H)'$,

Worked Example 5

Shading of region in Venn diagram

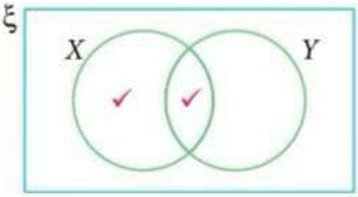
Shade the following regions on separate Venn diagrams.

(i) $X \cup Y'$
(ii) $X \cap Y'$
(iii) $(X \cap Y)'$

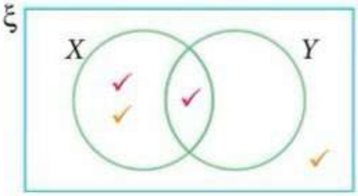


Solution

(i) Step 1: Put a tick in each of the two regions for X.

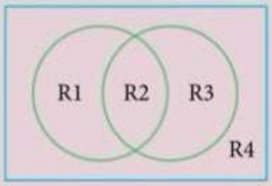


Step 2: Put a tick in each of the two regions for Y'.

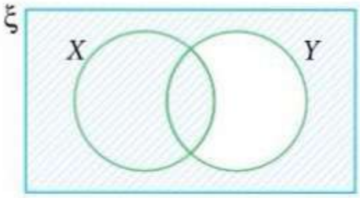


Problem-solving Tip

The diagram is divided into 4 disjoint regions, indicated as R1, R2, R3 and R4 below:



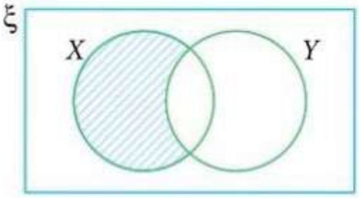
Step 3: As we want a union, shade all the regions with at least one tick.



Problem-solving Tip

$X \cup Y'$ is the set of all the elements belonging to X **or** to Y', so shade all the regions *with at least one tick*.

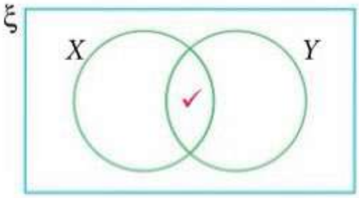
(ii) From part **(i)**, as we want an intersection, shade all the regions with exactly two ticks. Then erase the ticks in the unshaded region.



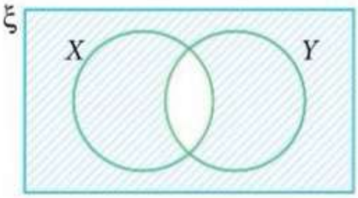
Problem-solving Tip

$X \cap Y'$ is the set of all the elements common to both X **and** Y', so shade all the regions *with two ticks*.

(iii) Step 1: Put a tick in the region for $X \cap Y$.



Step 2: As we want a complement of $X \cap Y$, shade all the regions without any tick. Then erase the tick.



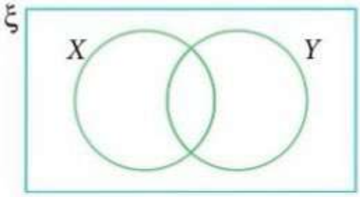
Problem-solving Tip

$(X \cap Y)'$ is the set of all the elements belonging to ξ but **not to** $X \cap Y$, so shade all the regions *without any tick*.

Practise Now 5

Similar and Further Questions
Exercise 1B
Questions 4, 11-13, 14(a)-(f), 24, 25

Shade the following regions on separate Venn diagrams.



(i) $X' \cup Y$ **(ii)** $X' \cap Y$ **(iii)** $(X \cup Y)'$
(iv) $X' \cup Y'$ **(v)** $X' \cap Y'$ **(vi)** $(X \cap Y')'$

Thinking Time

Using your answers in Practise Now 5, answer each of the following.

1. Is $(X \cup Y)'$ equal to $X' \cup Y'$ or $X' \cap Y'$?

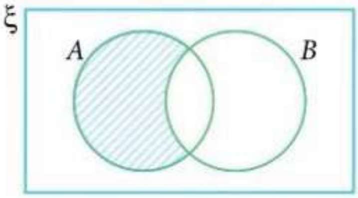
2. Is $(X \cap Y)'$ equal to $X' \cap Y'$ or $X' \cup Y'$?

Worked Example 6

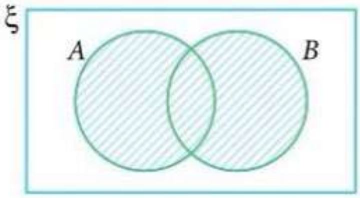
Describing shaded region in Venn diagram

Describe the shaded region in each of the following Venn diagrams in set notation.

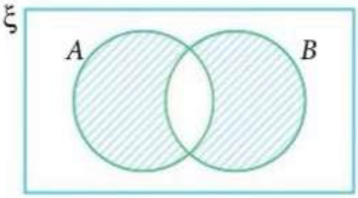
(a)



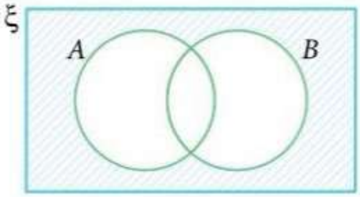
(b)



(c)



(d)



Solution

(a) Ask yourself the following questions:

1. "Is the shaded region inside A or outside A?"
   Answer: Inside A
2. "Is the shaded region inside B or outside B?"
   Answer: Outside B
3. "Is the shaded region 'inside A and outside B', or 'inside A or outside B'?"
   Answer: Inside A and outside B

Therefore, the shaded region is $A \cap B'$.

(b) Ask yourself the following questions:

1. "Is the shaded region inside A or outside A?"
   Answer: Inside A
2. "Is the shaded region inside B or outside B?"
   Answer: Inside B
3. "Is the shaded region 'inside A and inside B', or 'inside A or inside B'?"
   Answer: Inside A or inside B (or both)

Therefore, the shaded region is $A \cup B$.

(c) Ask yourself the following questions:

1. "Is the shaded region inside A or outside A?"
   Answer: Inside A
2. "Is the shaded region inside B or outside B?"
   Answer: Inside B

3. "Is the shaded region 'inside A and inside B', or 'inside A or inside B'?"
   Answer: Inside A or inside B (but not both)
   In other words, this is unlike part **(b)**, so it cannot be $A \cup B$.

The shaded region in part **(a)**, which is $A \cap B'$, is the left shaded region in part **(c)**. Thus the right shaded region in part **(c)** is $A' \cap B$. Therefore, the shaded region in part **(c)** is $(A \cap B') \cup (A' \cap B)$.

Problem-solving Tip

It is important to put brackets to indicate which operations in $(A \cap B') \cup (A' \cap B)$ we want to do first, because $(A \cap B) \cup C \neq A \cap (B \cup C)$.

(d) Ask yourself the following questions:

1. "Does the unshaded region look familiar? Can I find the unshaded region easily?"
   Answer: Yes, the unshaded region is $A \cup B$ (refer to part **(b)**).

Therefore, the shaded region is $(A \cup B)'$.

Reflection

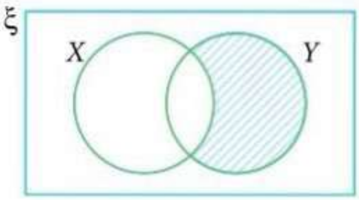
What useful strategies can you learn from the four parts in Worked Example 6 to help you describe any shaded region using set notation?

Practise Now 6

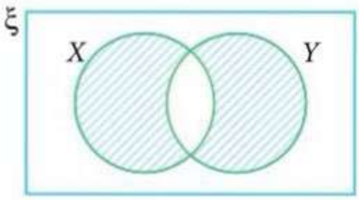
Similar and Further Questions
Exercise 1B
Questions 5(a)–(f), 15(a)–(j), 26

Describe the shaded region in each of the following Venn diagrams in set notation.

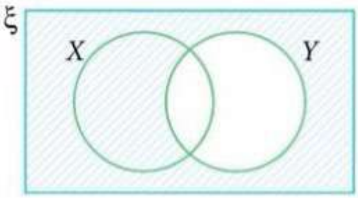
(a)



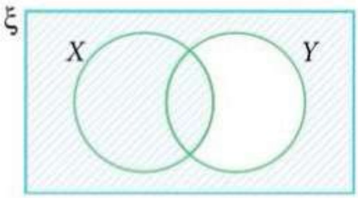
(b)



(c)



(d)



Worked Example 7

Solving problem using Venn diagram

In a class of 40 students, 16 students play soccer, 14 students play basketball, and 5 students play both. How many students in the class play neither soccer nor basketball?

Solution

We will use **Pólya's Problem Solving Model** to guide us in solving this problem.

Stage 1: Understand the problem

What information is given and what is not?

- What is given: total number of students in the class; number of students who play each of the two sports; number of students who play both sports
- What is not given: number of students who only play soccer; number of students who only play basketball

What are we supposed to find?

- Number of students who play neither of the two sports

Stage 2: Think of a plan

To find the number of students who play neither of the two sports, we need to find the number of students who play either soccer or basketball or both.

We cannot just add 16 and 14 to give 30 since there are 5 students who play both sports.

Since we want to find the number of students who only play soccer and those who only play basketball, can we use a Venn diagram to help us visualise and find the information easily?

Yes, we can. Draw a Venn diagram to represent the sets (see diagram in Stage 3).

Stage 3: Carry out the plan

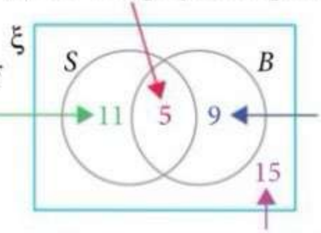
Let S and B denote the set of students who play soccer and basketball respectively.

The Venn diagram shows the number of students belonging to each region.

Attention

A Venn diagram can be used to represent the elements (as we have learnt earlier) or the number of elements (as shown in Worked Example 7).

Step 1: Fill in the number of students who play both sports = 5



Step 2: Fill in the number of students who only play soccer = 16 – 5 = 11

Step 3: Fill in the number of students who only play basketball = 14 – 5 = 9

Step 4: Fill in the number of students who play neither of the two sports = 40 – 11 – 5 – 9 = 15

Big Idea

Diagrams

A Venn diagram can be used to help us solve problems because it presents information in a way that allows us to obtain other information from it.

Stage 4: Look back

How can we check whether the answers are correct?

We can add up the numbers in each region to see if they tally with the given information:

- Number of students who play soccer = 11 + 5 = 16
- Number of students who play basketball = 5 + 9 = 14
- Total number of students in the class = 11 + 5 + 9 + 15 = 40

Are there other ways to solve this problem? For example, do we really need to do Step 3?

Step 3 is not necessary to solve the problem. Once we determine that there are 11 students who only play soccer, we can find the number of students who play neither of the two sports by doing this: 40 – 11 – 14 = 15. However, we complete the Venn diagram for purposes of illustration.

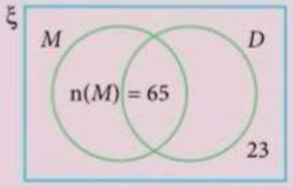
Practise Now 7

Similar and Further Questions
Exercise 1B
Questions 6, 7, 16, 17, 27–31

1. In a class of 38 students, 11 students study Physics, 20 students study Chemistry, and 8 students study both. Find the number of students in the class who study neither Physics nor Chemistry.
2. There are 117 participants in a conference, of whom 65 like movies and 23 like neither movies nor dramas. How many participants of the conference like dramas but not movies?

Problem-solving Tip

2. Let M and D denote the set of participants who like movies and dramas respectively.
We can label the Venn diagram like this:



Introductory Problem Revisited

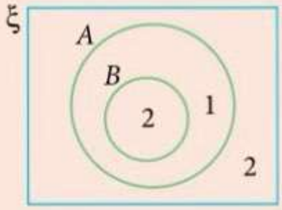
After learning how to use Venn diagrams to solve problems, can you solve the Introductory Problem?

Investigation

Exploring related sets

Part 1

The sets ξ, A and B satisfy the conditions $n(\xi) = 5$, $n(A) = 3$ and $n(B) = 2$.



1. Draw Venn diagrams to show all the ways in which the sets can be related, indicating clearly the number of elements in each region.
An example of such a diagram is shown.

2. Using your Venn diagrams in Question 1, complete the following:

(i) $n(A \cap B)$ is the largest when ____ $\subset$ ____.
The largest value of $n(A \cap B)$ is ____.

(ii) $n(A \cup B)$ is the smallest when ____ $\subset$ ____.
The smallest value of $n(A \cup B)$ is ____.

(iii) $n(A \cap B)$ is the smallest when $A \cap B =$ ____.
The smallest value of $n(A \cap B)$ is ____.

(iv) $n(A \cup B)$ is the largest when $A \cap B =$ ____.
The largest value of $n(A \cup B)$ is ____.

Part 2

The sets ξ, A and B satisfy the conditions $n(\xi) = 6$, $n(A) = 4$ and $n(B) = 3$.

3. Draw Venn diagrams to show all the ways in which the sets can be related, indicating clearly the number of elements in each region.

4. Using your Venn diagrams in Question 3, complete the following:

(i) $n(A \cap B)$ is the largest when ____ $\subset$ ____.
The largest value of $n(A \cap B)$ is ____.

(ii) $n(A \cup B)$ is the smallest when ____ $\subset$ ____.
The smallest value of $n(A \cup B)$ is ____.

(iii) $n(A \cap B)$ is the smallest when $A \cup B =$ ____.
The smallest value of $n(A \cap B)$ is ____.

(iv) $n(A \cup B)$ is the largest when $A \cup B =$ ____.
The largest value of $n(A \cup B)$ is ____.

From the Investigation on page 14, we observe the following:

> Suppose $n(A) \geqslant n(B)$.
> $n(A \cap B)$ is the largest and $n(A \cup B)$ is the smallest when $B \subseteq A$ (i.e. when B is a **subset** of A).
> $n(A \cap B)$ is the smallest and $n(A \cup B)$ is the largest when
> - $A \cap B = \varnothing$ (i.e. A and B are **disjoint sets**), if $n(A) + n(B) \leqslant n(\xi)$;
> - $A \cup B = \xi$, if $n(A) + n(B) > n(\xi)$ (i.e. A and B cannot be disjoint sets).

Worked Example 8

Related sets

The sets ξ, A and B satisfy the conditions $n(\xi) = 16$, $n(A) = 7$ and $n(B) = 14$. Find

(i) the largest possible value of $n(A \cup B)'$,

(ii) the smallest possible value of $n(A \cup B)'$.

Solution

(i) $n(A \cup B)'$ is largest when $n(A \cup B)$ is smallest.
This will occur when $A \subseteq B$.

$$\therefore \text{largest possible value of } n(A \cup B)' = 16 - 14$$
$$= 2$$

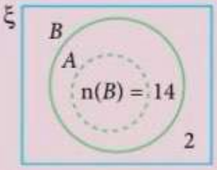
(ii) $n(A \cup B)'$ is smallest when $n(A \cup B)$ is largest.
This will occur when $A \cup B = \xi$ (since A and B cannot be disjoint sets), i.e. $n(A \cup B) = n(\xi) = 16$.

$$\therefore \text{smallest possible value of } n(A \cup B)' = 0$$

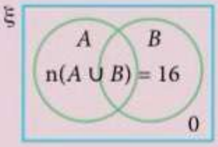
Problem-solving Tip

We can draw a Venn diagram to help us visualise.

(i)



(ii)



Practise Now 8

Similar and Further Questions
Exercise 1B
Questions 18–21, 32, 33

1. The sets ξ, A and B satisfy the conditions $n(\xi) = 23$, $n(A) = 15$ and $n(B) = 11$. Find
   - **(i)** the largest possible value of $n(A \cup B)'$,
   - **(ii)** the smallest possible value of $n(A \cup B)'$.
2. In a class of 36 students, 8 students enjoy reading comic books and 11 students enjoy reading adventure story books. Find
   - **(i)** the largest possible number of students who enjoy reading comic books or adventure story books,
   - **(ii)** the smallest possible number of students who do not enjoy reading comic books or adventure story books,
   - **(iii)** the largest possible number of students who enjoy reading comic books and adventure story books,
   - **(iv)** the smallest possible number of students who enjoy reading adventure story books but not comic books.

Performance Task

Find out from your classmates how they usually travel to school every morning.

Present your findings on a vanguard sheet by drawing a Venn diagram to display the following sets where appropriate.

$A = \{x : x$ is a student in your class who travels to school by public bus$\}$
$B = \{x : x$ is a student in your class who travels to school by train$\}$
$C = \{x : x$ is a student in your class who travels to school by chartered bus$\}$
$D = \{x : x$ is a student in your class who travels to school by taxi$\}$
$E = \{x : x$ is a student in your class who travels to school by car$\}$
$F = \{x : x$ is a student in your class who travels to school by foot only$\}$
$G = \{x : x$ is a student in your class who travels to school by helicopter$\}$
$H = \{x : x$ is a student in your class who travels to school by other modes of transport not stated above$\}$

1. Do you want to include yourself in the above survey? Explain.
2. What would you take as the universal set ξ?
3. Are there any empty sets? Do you want to include an empty set in your Venn diagram?
4. Every student will have to walk a certain distance, e.g. from the bus stop to the school. Do you want to include the option of walking in the above survey? Explain.
5. Do some sets intersect each other? How would you represent these sets in the Venn diagram?

Reflection

1. How do I decide on which region in a Venn diagram to shade when given the corresponding set notation?
2. What are some guiding questions that can help me to describe a region in a Venn diagram using set notation?
3. What have I learnt in this section or chapter that I am still unclear of?

Basic Intermediate Advanced

Exercise 1B

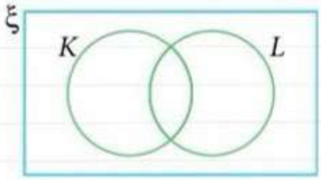
1. $\xi = \{x : x$ is a positive integer less than $16\}$
$I = \{x : x$ is a multiple of $4\}$
$J = \{x : x$ is a factor of $8\}$
(i) List all the elements of ξ, I and J in set notation.
(ii) Draw a Venn diagram to represent the sets ξ, I and J.
(iii) From the Venn diagram, find
(a) $(I \cup J)'$, **(b)** $I \cap J'$.

2. $\xi = \{x : x$ is a positive integer such that $3 < x \leqslant 18\}$
$Y = \{x : x$ is a multiple of $3\}$
$Z = \{x : x$ is a multiple of $9\}$
(i) List all the elements of ξ, Y and Z in set notation.
(ii) Draw a Venn diagram to represent the sets ξ, Y and Z.
(iii) From the Venn diagram, find
(a) $(Y \cup Z)'$, **(b)** $Y \cap Z'$.

Exercise 1B

3. $\xi = \{x : x \text{ is a non-negative integer less than } 12\}$
$P = \{x : x \text{ is a prime number}\}$
$Q = \{x : x \text{ is a composite number}\}$
(i) List all the elements of ξ, P and Q in set notation.
(ii) Draw a Venn diagram to represent the sets ξ, P and Q.
(iii) From the Venn diagram, find
(a) $P \cup Q$, **(b)** $(P \cup Q)'$,
(c) $P' \cap Q$.

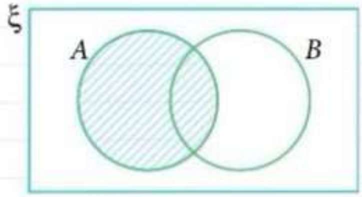
4. Shade the following regions on separate Venn diagrams.



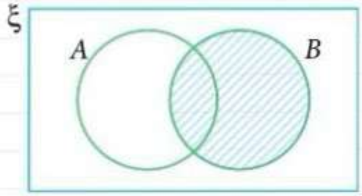
(i) $K \cap L$ **(ii)** $K \cup L$
(iii) $(K \cup L)'$

5. Describe the shaded region in each of the following Venn diagrams in set notation.

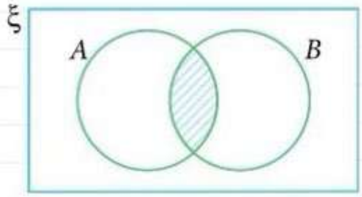
(a)



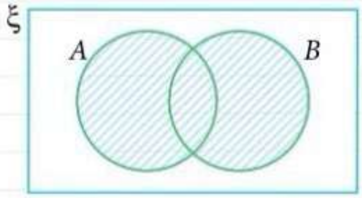
(b)



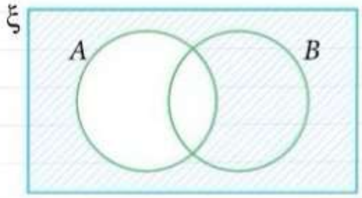
(c)



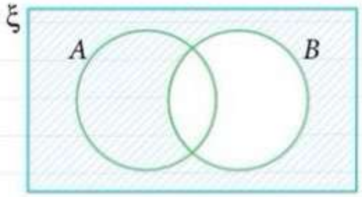
(d)



(e)



(f)

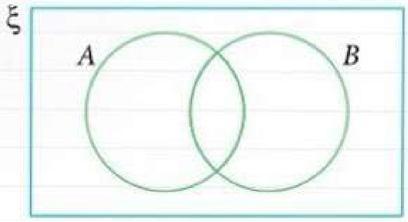


6. 253 participants were asked to complete a survey about their fruit preferences. 179 of them like durian while 43 of them like neither durian nor apple. How many participants of the survey like apple but not durian?

7. In a class of 38 students, 19 students have travelled to Malaysia while 4 students have neither travelled to Malaysia nor Indonesia. Find the number of students in the class who have travelled to Indonesia but not Malaysia.

8. $\xi = \{x : x \text{ is a quadrilateral}\}$
$A = \{x : x \text{ is a rhombus}\}$
$B = \{x : x \text{ is a quadrilateral with four equal angles}\}$
Q is a rectangle of sides 12 cm by 10 cm.
R is a rhombus such that two of its angles are 30° and 150°.
S is a square of sides 8 cm.
T is a trapezium of sides 5 cm, 5 cm, 5 cm and 9 cm.
On the Venn diagram, write Q, R, S and T in the appropriate subsets.

Exercise 1B



9. $\xi = \{a, b, d, e, g, i, n, s, t, u\}$
$A = \{a, g, i, s\}$
$B = \{e, i, n, s, t\}$
$C = \{e, g, s, u\}$

(i) Draw a Venn diagram to represent the sets ξ, A, B and C.

(ii) Find

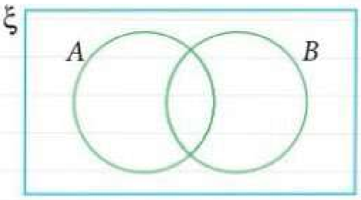
(a) $(A \cup B) \cap C$, **(b)** $(A \cap B) \cup C$.

10. **(a)** If $A = \{1, 2, 3, 4, 5\}$, $B = \{2, 4, 6, 8\}$ and $C = \{1, 3, 5, 7\}$, list the elements of the following.

(i) $(A \cup B) \cup C$ (ii) $(A \cup C) \cap B$

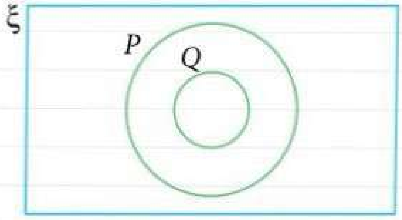
(b) Given that $D = \{a, b, c, d\}$, $E = \{a, b, c\}$ and $F = \{a, e\}$, find $D \cap (E \cup F)$.

11. Shade the following regions on separate Venn diagrams.



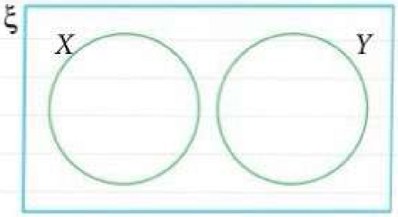
(i) $A \cup B'$ **(ii)** $A \cap B'$
(iii) $(A \cup B)'$ **(iv)** $(A \cap B)'$
(v) $A' \cup B'$ **(vi)** $A' \cap B'$
(vii) $(A' \cup B)'$ **(viii)** $(A \cup B')'$

12. Shade the following regions on separate Venn diagrams.



(i) $P \cup Q'$ **(ii)** $P \cap Q'$
(iii) $(P \cup Q)'$ **(iv)** $(P \cap Q)'$
(v) $P' \cup Q'$ **(vi)** $P' \cap Q'$
(vii) $(P' \cup Q)'$ **(viii)** $(P \cup Q')'$

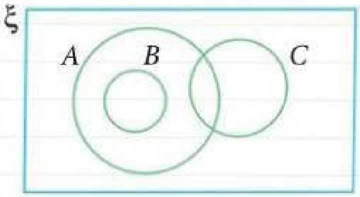
13. Shade the following regions on separate Venn diagrams.



(i) $X \cup Y'$ **(ii)** $X \cap Y'$
(iii) $(X \cup Y)'$ **(iv)** $(X \cap Y)'$
(v) $X' \cup Y'$ **(vi)** $X' \cap Y'$
(vii) $(X' \cup Y)'$ **(viii)** $(X \cup Y')'$

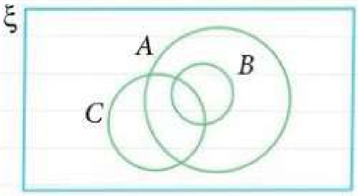
14. For each of the following Venn diagrams, shade the regions representing each set.

(a)



$C \cap B' \cap A$

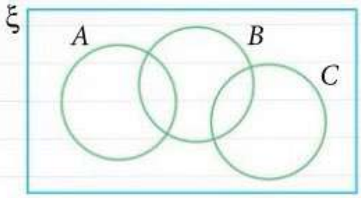
(b)



$C \cup (A \cap B)$

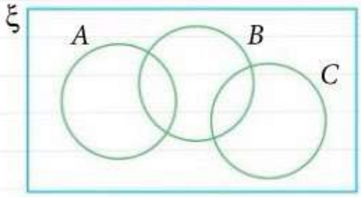
Exercise 1B

(c)



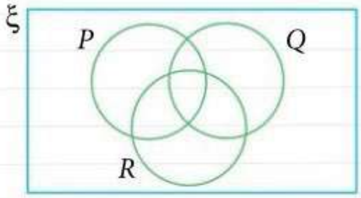
(A ∩ B) ∪ C

(d)



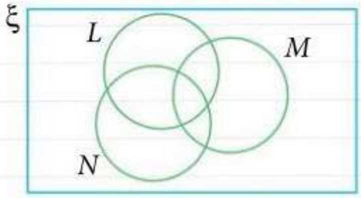
A′ ∩ C′ ∩ B

(e)



(P ∪ Q) ∩ R

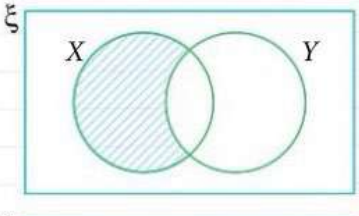
(f)



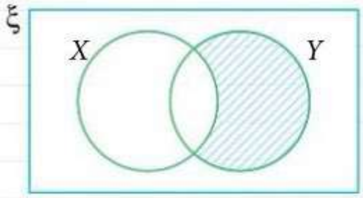
L′ ∩ (M ∪ N)

15. Describe the shaded region in each of the following Venn diagrams in set notation.

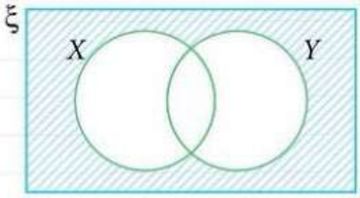
(a)



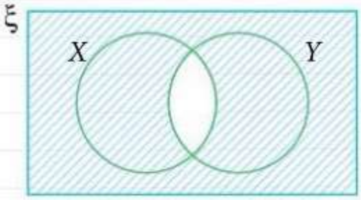
(b)



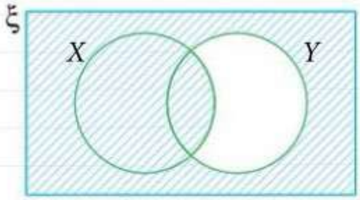
(c)



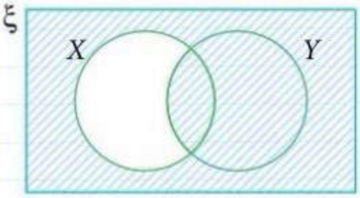
(d)



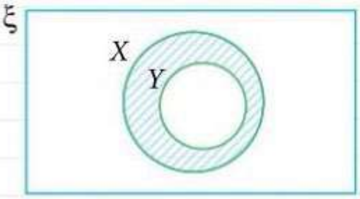
(e)



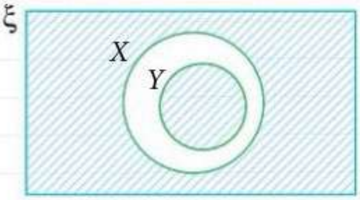
(f)



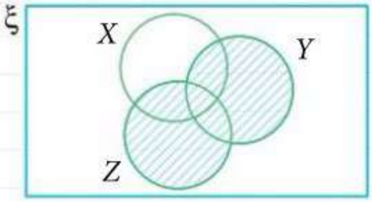
(g)



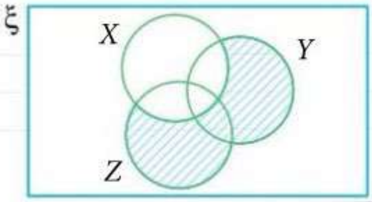
(h)



(i)



(j)



Exercise 1B

16. In a class of 37 students, 12 students like novels, 19 students like comic books, and 8 students like both. How many students in the class like neither novels nor comic books?

17. In a group of people surveyed, 23 of them own laptops, 17 of them own desktop computers, and 9 of them own both types of computers. How many people in the group own either laptops or desktop computers, but not both?

18. The sets ξ, A and B satisfy the conditions $n(\xi) = 17$, $n(A) = 8$ and $n(B) = 12$. Find

(i) the largest possible value of $n(A \cup B)'$,

(ii) the smallest possible value of $n(A \cup B)'$.

19. The sets ξ, X and Y satisfy the conditions $n(\xi) = 25$, $n(X) = 17$ and $n(Y) = 9$. Find

(i) the smallest possible value of $n(X \cap Y)'$,

(ii) the largest possible value of $n(X \cap Y)'$.

20. The sets ξ, A and B satisfy the conditions $n(\xi) = 25$, $n(A) = 15$ and $n(B) = 9$. Find

(i) the largest possible value of $n(A \cup B)'$,

(ii) the smallest possible value of $n(A \cup B)'$.

21. The sets ξ, X and Y satisfy the conditions $n(\xi) = 11$, $n(X) = 4$ and $n(Y) = 7$. Find

(i) the smallest possible value of $n(X \cap Y)'$,

(ii) the largest possible value of $n(X \cap Y)'$.

22. Illustrate on a Venn diagram, the sets P, Q and R if $Q \subseteq P$, $R \subseteq Q'$ and $P \cap R \neq \varnothing$.

23. The universal set ξ is the set of all triangles. Given that $S = \{x : x \text{ is an isosceles triangle}\}$, $T = \{x : x \text{ is an equilateral triangle}\}$ and $U = \{x : x \text{ is an obtuse-angled triangle}\}$, draw a Venn diagram to represent the sets ξ, S, T and U.

24. For any set A, simplify the following if possible.

(i) $A \cap \xi$ **(ii)** $A \cup \xi$

(iii) $A \cap \varnothing$ **(iv)** $A \cup \varnothing$

25. If $A \subset B$ and $A \cap C = \varnothing$, simplify the following if possible.

(i) $A \cap B$ **(ii)** $A \cup B$

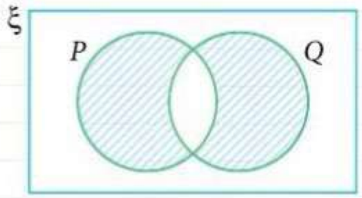
(iii) $B \cap C$ **(iv)** $A \cup C$

(v) $(B \cup C) \cap A$ **(vi)** $(B \cap C) \cap A$

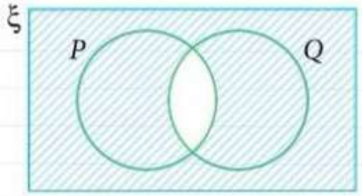
(vii) $(A \cup C) \cap B$ **(viii)** $(A \cap C) \cup B$

Hint: Draw a Venn diagram to visualise.

26. **(i)** Describe the shaded region in the Venn diagram in set notation.



(ii) Using your answer to part **(i)**, describe the shaded region in the Venn diagram in set notation.



27. In a class of 35 students, 11 students like noodles, 21 students like rice, and 7 like neither. Find the number of students who like both noodles and rice.

28. 40 working adults were asked to complete a survey about their modes of transport to work. 12 of them commute to work by bus, 25 of them commute to work by train, and 8 of them do not commute to work by bus or by train. Find the number of working adults who commute to work either by bus or by train, but not both.

Exercise 1B

29. A group of 90 people is surveyed on their hobbies and the results are shown below.

Dancing: 43
Sewing: 42
Swimming: 48
Dancing and sewing: 16
Sewing and swimming: 17
Swimming and dancing: 22

Each youth has at least one hobby. It is given that x youths have three hobbies.

(i) Draw a Venn diagram to represent the above information, indicating the number in each separate region, in terms of x where necessary.

(ii) Write down an equation in x and show that it simplifies to $x + 78 = 90$.

(iii) Hence, solve the equation $x + 78 = 90$.

30. In a medical test, 68 elderly men were diagnosed with least one of the following diseases: heart disease, lung disease and kidney disease. The information below shows the numbers of men with the diseases.

Heart disease: 30
Lung disease: 30
Kidney disease: 33
Heart and lung diseases: 7
Lung and kidney diseases: 10
Heart and kidney diseases: 11

Let x be the number of men diagnosed with all three diseases.

(i) Draw a Venn diagram to represent the above information, indicating the number in each separate region, in terms of x where necessary.

(ii) By forming an equation in x and solving it, find the value of x.

(iii) Hence, how many men were diagnosed with only lung disease?

31. In a certain school, students must study at least one of the following subjects: Mathematics, Science or Geography. Among a group of 40 students, the numbers of students who study the subjects are given below.

Mathematics: 20
Science: 22
Geography: 28
Mathematics and Science: 12
Science and Geography: 14
Mathematics and Geography: 15

How many students study all three subjects?

32. In a class of 38 students, 15 students enjoy cycling and 9 students enjoy swimming. Find

(i) the largest possible number of students who enjoy cycling or swimming,

(ii) the smallest possible number of students who do not enjoy cycling or swimming,

(iii) the largest possible number of students who enjoy cycling and swimming,

(iv) the smallest possible number of students who enjoy swimming but not cycling.

33. In a group of 35 children, 13 of them enjoy playing board games and 28 of them enjoy playing card games. Find

(i) the smallest possible number of children who enjoy playing board games and card games,

(ii) the smallest possible number of children who do not enjoy playing board games or card games,

(iii) the smallest possible number of children who enjoy playing board games or card games,

(iv) the smallest possible number of children who enjoy playing card games but not board games.

Looking Back

In this chapter, we have explored more ideas related to set theory, which is largely based on the work of Georg Cantor, a famous mathematician, in the 1870s. Using set language and set notations, he was able to show that there are more real numbers than there are natural numbers, which opened the door to the investigation of the different sizes of infinity. The development of set theory helped lay the foundations of abstract mathematics, which provide ways for mathematicians to communicate their ideas clearly, without any ambiguity. It is the high levels of precision and accuracy of ideas communicated via set language and notation that make mathematics useful for modelling relationships and structures in the real world.

Summary

1. **(a)** The **intersection** of sets A and B is the set of all the elements which are *common* to both A and B. It is denoted by $A \cap B$.

(b) The **union** of sets A and B is the set of all the elements belonging to A or to B. It is denoted by $A \cup B$. In mathematics, all the elements belonging to A or to B *include* the elements belonging to both A and B.

- $\xi = \{1, 2, 3, 4, 5, 6, 7\}$
  $A = \{2, 3, 5, 7\}$
  $B = \{1, 3, 5\}$
  Draw a Venn diagram to represent the sets ξ, A and B, and list all the elements of $A \cap B$ and of $A \cup B$ in set notation.

2. Suppose $n(A) \geqslant n(B)$.
$n(A \cap B)$ is the largest and $n(A \cup B)$ is the smallest when $B \subseteq A$ (i.e. when B is a **subset** of A).
$n(A \cap B)$ is the smallest and $n(A \cup B)$ is the largest when

- $A \cap B = \varnothing$ (i.e. A and B are **disjoint sets**), if $n(A) + n(B) \leqslant n(\xi)$;
- $A \cup B = \xi$, if $n(A) + n(B) > n(\xi)$ (i.e. A and B cannot be disjoint sets).

CHAPTER 2

Probability of Combined Events

In Book 2, we explored the idea of probability as a **measure** of chance, or more specifically as a way to quantify the likelihood of an event happening. In this chapter, we are going to explore further how we can apply our basic idea of probability to calculate the probability of combined events and use **diagrams** to represent these combined events. This opens new possibilities for us to analyse the likelihood of different outcomes in games, sports, and family trees! Beyond games, the ideas in this chapter lay the foundation for us to **model** real-world situations such as the population of endangered animals using probability. For example, environmentalists often use the method of capture-recapture to estimate the population of an animal species. This method involves capturing and tagging a certain number of animals of a particular species, and then releasing them back into their habitat. High-speed cameras are then set up at different points in the habitat to capture images of the animals. By observing the number of animals of that species recorded by the cameras and the number of tagged animals among them, their population in the habitat can be estimated. Can you figure out how this can be done? What is the assumption behind this method?

Learning Outcomes

What will we learn in this chapter?

- What sample space diagrams and tree diagrams are
- How to calculate the probability of combined events using sample space diagrams and tree diagrams
- How to use the Addition Law of Probability to solve problems involving mutually exclusive events
- How to use the Multiplication Law of Probability to solve problems involving independent events
- Why probability has useful applications in real life

Introductory Problem

Two fair 6-sided dice were rolled together and the difference between the resulting numbers on their faces was calculated. Find the probability that the difference between the two numbers is a prime number.

In the Introductory Problem, how did you display the sample space? Other than listing the sample space using the braces { }, is there a better way to display the sample space together with the differences between any two numbers? In this chapter, we will learn how to do that and to find the probabilities of combined events. But first, let us recap what we have learnt about finding the probabilities of single events.

2.1 Probability of single event

In this section, we will revise what we learnt in Book 2 on finding the probability of *single events*, and the use of *set notations* to describe the sample space, events and probability.

A. Sample space and event

We learnt in Book 2 that a sample space is the collection of all the possible outcomes of a probability experiment.
In set language, a **sample space** is the *set* of all the possible outcomes of a probability experiment, i.e. it is the *universal set*. We usually denote the sample space by S.
For example, consider a probability experiment in which a *fair* die is rolled.
Since every number from 1 to 6 is a possible outcome of this experiment, the sample space is $S = \{1, 2, 3, 4, 5, 6\}$.
In set language, the total number of possible outcomes is the total number of elements in S and is denoted by $\mathbf{n(S)}$. Therefore, $n(S) = 6$ for the above probability experiment.

Reflection

We learnt in Book 2 that a die is fair if all the 6 possible outcomes are equally likely to occur. What if the die is biased or not fair? What will it affect?

Suppose E is the event that the number shown on the die is a multiple of 3.
Since the multiples of 3 in the sample space S are 3 and 6, the favourable outcomes are 3 and 6, and the event E can be written as $E = \{3, 6\}$.
In other words, an **event** is a *subset* of the sample space.
The number of favourable outcomes is the number of elements in the event E and is denoted by $\mathbf{n(E)}$. Therefore, $n(E) = 2$ for the above event E.

Fig. 2.1 shows the representation of S and E in a Venn diagram.
The set E', the *complement* of the set E, is also represented in Fig. 2.1.
For this experiment, $E' = \{1, 2, 4, 5\}$ and $n(E') = 4$. What does the event E' represent?

Recall

E' is the complement of the set E, i.e. the set of elements in the universal set which are not members of E.

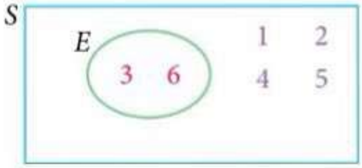


Fig. 2.1

B. Probability

We learnt in Book 2 that probability is a measure of chance.
In a probability experiment with a finite number of *equally likely outcomes*, the probability, P(E), of an event E occurring is given by:

$$P(E) = \frac{\text{number of favourable outcomes for event } E}{\text{total number of possible outcomes}}.$$

In set language, the probability, P(E), of an event E occurring is given by:

$$P(E) = \frac{\text{number of elements in event } E}{\text{number of elements in sample space } S} = \frac{n(E)}{n(S)}.$$

Using the example of rolling a fair die in Section 2.1A, the probability of the event E occurring is $P(E) = \frac{n(E)}{n(S)} = \frac{2}{6} = \frac{1}{3}$.

Big Idea

Measures
In Book 2, we learnt that the probability of the occurrence of an event is a measure of the chance or likelihood of the event occurring. This measure is given by the ratio of the number of favourable outcomes to the number of all possible outcomes. Hence, the probability of the occurrence of an event does not have a unit of measure.

In Book 2, we also learnt that:

$$P(\text{not } E) = 1 - P(E),$$

where P(not E) is the probability of all outcomes except those in E occurring.
In the above example, since E is the event that the number on the die is a multiple of 3, then "not E" is the event that the number on the die is not a multiple of 3.
As we have observed in Fig. 2.1, the complement of E, i.e. E', is the event "not E".
Therefore, in set notation, for any event E,

$$P(E') = 1 - P(E).$$

Attention

For an event E, $P(E) + P(E') = 1$. Explain why the sum of P(E) and P(E') must be equal to 1.

Class Discussion

In Book 2, we learnt that for any event E, $0 \leqslant P(E) \leqslant 1$.
Explain why P(E) must lie between 0 and 1 inclusive.

Worked Example 1

Calculating probability of single event (with countable outcomes)

There are 3 blue balls and 1 red ball in a bag. The balls are identical except for their colour. A ball is drawn at random from the bag.

(i) List the sample space in set notation.

(ii) By using set notation for the respective events, find the probability that the ball drawn is

(a) blue, **(b)** not blue.

(iii) What is the probability that the ball drawn is

(a) green, **(b)** blue or red?

***Solution**

(i) Let S represent the sample space. Then $S = \{B_1, B_2, B_3, R\}$.

(ii) (a) Let E be the event that the ball drawn is blue.

Then $E = \{B_1, B_2, B_3\}$.

$$\therefore \mathrm{P}(E) = \frac{\mathrm{n}(E)}{\mathrm{n}(S)}$$

$$= \frac{3}{4}$$

(b) **Method 1:**

$$E' = \{R\}$$

$$\mathrm{P}(E') = \frac{\mathrm{n}(E')}{\mathrm{n}(S)}$$

$$= \frac{1}{4}$$

Method 2:

$$\mathrm{P}(E') = 1 - \mathrm{P}(E)$$

$$= 1 - \frac{3}{4}$$

$$= \frac{1}{4}$$

(iii) (a) P(green) = 0 why?

(b) P(blue or red) = 1 why?

Problem-solving Tip

(i) Although the 3 blue balls are identical, they are still distinct. Hence, we need to distinguish among them by labelling them as B_1, B_2 and B_3. We cannot write $S = \{B, B, B, R\}$ because in set notation, this will become $S = \{B, R\}$ and the probability calculated, $\mathrm{P}(B) = \frac{1}{2}$, will be wrong.

Reflection

(ii) (b) Which method do you prefer? Why?

Practise Now 1

Similar and Further Questions

Exercise 2A

Questions 1(a)–(d), 2–4, 15, 16, 23

A letter is chosen at random from the word 'CLEVER'.

(i) List the sample space in set notation.

(ii) What is the probability that the letter chosen is

(a) an 'E', **(b)** a 'C',

(c) a 'C' or an 'R', **(d)** a 'T'?

(iii) By using set notation for the respective events, find the probability that the letter chosen is

(a) a vowel, **(b)** a consonant.

Attention

On page 358 of Book 2 Chapter 12, $S = \{x : x \text{ is a letter of the word 'CLEVER'}\} = \{\text{C, L, E, V, R}\}$ because we do not distinguish between the two 'E's as they are the same letter. However, in probability, we treat the two 'E's as distinct and there is a higher chance of choosing 'E' than each of the other letters.

We also learnt in Book 2 that if the outcomes cannot be counted, then the probability, P(E), of an event E occurring is given by:

$$\mathrm{P}(E) = \frac{\text{measure of favourable outcomes for event } E}{\text{measure of all possible outcomes for sample space } S}.$$

Worked Example 2

Calculating probability of single event (with uncountable outcomes)

A spinner consists of a circular board that is divided into sectors of different colours and a spinning pointer. The pointer is spun at random.

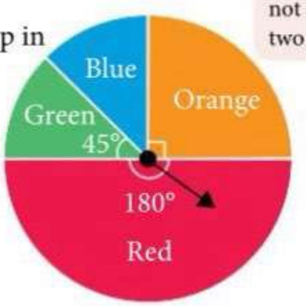
Find the probability that the pointer will stop in

(i) the orange sector,

(ii) a sector that is not orange,

(iii) the blue sector,

(iv) a sector that is either orange or blue.



Attention

We assume that the pointer will not stop on the line between an two adjacent sectors.

Solution

(i) Let O be the event that the pointer will stop in the orange sector.

$$\text{Then P}(O) = \frac{\text{area of orange sector}}{\text{area of circle}}$$

$$= \frac{\text{angle of orange sector}}{\text{angle of circle}}$$

$$= \frac{90°}{360°}$$

$$= \frac{1}{4}$$

Problem-solving Tip

(i) The area of a sector is proportional to its angle. Therefore, $\frac{\text{area of sector}}{\text{area of circle}} = \frac{\text{angle of sector}}{\text{angle of circle}}$, and the angle of the circle is 360°. Visually, we can also see that the orange sector makes up $\frac{1}{4}$ of the circle.

(ii) **Method 1:**

Sum of angles of all sectors that are not orange = 360° – 90°
= 270°

$$\therefore \text{P}(O') = \frac{\text{sum of areas of all sectors that are not orange}}{\text{area of circle}}$$

$$= \frac{\text{sum of angles of all sectors that are not orange}}{\text{angle of circle}}$$

$$= \frac{270°}{360°}$$

$$= \frac{3}{4}$$

Method 2:

$$\text{P}(O') = 1 - \text{P}(O)$$

$$= 1 - \frac{1}{4}$$

$$= \frac{3}{4}$$

Reflection

(ii) Which method do you prefer? Why?

(iii) Angle of blue sector = 360° – 180° – 90° – 45°
= 45°

Let B be the event that the pointer will stop in the blue sector.

$$\text{Then P}(B) = \frac{\text{area of blue sector}}{\text{area of circle}}$$

$$= \frac{\text{angle of blue sector}}{\text{angle of circle}}$$

$$= \frac{45°}{360°}$$

$$= \frac{1}{8}$$

(iv) Let E be the event that the pointer will stop in either the orange or the blue sector.

Method 1:

Sum of angles of orange and blue sectors $= 90° + 45°$
$= 135°$

$$\therefore \text{P}(E) = \frac{\text{sum of areas of orange and blue sectors}}{\text{area of circle}}$$
$$= \frac{\text{sum of angles of orange and blue sectors}}{\text{angle of circle}}$$
$$= \frac{135°}{360°}$$
$$= \frac{3}{8}$$

Method 2:

$$\text{P}(E) = \text{P}(O) + \text{P}(B)$$
$$= \frac{1}{4} + \frac{1}{8}$$
$$= \frac{3}{8}$$

Reflection

(iv) Why does Method 2 work? Will it always work? You will learn more about Method 2 in Section 2.3 of this chapter.

Practise Now 2

Similar and Further Questions
Exercise 2A
Questions 5, 6, 17

A circle is divided into sectors of different colours.
A point is selected at random in the circle.
Find the probability that the point lies in

(i) the red sector,
(ii) a sector that is not red,
(iii) the orange sector,
(iv) a sector that is either red or orange.

Reflection

1. What have I learnt in Book 2 that are similar or different from the concepts taught in this section?
2. What have I learnt in this section that I am still unclear of?

2.2 Probability of combined events

In Book 2 and the previous section, we learnt how to calculate the probability of a single event occurring in a one-stage probability experiment, such as tossing a coin or rolling a die.

In this section, we will learn how to calculate the probability of combined events occurring in a multiple-stage probability experiment, such as tossing a coin twice or rolling two dice.

But first, how do we list the sample space of a multiple-stage probability experiment?

Attention

Although rolling two dice seems like a one-stage experiment, we can view it as rolling one die and then rolling another die in a two-stage experiment.

A. Sample space diagram

The possible outcomes for rolling a fair die are 1, 2, 3, 4, 5 and 6, and we write the sample space as {1, 2, 3, 4, 5, 6}.

But how do we list the possible outcomes for rolling two fair dice?
We can represent a possible outcome by using an **ordered pair**, e.g. (2, 3) means that the first die shows a '2' and the second die shows a '3'.
What does the outcome (3, 2) mean? Is it the same as the outcome (2, 3)?

But how do we write the sample space for rolling two fair dice? Is {(1, 1), (1, 2), (1, 3), …, (6, 6)} clear enough?
Do we know what the ellipsis … represents? It is not clear that the outcome after (1, 6) is (2, 1) and not (1, 7).
Hence, we need to represent the sample space differently. Fig. 2.2 shows one way of drawing a **sample space diagram** to represent the sample space for rolling two fair dice.

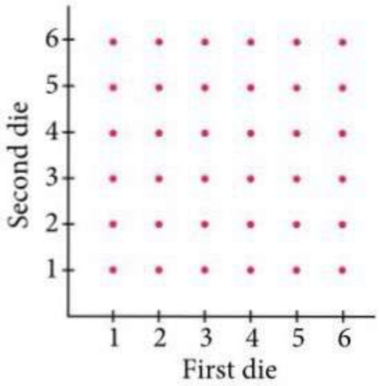


Fig. 2.2

Big Idea

Diagrams
A **sample space diagram** is used to represent the sample space of a *two-stage* probability experiment where each outcome has two components. For example, in Fig. 2.2, an outcome (represented by a red dot •) is determined by the values displayed by the first and the second die. Thus, we see the usefulness of a sample space diagram in representing the outcomes of a two-stage probability experiment succinctly.
By convention, we draw the outcomes of the first die on the horizontal axis and the outcomes of the second die on the vertical axis.

From the above sample space diagram, we observe that the total number of possible outcomes is 6 × 6 = 36, i.e. the size or *measure of the sample space* is 36.
We can also calculate the probability of **combined events** using a sample space diagram, as shown in Worked Example 3.

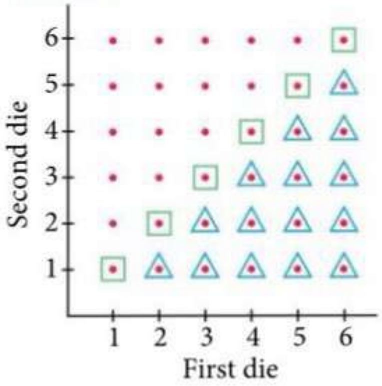
Thinking Time

In Fig. 2.2,
(i) circle the dot that represents the outcome (2, 3),
(ii) draw a triangle over the dot that represents the outcome (3, 2).
Do the two dots represent the same outcome? Explain.

Worked Example 3

Calculating probability of combined events using sample space diagram
Two fair dice are rolled. What is the probability that
(i) both dice show the same number,
(ii) the number shown on the first die is greater than the number shown on the second die?

Solution



Problem-solving Tip

Mark out the favourable outcomes on the sample space diagram.
Count the number of □ for **(i)** and the number of △ for **(ii)**.

(i) P(both dice show the same number) $= \frac{6}{36}$

$= \frac{1}{6}$

(ii) P(number shown on the first die is greater than the number shown on the second die)

$= \frac{15}{36}$

$= \frac{5}{12}$

Practise Now 3

Similar and Further Questions
Exercise 2A
Questions 7, 8, 18, 24

1. A fair tetrahedral die (4-sided die) and a fair 6-sided die are rolled simultaneously. The numbers on the tetrahedral die are 1, 2, 5 and 6 while the numbers on the 6-sided die are 1, 2, 3, 4, 5 and 6.

- **(i)** Display all the outcomes of the experiment using a sample space diagram.
- **(ii)** Using the sample space diagram, find the probability that
  - **(a)** both dice show the same number,
  - **(b)** the number shown on the tetrahedral die is greater than the number shown on the 6-sided die,
  - **(c)** the numbers shown on both dice are prime numbers.

Information

A tetrahedral die has 4 faces. When it is rolled, the number shown is actually the number on the face of the die that is facing down. As it is very troublesome to lift up the die to see what that number is, the number is actually indicated at the bottom of each face. In the picture below, do you see that the number 4 is indicated at the bottom of the two faces? In fact, the number 4 is also indicated at the bottom of the other face behind the die. Therefore, the number for this die is 4.

2. A bag contains five cards and the cards are numbered 1, 2, 3, 4 and 5. A card is drawn at random from the bag and its number is noted. The card is then replaced and a second card is drawn at random from the bag. Using a sample space diagram, find the probability that

- **(i)** the number shown on the second card is greater than the number shown on the first card,
- **(ii)** the sum of the two numbers shown is greater than 7,
- **(iii)** the product of the two numbers shown is greater than 10.

There is another way to draw a sample space diagram to represent the sample space for rolling two fair dic as own in Worked Example 4.

We use this kind of sample space diagram when we are interested in the final value obtained from performing arithmeti operations on the values obtained from the two events. For example, in Worked Example 4, we are interested in th sum of the numbers shown on the two dice.

Worked Example 4

Calculating probability using another type of sample space diagram

Two fair dice are rolled. Find the probability that the sum of the numbers shown on the dice is

(i) equal to 5, **(ii)** even.

Solution

First die

Second die

+	1	2	3	4	5	6
1	$\boxed{2}$	3	$\boxed{4}$	⑤	$\boxed{6}$	7
2	3	$\boxed{4}$	⑤	$\boxed{6}$	7	$\boxed{8}$
3	$\boxed{4}$	⑤	$\boxed{6}$	7	$\boxed{8}$	9
4	⑤	$\boxed{6}$	7	$\boxed{8}$	9	$\boxed{10}$
5	$\boxed{6}$	7	$\boxed{8}$	9	$\boxed{10}$	11
6	7	$\boxed{8}$	9	$\boxed{10}$	11	$\boxed{12}$

(i) $$\text{P(sum is equal to 5)} = \frac{4}{36} = \frac{1}{9}$$

(ii) $$\text{P(sum is even)} = \frac{18}{36} = \frac{1}{2}$$

Attention

The sum of the numbers is shown in each cell. In your own sample space diagram, you can draw double lines to avoid accidentally counting the numbers in the first row and in the first column, when counting the number of favourable outcomes.

Problem-solving Tip

Mark out the favourable outcomes on the sample space diagram.
Count the number of ○ for **(i)** and the number of □ for **(ii)**.

Reflection

Is the sample space diagram from Worked Example 3 helpful here? Explain.

Practise Now 4

Similar and Further Questions
Exercise 2A
Questions 9, 10, 19, 20, 25

1. The numbers on a fair tetrahedral die are 1, 2, 5 and 6 while the numbers on a fair 6-sided die are 1, 2, 3, 4, 5 and 6. The two dice are rolled at the same time and the scores on both dice are recorded. The sample space diagrams display separately some of the values of the sum and product of the two scores.

Tetrahedral die

6-sided die

+	1	2	5	6
1		3		
2				
3			8	
4				
5				
6	7			

Tetrahedral die

6-sided die

×	1	2	5	6
1				
2		4		
3				18
4				
5				
6			30	

(i) Copy and complete the sample space diagrams.

(ii) Using the appropriate sample space diagram, find the probability that the sum of the scores is

(a) even, **(b)** divisible by 3,

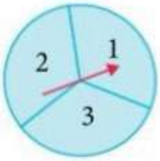
(c) a perfect square, **(d)** less than 2.

(iii) Using the appropriate sample space diagram, find the probability that the product of the scores is

(a) odd, **(b)** larger than 12,

(c) a prime number, **(d)** less than 37.

2. A circular card is divided into 3 equal sectors with scores of 1, 2 and 3. The card has a spinning pointer pivoted at its centre. The pointer is spun twice. Each time the pointer is spun, it is equally likely to stop at any of the sectors.



(i) With the help of a sample space diagram, find the probability that

(a) each score is a '1',

(b) at least one of the scores is a '3'.

(ii) In a game, a player spins the pointer twice. His final score is the larger of the two individual scores if they are different and their common value if they are the same. The sample space diagram below shows the player's final score.

	1	2	3
1	1		
2			
3		3	

(a) Copy and complete the sample space diagram.

Using the sample space diagram, find the probability that his final score is

(b) even,

(c) a prime number.

B. Tree diagram

The sample space for tossing a fair coin is $\{H, T\}$.
The sample space for tossing two fair coins can be represented by a sample space diagram, as shown in Fig. 2.3.
How can we represent the sample space for tossing three fair coins?
Since it is difficult to draw a 3-dimensional sample space diagram, we have to use a different type of diagram called a **tree diagram** to represent the sample space, as shown in Fig. 2.4. The following steps show how the tree diagram is constructed.

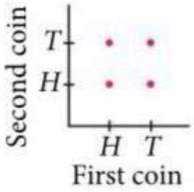
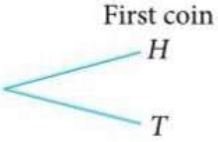
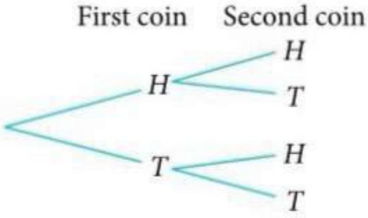


Fig. 2.3

1. When the first coin is tossed, there are two possible outcomes, head (H) or tail (T), so we start with a point and draw two branches H and T. We indicate 'First coin' on top of the two branches as shown.



2. The second coin is then tossed. Regardless of the outcome of the first toss, the second coin would also yield either a H or a T, thus we draw two branches after the H and the T from the first toss as shown below. There are a total of $2 \times 2 = 4$ branches, i.e. there are 4 possible outcomes at this stage.
   We indicate 'Second coin' on top of the four branches as shown.

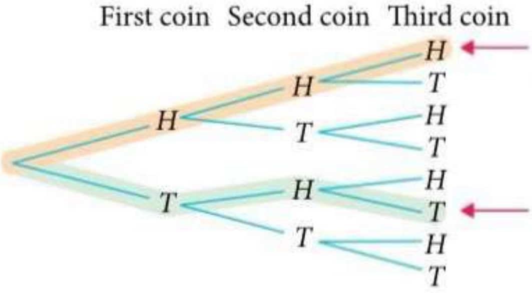


Big Idea

Diagrams

A **tree diagram** is used to represent the sample space of a *multiple-stage* probability experiment where each outcome has at least two components.
For example, in Fig. 2.4, the first branch represents the outcome HHH, i.e. all the 3 coins show a head. Thus we see the usefulness of a tree diagram in representing the outcomes of a multiple-stage probability experiment succinctly.
By convention, we draw a tree diagram horizontally from left to right.

3. The third coin could also yield two outcomes when the first two outcomes are HH, HT, TH or TT. Thus we obtain the tree diagram as shown in Fig. 2.4.
   We observe that there are a total of $2 \times 2 \times 2 = 8$ branches, i.e. the total number of possible outcomes or the *measure of the sample space* is 8.



This branch represents the outcome HHH.

This branch represents the outcome THT.

Fig. 2.4

Thinking Time

In Fig. 2.4, the branch representing the outcome *THT* has been highlighted in green. Use a highlighter to highlight the branch representing the outcome

(i) *TTH*, **(ii)** *HTT*.

Do these three branches represent the same outcome? Explain.

In summary,

Number of stages of probability experiment	Example(s)	Representation of sample space
One-stage	Tossing 1 coin	List of outcomes in set notation
Two-stage	Tossing 2 coins; tossing a coin 2 times	Sample space diagram or tree diagram
Three-stage	Tossing 3 coins; tossing a coin 3 times	Tree diagram

Worked Example 5

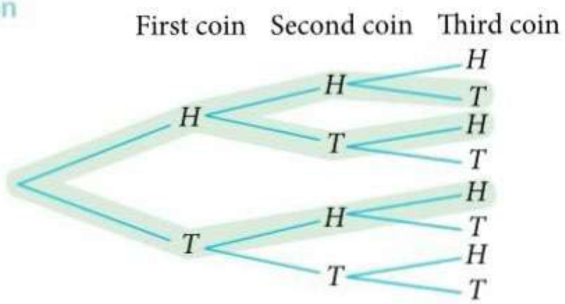
Calculating probability of combined events using tree diagram

Three fair coins are tossed. By using a tree diagram, find the probability that

(i) there are two heads and one tail,

(ii) there is at least one tail.

Solution



(i) P(two heads and one tail) = $\frac{3}{8}$ see highlighted branches: *HHT*, *HTH* and *THH*

(ii) P(at least one tail) = 1 – P(no tail)

$$= 1 - \text{P(three heads)}$$

$$= 1 - \frac{1}{8}$$

$$= \frac{7}{8}$$

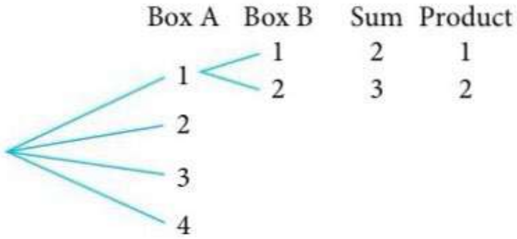
Practise Now 5

Similar and Further Questions
Exercise 2A
Questions 11–13, 21

1. Imran is a darts player. There is an equal probability that he will hit or miss the bullseye. He aims for the bullseye and attempts 3 throws. By using a tree diagram, find the probability that
   (i) he misses the bullseye once,
   (ii) he hits the bullseye at least once.

2. Box A contains 4 pieces of paper numbered 1, 2, 3 and 4. Box B contains 2 pie o paper numbered 1 and 2. One piece of paper is removed at random from each box.

(i) Copy and complete the following tree diagram.



(ii) Find the probability that

- **(a)** at least one '1' is obtained,
- **(b)** the sum of the two numbers is 3,
- **(c)** the product of the two numbers is at least 4,
- **(d)** the sum is equal to the product.

Reflection

How do you use *one* sample space diagram to help you solve Question 2(**ii**)?

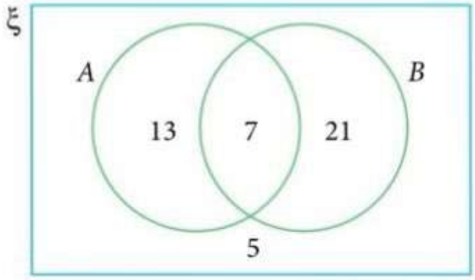
C. Venn diagram

In this section, we shall apply what we have learnt about Venn diagrams in Book 2 and in Chapter 1 of this book to solve problems involving the probability of combined events.

Worked Example 6

Calculating probability using a Venn diagram

The Venn diagram shows the survey results of a group of adults who like either brand of coffee, *A* or *B*. They may choose either brand, both brands, or none. The Venn diagram shows the survey results.



An adult is selected at random.
Find the probability that this adult

- **(a)** likes brand *A*,
- **(b)** likes brand *A* or *B*,
- **(c)** likes brand *B* but not brand *A*,
- **(d)** does not like either brand.

Solution

(a) $$\text{P(likes brand } A) = \frac{n(A)}{n(\xi)} = \frac{13+7}{13+7+21+5} = \frac{20}{46} = \frac{46}{23}$$

(b) $$\text{P(likes brand } A \text{ or } B) = \frac{n(A \cup B)}{n(\xi)} = \frac{13+7+21}{46} = \frac{41}{46}$$

(c) $$\text{P(likes brand } B \text{ but not brand } A) = \frac{n(B \cap A')}{n(\xi)} = \frac{21}{46}$$

(d) $$\text{P(does not like either brand)} = \frac{n(A' \cap B')}{n(\xi)} = \frac{5}{46}$$

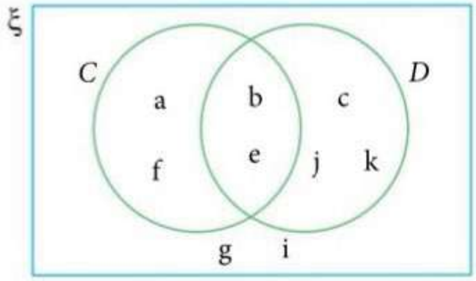
Practise Now 6

Similar and Further Questions

Exercise 2A

Questions 14, 22, 26

The Venn diagram shows the elements of ξ, of *C* and of *D*.



An element is selected at random. Find the probability that the selected element is

(a) an 'a',
(b) a 'g' or a 'j',
(c) a vowel,
(d) an element belonging to *C* and *D*.

Reflection

1. Do I know when to use set notation, a sample space diagram, tree diagram or a Venn diagram to represent the sample space of a probability experiment? If yes, elaborate.
2. What have I learnt in this section that I am still unclear of?

Exercise 2A

1. List the sample space of each of the following probability experiments.

- **(a)** A fair 12-sided die, numbered from 1 to 12, is rolled.
- **(b)** A fair coin is tossed.
- **(c)** A ball is drawn randomly from a bag containing 4 black balls and 2 white balls, where the balls are identical except for their colour.
- **(d)** A letter is chosen at random from the word ‘STUDENTS’.

2. There are 2 green marbles and 3 yellow marbles in a bag. The marbles are identical except for their colour. A marble is drawn at random from the bag.

- **(i)** List the sample space in set notation.
- **(ii)** By using set notation for the respective events, find the probability that the marble drawn is
  - **(a)** yellow,
  - **(b)** not yellow.
- **(iii)** What is the probability that the marble drawn is
  - **(a)** black,
  - **(b)** green or yellow?

3. Each letter of the word ‘PROBABILITY’ is written on identical cards. One card is chosen at random.

- **(i)** List the sample space in set notation.
- **(ii)** What is the probability that the letter on the chosen card is
  - **(a)** a ‘B’,
  - **(b)** a ‘T’,
  - **(c)** an ‘I’ or an ‘O’,
  - **(d)** an ‘E’?
- **(iii)** By using set notation for the respective events, find the probability that the letter chosen is
  - **(a)** a vowel,
  - **(b)** a consonant.

4. All eight pangolin species are protected under international laws. In Singapore, the method of capture-recapture (as described in the Chapter Opener) is used to monitor the pangolin population in the nature reserves. In a particular year, 22 injured pangolins were rescued and tagged. These pangolins were then released back into the nature reserves after they recovered.

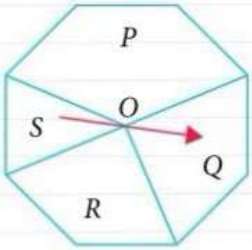
Over the period of one year, five pangolins were captured on camera, of which only one was tagged. Estimate the number of pangolins that lived in the nature reserves. State the assumptions you have made to solve this problem.

5. A survey is conducted to find out which of the four activities, soccer, basketball, chess and reading, students in a class prefer. The pie chart shows the results of the survey.

A student is selected at random. Find the probability that the student prefers

- **(i)** soccer,
- **(ii)** reading,
- **(iii)** basketball or chess.

6. A spinner consists of a board in the shape of a regular octagon with centre O and a spinning pointer. The board is divided into 4 regions and the spinning pointer is spun at random.



Find the probability that the spinning pointer will stop in

- **(i)** region S,
- **(ii)** region Q,
- **(iii)** region P or R.

Exercise 2A

7. A box contains three cards bearing the numbers 1, 2 and 3. A second box contains four cards bearing the numbers 2, 3, 4 and 5. A card is chosen at random from each box.

(i) Display all the possible outcomes of the experiment using a sample space diagram.

(ii) With the help of the sample space diagram, calculate the probability that

- **(a)** the cards bear the same number,
- **(b)** the numbers on the cards are different,
- **(c)** both numbers are prime numbers,
- **(d)** exactly one number is a multiple of 2.

8. Bag P contains a red, a blue and a white marble while bag Q contains a blue and a red marble. The marbles are identical except for their colour. A marble is picked at random from each bag.

(i) Display all the possible outcomes of the experiment using a sample space diagram.

(ii) With the help of the sample space diagram, find the probability that the two marbles selected are

- **(a)** of the same colour,
- **(b)** blue and red,
- **(c)** of different colours.

9. Six cards numbered 0, 1, 2, 3, 4 and 5 are placed in a box. A card is drawn at random from the box and the number on the card is noted before it is replaced in the box. A second card is then drawn at random from the box and the sum of the two numbers is obtained. The sample space diagram below shows some of the values of the sum of the two numbers.

First number

Second number

+	0	1	2	3	4	5
0						
1	1			4		
2						
3						
4		5				
5						

(i) Copy and complete the sample space diagram.

(ii) How many possible outcomes are there in the sample space of this experiment?

(iii) What is the probability that the sum of the two numbers is

- **(a)** 7,
- **(b)** a prime number,
- **(c)** not a prime number,
- **(d)** even,
- **(e)** not even?

(iv) Which of the two sums of the two numbers is more likely to occur, 7 or 8? Explain.

10. It is given that $X = \{4, 5, 6\}$ and $Y = \{7, 8, 9\}$. An element x is selected at random from X and an element y is selected at random from Y. The sample space diagrams display separately some of the values of $x + y$ and xy.

x

y

+	4	5	6
7	11		
8			
9		14	

x

y

×	4	5	6
7			42
8		40	
9			

(i) Copy and complete the sample space diagrams.

(ii) Find the probability that the sum $x + y$ is

- **(a)** prime,
- **(b)** greater than 12,
- **(c)** at most 14.

(iii) Find the probability that the product xy is

- **(a)** odd,
- **(b)** even,
- **(c)** at most 40.

11. A fair coin is tossed three times.

(i) Display all the possible outcomes of the experiment using a tree diagram.

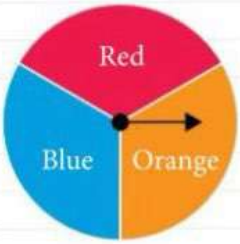
(ii) With the help of the tree diagram, find the probability of obtaining

- **(a)** three tails,
- **(b)** exactly two tails,
- **(c)** at least two tails,
- **(d)** no tails.

Exercise 2A

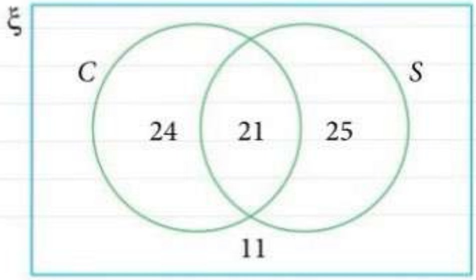
12. Three ladies are happily awaiting the arrival of their bundles of joy within the year. Display the sample space of the genders of the three babies using an appropriate diagram, assuming that the babies are equally likely to be either a boy or a girl.
Hence, find the probability that there will be a total of
- **(i)** three baby boys,
- **(ii)** two baby boys and one baby girl,
- **(iii)** one baby boy and two baby girls.

13. A spinner, with three equal sectors and a spinning pointer as shown in the diagram, and a fair coin are used in a game. The pointer is spun once and the coin is tossed once. Each time the pointer is spun, it is equally likely to stop at any sector.



- **(i)** Display all the possible outcomes of the experiment using a tree diagram.
- **(ii)** With the help of the tree diagram, calculate the probability of getting
  - **(a)** red on the spinner and tail on the coin,
  - **(b)** blue or orange on the spinner and head on the coin.

14. The Venn diagram shows the number of teenagers who play cricket and the number of teenagers who play soccer in a neighbourhood, where *C* is the set of teenagers who play cricket and *S* is the set of teenagers who play soccer.



A teenager is chosen at random. What is the probability that this teenager
- **(i)** plays either cricket or soccer,
- **(ii)** only plays cricket,
- **(iii)** plays cricket,
- **(iv)** does not play either sport.

15. A card is drawn at random from a standard pack of 52 playing cards. Find the probability of drawing
- **(i)** the Queen of diamonds,
- **(ii)** a black card,
- **(iii)** a spade,
- **(iv)** a card which is not a spade.

Hint: There are 4 suits in a standard pack of 52 playing cards, i.e. spade ♠, heart ♥, club ♣ and diamond ♦.

Each suit has 13 cards, i.e. Ace, 2, 3, 4, …, 10, Jack, Queen and King.

All the clubs and spades are black in colour.

All the diamonds and hearts are red in colour.

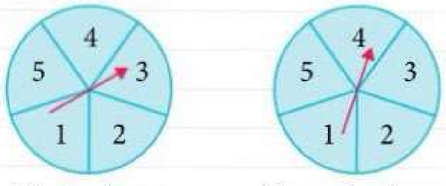
All the Jack, Queen and King cards are also known as picture cards.

16. A two-digit number is randomly formed using the digits 1, 4 and 6. Repetition of digits is allowed.
- **(i)** List the sample space in set notation.
- **(ii)** By using set notation for the respective events, find the probability that the two-digit number formed is
  - **(a)** divisible by 3,
  - **(b)** a perfect square,
  - **(c)** a prime number,
  - **(d)** a composite number.

17. *ABCDEF* is a regular hexagon with centre *O*. *M* is the midpoint of *AB* and *N* is the midpoint of *BC*. A point is selected at random in the regular hexagon. Find the probability that the point lies in the kite *MBNO*.

Exercise 2A

18. In an experiment, two spinners are constructed with spinning pointers as shown in the diagrams below. Both pointers are spun. Each time the pointer is spun, it is equally likely to stop at any sector.



First spinner Second spinner

(i) Find the probability that the pointers will point at
  - **(a)** numbers on the spinners whose sum is 6,
  - **(b)** the same number on both spinners,
  - **(c)** different numbers on the spinners,
  - **(d)** two different prime numbers.

(ii) What is the probability that the number on the first spinner will be less than the number on the second spinner?

(iii) What is the probability that the larger of the two numbers on the spinners is 3?

19. In a game, the player tosses a fair coin and rolls a fair 6-sided die simultaneously. If the coin shows a head, the player's score is the score on the die. If the coin shows a tail, then the player's score is twice the score on the die. Some of the player's possible scores are shown in the sample space diagram.

Die

		1	2	3	4	5	6
Coin	*H*	1					
	T			6			

(i) Copy and complete the sample space diagram.

(ii) Using the sample space diagram, find the probability that the player's score is
  - **(a)** odd, **(b)** even,
  - **(c)** a prime number,
  - **(d)** less than or equal to 8,
  - **(e)** a multiple of 3.

20. Two fair 6-sided dice were rolled together and the difference between the numbers on their faces was calculated. Some of the differences are shown in the sample space diagram below.

First die

Second die

–	1	2	3	4	5	6
1	0					
2			1			4
3						
4						
5						
6		4				0

(i) Copy and complete the sample space diagram.

(ii) Using the sample space diagram, find the probability that the difference between the two numbers is
  - **(a)** 1, **(b)** non-zero,
  - **(c)** odd,
  - **(d)** a prime number,
    Note: This is the Introductory Problem.
  - **(e)** greater than 2.

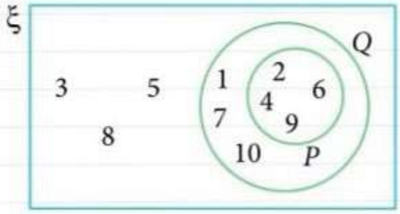
21. A bag contains 3 cards numbered 1, 3 and 5. A second bag contains 3 cards numbered 1, 2 and 7. One card is drawn at random from each bag.

(i) Display all the possible outcomes of the experiment using a tree diagram.

(ii) With the help of the tree diagram, calculate the probability that the two numbers obtained
  - **(a)** are both odd,
  - **(b)** are both prime,
  - **(c)** have a sum greater than 4,
  - **(d)** have a sum that is even,
  - **(e)** have a product that is prime,
  - **(f)** have a product that is greater than 20,
  - **(g)** have a product that is divisible by 7.

Exercise 2A

22. The elements of ξ, of P and of Q are shown in the Venn diagram below.



An element is selected at random. Find the probability that the selected element is

(i) an even number,
(ii) a number not in P or Q,
(iii) a prime number,
(iv) a prime number in Q.

23. A fair coin and a fair 6-sided die are tossed and rolled respectively. Using set notation, list the sample space of the experiment.

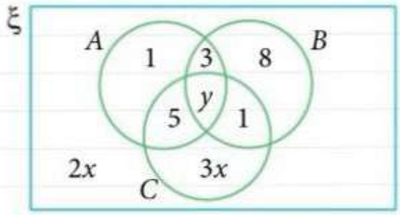
24. Hotel Y is a two-storey hotel, with rooms (and their respective room numbers) arranged as shown in the diagram below. Rooms are allocated at random when guests arrive and each guest is allocated one room. Cheryl and Joyce arrive at Hotel Y on a particular day. Upon their arrival, none of the rooms in Hotel Y are occupied.

(i) With the help of a sample space diagram, find the probability that Cheryl and Joyce
  (a) stay next to each other,
  (b) stay on different storeys,
  (c) do not stay next to each other.

(ii) Suppose the hotel accepts Cheryl's request that she only wants to be allocated rooms on the second floor, what is the probability of her staying next to Joyce?

25. Two fair tetrahedral dice and a fair 6-sided die are rolled simultaneously. The numbers on the tetrahedral dice are 1, 2, 3 and 4 while the numbers on the 6-sided die are 1, 2, 3, 4, 5 and 6. What is the probability that the score on the 6-sided die is greater than the sum of the scores on the two tetrahedral dice?

26. The Venn diagram shows the number of elements of ξ, of A, of B and of C.



It is given that $n(\xi) = 48$ and $n(A \cap B) = n(C')$.

(i) Find the value of x and of y.

(ii) An element is selected at random. Find
  (a) $P(C)$,   **(b)** $P(A \cup B)$.

2.3 Addition Law of Probability and mutually exclusive events

Previously in Worked Example 2 part **(iv)**, we have learnt that probabilities can be added. But can we always add probabilities?
In this section, we will learn the conditions for adding probabilities.

Investigation Mutually exclusive and non-mutually exclusive events

Eight cards numbered 1 to 8 are placed in a box. A card is drawn at random.
Let A be the event of drawing a card that shows a prime number.
Let B be the event of drawing a card that shows a multiple of 4.
Let C be the event of drawing a card that shows an odd number.

1. List the sample space.

Part 1: Mutually exclusive events

2. List the favourable outcomes for event A and find the probability that A will occur, i.e. $P(A)$.
3. List the favourable outcomes for event B and find the probability that B will occur, i.e. $P(B)$.
4. Are there any *overlaps* between the favourable outcomes for event A and the favourable outcomes for event B? That is, are there any outcomes that favour the occurrence of *both* event A and event B?
   Since there are no overlaps, these two events are said to be **mutually exclusive**.
5. List the favourable outcomes for the combined event **A or B** (i.e. **$A \cup B$**), and find the probability that the combined event will occur, i.e. find $P(A \text{ or } B)$ or $P(A \cup B)$.
6. Is $P(A \cup B) = P(A) + P(B)$ in this case? Can you explain why?

Part 2: Non-mutually exclusive events

7. List the favourable outcomes for event C and find the probability that C will occur, i.e. $P(C)$.
8. Are there any *overlaps* between the favourable outcomes for event A and the favourable outcomes for event C? That is, are there any outcomes that favour the occurrence of *both* event A and event C?
   Since there are overlaps, these two events are said to be **non-mutually exclusive.**
9. List the favourable outcomes for the combined event A or C (i.e. $A \cup C$), and find the probability that the combined event will occur, i.e. find $P(A \text{ or } C)$ or $P(A \cup C)$.
10. Is $P(A \cup C) = P(A) + P(C)$ in this case? Can you explain why?

From the above Investigation,, we observe that if two events A and B cannot occur at the same time (i.e. the events are *mutually exclusive*), then $P(A \text{ or } B)$ or $P(A \cup B) = P(A) + P(B)$.
On the other hand, if two events A and C can occur at the same time (i.e. the events are *not mutually exclusive*), then $P(A \text{ or } C)$ or $P(A \cup C) \neq P(A) + P(C)$.

In general, the **Addition Law of Probability** states that:

If A and B are **mutually exclusive events**,
$P(A \cup B)$ or **$P(A \text{ or } B) = P(A) + P(B)$**.

Information

In higher level mathematics, we will learn a more general Addition Law of Probability that applies to any two events, regardless of whether they are mutually exclusive.

Worked Example 7

Calculating probability using Addition Law of Probability

A card is drawn at random from a standard pack of 52 playing cards. Find the probability tha the card is

(i) an Ace,
(ii) a King,
(iii) an Ace or a King,
(iv) neither an Ace nor a King,
(v) a spade,
(vi) an Ace or a spade.

Attention

See **Hint** for Exercise 2A Question 15 on page 39 for the types of cards in a standard pack of playing cards.

Solution

(i)
$$\text{P(drawing an Ace)} = \frac{4}{52} = \frac{1}{13}$$

(ii)
$$\text{P(drawing a King)} = \frac{4}{52} = \frac{1}{13}$$

(iii) **Method 1:**

$$\text{P(drawing an Ace or a King)} = \text{P(Ace)} + \text{P(King)} = \frac{1}{13} + \frac{1}{13} = \frac{2}{13}$$

Method 2:

Number of Aces in the pack = 4
Number of Kings in the pack = 4

$$\therefore \text{P(drawing an Ace or a King)} = \frac{8}{52} = \frac{2}{13}$$

Problem-solving Tip

(iii) Since only *one* card is drawn, the events of drawing an Ace and a King cannot occur at the same time, i.e. the events are *mutually exclusive*. Then we can apply the **Addition Law of Probability** to obtain the answer.

(iv) **Method 1:**

$$\text{P(drawing neither an Ace nor a King)} = 1 - \text{P(drawing an Ace or a King)} = 1 - \frac{2}{13} = \frac{11}{13}$$

Method 2:

Number of cards in the pack that are neither an Ace nor a King
= 52 − 8
= 44

$$\text{P(drawing neither an Ace nor a King)} = \frac{44}{52} = \frac{11}{13}$$

(v)
$$\text{P(drawing a spade)} = \frac{13}{52} = \frac{1}{4}$$

(vi) Number of Aces in the pack = 4

Number of spades in the pack = 13

$$\text{P(drawing an Ace or a spade)} = \frac{4 + 13 - 1}{52} = \frac{16}{52} = \frac{4}{13}$$

Problem-solving Tip

(vi) Drawing an Ace of spades is a favourable outcome for both the events. We must subtract 1 from the sum of 4 and 13 because the Ace of spades is counted twice.

Reflection

(vi) Is P(drawing an Ace or a spade) = P(drawing an Ace) + P(drawing a spade)? Why or why not?

Practise Now 7

Similar and Further Questions
Exercise 2B
Questions 1, 2, 6, 7, 10

A card is drawn at random from a standard pack of 52 playing cards. Find the probability of drawing

(i) a '7',
(ii) a picture card,
(iii) a '7' or a picture card,
(iv) neither a '7' nor a picture card,
(v) a diamond,
(vi) a '7' or a diamond.

Attention

All the Jack, Queen and King cards are also known as picture cards.

Worked Example 8

Calculating probability of combined events when sample space is not given

The probabilities of three teams, L, M and N, winning a football competition are $\frac{1}{4}$, $\frac{1}{8}$ and $\frac{1}{10}$ respectively. Only one team can win the competition.

(i) Explain why there are more than three teams in the competition.
(ii) Calculate the probability that
  (a) either L or M wins, **(b)** L does not win,
  (c) N does not win, **(d)** neither L nor N wins.
(iii) Is P(neither L nor N wins) = P(L does not win) + P(N does not win)? Why or why not?

Solution

(i) Since only one team can win, the events of each of the teams L, M and N, winning are *mutually exclusive*.

But $\text{P(L wins)} + \text{P(M wins)} + \text{P(N wins)} = \frac{1}{4} + \frac{1}{8} + \frac{1}{10} = \frac{19}{40} < 1.$

Therefore, there are more than three teams in the competition.

(ii) (a) $\text{P(either L or M wins)} = \text{P(L wins)} + \text{P(M wins)}$

$$= \frac{1}{4} + \frac{1}{8} = \frac{3}{8}$$

Reflection

(ii) (a) Can you use Method 2 in Worked Example 7**(iii)** to solve this part-question? Explain.

(b) $\text{P(L does not win)} = 1 - \text{P(L wins)}$

$$= 1 - \frac{1}{4} = \frac{3}{4}$$

(c) P(N does not win) = 1 – P(N wins)

$$= 1 - \frac{1}{10}$$

$$= \frac{9}{10}$$

(d) P(either L or N wins) = P(L wins) + P(N wins)

$$= \frac{1}{4} + \frac{1}{10}$$

$$= \frac{7}{20}$$

P(neither L nor N wins) = 1 – P(either L or N wins)

$$= 1 - \frac{7}{20}$$

$$= \frac{13}{20}$$

(iii) P(L does not win) + P(N does not win) $= \frac{3}{4} + \frac{9}{10} > 1$

$\therefore$ P(neither L nor N wins) $\neq$ P(L does not win) + P(N does not win)

The Addition Law of Probability does not apply because the events of each team not winning are not mutually exclusive since there will be more than one team that will not win. In other words, 'L does not win' does not exclude 'N does not win', and similarly, 'N does not win' does not exclude 'L does not win'.

Practise Now 8

Similar and Further Questions
Exercise 2B
Questions 3–5, 8, 9, 11

The probabilities of four teams, P, Q, R and S, winning the National Volleyball Championship are $\frac{1}{5}, \frac{1}{6}, \frac{1}{7}$ and $\frac{1}{8}$ respectively. Only one team can win the championship.

(i) Explain why there are more than four teams in the competition.

(ii) Calculate the probability that

(a) either P or Q wins,
(b) Q, R or S wins,
(c) Q does not win,
(d) Q, R and S do not win.

(iii) Without doing any calculations, explain why P(neither Q, R nor S wins) $\neq$ P(Q does not win) + P(R does not win) + P(S does not win).

Reflection

1. How do I decide whether two events are mutually exclusive?
2. How do I decide when to use the Addition Law of Probability?
3. What have I learnt in this section that I am still unclear of?

Exercise 2B

1. Eleven cards numbered 11, 12, 13, 14, …, 21 are placed in a box. A card is removed at random from the box. Find the probability that the number on the card is
- **(i)** even,
- **(ii)** prime,
- **(iii)** either even or prime,
- **(iv)** neither even nor prime.

2. A bag contains 7 red marbles, 5 green marbles and 3 blue marbles. A marble is selected at random from the bag. Find the probability of selecting
- **(i)** a red marble,
- **(ii)** a green marble,
- **(iii)** either a red or a green marble,
- **(iv)** neither a red nor a green marble.

3. The probability of a football team winning any match is $\frac{7}{10}$, and the probability of them losing any match is $\frac{2}{15}$. What is the probability that the team
- **(i)** wins or loses a match,
- **(ii)** neither wins nor loses a match?

4. The probabilities of three teams, A, B and C, winning a basketball competition are $\frac{1}{14}$, $\frac{2}{7}$ and $\frac{3}{7}$ respectively. Only one team can win the competition. Calculate the probability that
- **(i)** C does not win,
- **(ii)** either A or B wins,
- **(iii)** none of these three teams win.

5. Every year, only one student can win the 'Student of the Year' award. The probabilities of Vasi, Waseem and Nadia winning the award are $\frac{1}{3}$, $\frac{1}{8}$ and $\frac{1}{20}$ respectively. What is the probability that
- **(i)** one of them will win the award,
- **(ii)** none of them will win the award,
- **(iii)** Vasi and Waseem will not win the award?

6. The letters of the word 'MUTUALLY' and the word 'EXCLUSIVE' are written on individual cards and the cards are then put into a box. A card is picked at random. What is the probability that the letter on the card is
- **(i)** a 'U',
- **(ii)** an 'E',
- **(iii)** a 'U' or an 'E',
- **(iv)** a consonant,
- **(v)** a 'U' or a consonant,
- **(vi)** a 'U' or a vowel?

7. A card is drawn at random from a standard pack of 52 playing cards. Find the probability of drawing
- **(i)** a King or a Jack,
- **(ii)** neither a King nor a Jack,
- **(iii)** a Queen or a card bearing a prime number,
- **(iv)** a card bearing a number that is divisible by 3 or by 5,
- **(v)** a card bearing a number that is divisible by 2 or by 3.

8. In a basketball tournament, three of the participating teams are Alpha, Beta and Gamma. The probabilities of each of these three teams winning the tournament are $\frac{4}{15}$, $\frac{1}{10}$ and $\frac{1}{5}$ respectively. Only one team can win the tournament.
- **(i)** Explain why there are more than three teams in the tournament.
- **(ii)** Calculate the probability that
  - **(a)** either Alpha or Gamma will win the tournament,
  - **(b)** Alpha, Beta or Gamma will win the tournament,
  - **(c)** neither Alpha nor Gamma will win the tournament,
  - **(d)** none of these three teams will win the tournament.
- **(iii)** Without doing any calculations, explain why P(neither Alpha nor Gamma wins) $\neq$ P(Alpha does not win) + P(Gamma does not win).

Exercise 2B

9. The probabilities of four teams, E, F, G and H, winning a hockey competition are $\frac{1}{3}, \frac{1}{5}, \frac{1}{7}$ and $\frac{1}{11}$ respectively. Only one team can win the competition.

- **(i)** Explain why there are more than four teams in the competition.
- **(ii)** Calculate the probability that
  - **(a)** either E or F wins,
  - **(b)** F, G or H wins,
  - **(c)** F does not win,
  - **(d)** F, G and H do not win.
- **(iii)** Without doing any calculations, explain why P(F, G and H do not win) ≠ P(F does not win) + P(G does not win) + P(H does not win).

10. In a probability experiment, three fair coins are tossed, one after another.

- **(i)** Display all the possible outcomes of the experiment using a tree diagram.
- **(ii)** For the experiment, the events *A*, *B*, *C* and *D* are defined as follows:

  A: All three coins show heads.
  B: At least two coins show tails.
  C: Exactly one coin shows a head.
  D: The faces appear alternately.

  For each part, identify if the following events are mutually exclusive.
  - **(a)** *A*, *B*
  - **(b)** *C*, *D*
  - **(c)** *B*, *C*
  - **(d)** *A*, *C*
  - **(e)** *B*, *D*
  - **(f)** *A*, *B*, *C*

11. In a game, Yasir attempts to score a penalty kick against a goalkeeper who will try to save his shot. There is an equal chance that he will score or miss his penalty kick. Yasir has three chances to score, and the game ends once Yasir scores a penalty kick.

- **(i)** Draw a tree diagram to show all the possible outcomes. What is the total number of outcomes?
- **(ii)** Events *A* and *B* are defined as follows:

  A: Exactly two penalty kicks are attempted.
  B: At most two penalty kicks are attempted.

  Are *A* and *B* mutually exclusive events? Explain.

2.4 Multiplication Law of Probability and independent events

In this section, we will learn another type of tree diagram to represent the sample space, and the conditions for multiplying probabilities.

A. Another type of tree diagram

Class Discussion

Choosing an appropriate diagram to represent sample space

There are 11 blue balls and 9 red balls in a bag. The balls are identical except for their colour. A ball is chosen at random and is then *replaced*. A second ball is then chosen at random.

1. Try representing the sample space for this probability experiment using
   (a) a sample space diagram, **(b)** a tree diagram.
2. Is it easy or tedious to represent the sample space in each diagram?

From the Class Discussion on page 47, we observe that it is very tedious to draw a 20 by 20 sample space diagram or a tree diagram with 20 × 20 = 400 branches.

However, we can simplify the tree diagram to represent the sample space as shown in Fig. 2.5, where *B* represents 'blue ball' and *R* represents 'red ball', and the probability on each branch represents the probability of the outcome at the end of the branch occurring.

Attention

In some countries, the terms 'tree diagram', 'probability tree' and 'probability tree diagram' are used interchangeably.

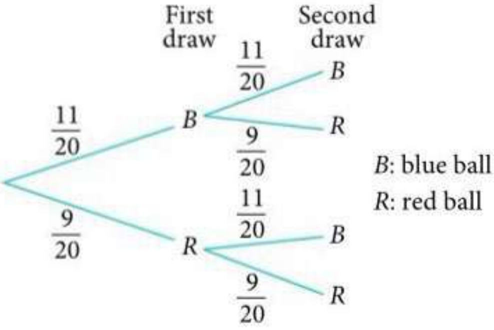


Fig. 2.5

Reflection

What is the difference between the tree diagram in Fig. 2.5 and the tree diagram that we have learnt in Section 2.2B?

How do we calculate the probability of a combined event, such as (*B*, *R*), i.e. obtaining a blue ball in the first draw, followed by a red ball in the second draw?

Let us first look at a probability experiment with fewer outcomes.

A fair coin is first tossed. Then a ball is drawn randomly from a bag which contains 1 blue ball, 1 red ball and 1 green ball. The balls are identical except for their colour.

Fig. 2.6 shows a tree diagram representing the sample space and the probability for each branch. It also shows all the 6 possible outcomes.

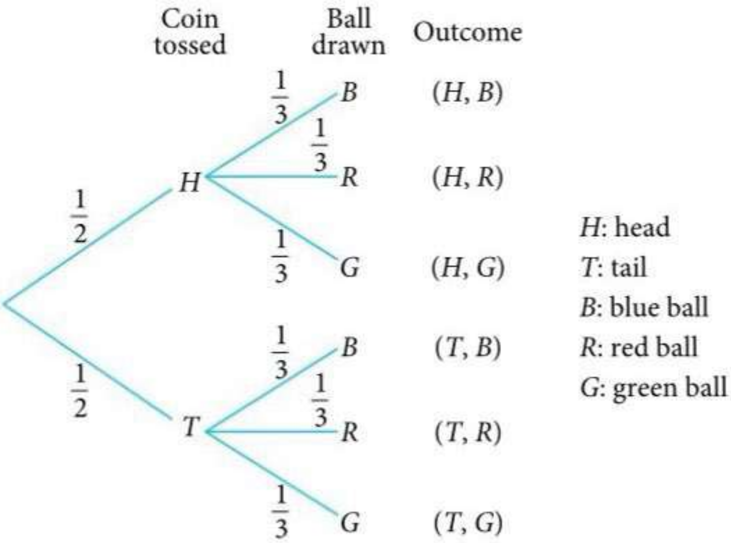


Fig. 2.6

What is the probability of obtaining the outcome (*H*, *B*)?

By counting the favourable and possible outcomes, the answer is $\frac{1}{6}$.

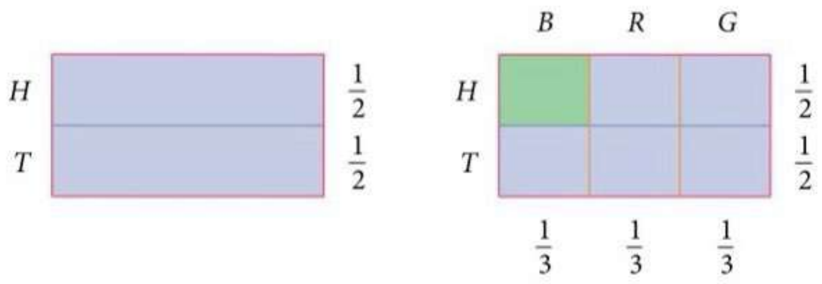
Can we obtain the answer from the tree diagram without listing all the possible outcomes?

Recall that the probability of all possible outcomes will add up to 1.

Let each of the largest rectangles in Fig. 2.7**(a)** and **(b)** (with a red outline) represent the total probability of 1.

Consider the first event of tossing a fair coin. Since obtaining a head or a tail is equally likely, we can divid rectangle into two equal parts such that each represents a probability of $\frac{1}{2}$ as shown in Fig. 2.7(**a**).

Next, consider the second event of drawing a ball from the bag. Since drawing a blue ball, a red ball and a green bal are three equally likely outcomes, we can divide each of the smaller rectangles into three equal parts as shown in Fig. 2.7(**b**).



(a) **(b)**

Fig. 2.7

The probability of obtaining the outcome (H, B) is represented by the green shaded rectangle in Fig. 2.7(**b**), which is $\frac{1}{3}$ of the rectangle representing a probability of $\frac{1}{2}$. Therefore, the probability of (H, B) is $\frac{1}{3}$ of $\frac{1}{2}$, i.e. $\frac{1}{3} \times \frac{1}{2}$ or $\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$.

This will help us understand why we can *multiply the probabilities along the respective branches* of a tree diagram to obtain the probability of a combined event.

To find the probability of (H, B) occurring, we multiply the probabilities along the respective branches of the tree diagram as shown in Fig. 2.8.

In other words,

$$\begin{aligned} P(H, B) &= P(H) \times P(\text{ball drawn is } B\text{, given that the coin toss obtains } H) \\ &= \frac{1}{2} \times \frac{1}{3} \qquad \text{see ○ in Fig. 2.8} \\ &= \frac{1}{6} \end{aligned}$$

Attention

For dependent events, $P(H, B) \neq P(H) \times P(B)$, which we will learn later in Section 2.4C. When we multiply the probabilities along the respective branches, we are actually doing this: $P(H, B) = P(H) \times P(B, \text{given } H)$.

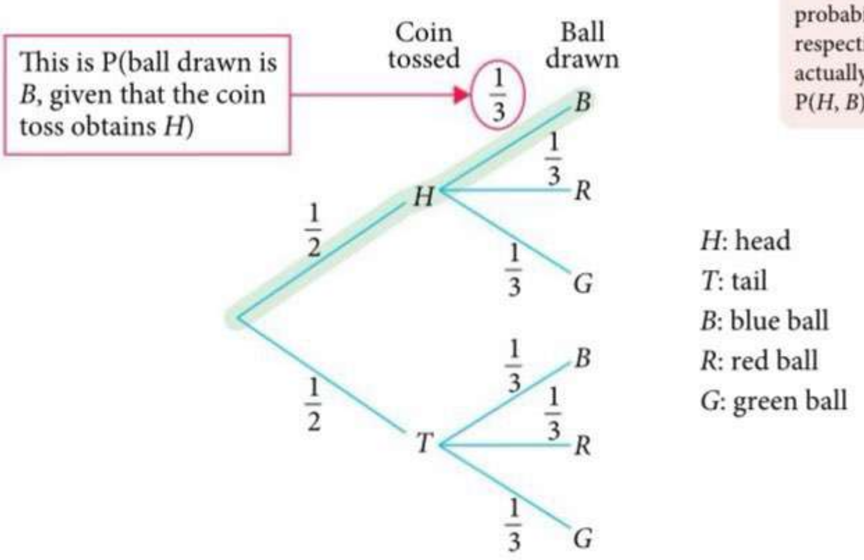

This is P(ball drawn is B, given that the coin toss obtains H)

Fig. 2.8

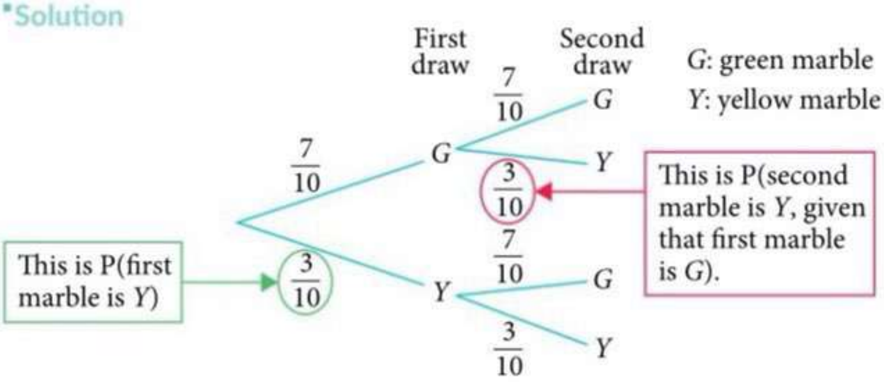
Worked Example 9

Calculating probability using tree diagram

There are 7 green marbles and 3 yellow marbles in a bag. The marbles are identical except for their colour. A marble is drawn at random from the bag and is replaced in the bag. A second marble is then drawn at random from the bag. Find the probability that

(i) the first marble drawn is yellow,
(ii) the second marble drawn is yellow given that the first marble drawn is green,
(iii) the first marble drawn is green and the second marble drawn is yellow,
(iv) the second marble drawn is yellow,
(v) one marble is green and the other is yellow.

Solution



G: green marble
Y: yellow marble

This is P(first marble is Y)

This is P(second marble is Y, given that first marble is G).

(i) P(first marble is Y) = $\frac{3}{10}$ see ○ – $\frac{\text{Number of yellow marbles in bag for first draw}}{\text{Total number of marbles in bag for first draw}}$

(ii) P(second marble is Y, given that first marble is G)

$= \frac{3}{10}$ see ○ – $\frac{\text{Number of yellow marbles in bag for second draw, given that first marble is green}}{\text{Total number of marbles in bag for second draw}}$

(iii) P(first marble is G and second marble is Y) or P(GY) $= \frac{7}{10} \times \frac{3}{10}$

$$= \frac{21}{100}$$

(iv) P(second marble is Y) = P(GY) + P(YY)

$$= \left(\frac{7}{10} \times \frac{3}{10}\right) + \left(\frac{3}{10} \times \frac{3}{10}\right)$$

$$= \frac{21}{100} + \frac{9}{100}$$

$$= \frac{30}{100}$$

$$= \frac{3}{10}$$

Reflection

Why is P(second marble is Y) = P(first marble is Y)?

(v) P(one marble is G and the other is Y) = P(GY) + P(YG)

$$= \left(\frac{7}{10} \times \frac{3}{10}\right) + \left(\frac{3}{10} \times \frac{7}{10}\right)$$

$$= \frac{21}{50}$$

Practise Now 9

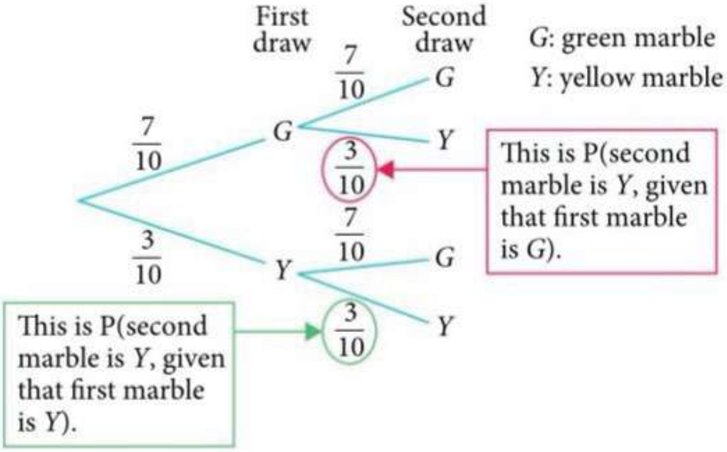
Similar and Further Questions
Exercise 2C
Questions 1, 2, 10, 11

A box contains 5 blue pens and 7 red pens. The pens are identical except for their colour. A pen is selected at random from the box and its colour is noted. The pen is replaced in the box. A second pen is then selected at random from the box. Find the probability that

(i) the first pen selected is red,
(ii) the second pen selected is blue, given that the first pen selected is red,
(iii) the first pen selected is red and the second pen selected is blue,
(iv) the second pen selected is blue,
(v) one pen is red and the other is blue.

B. Independent events

Let us revisit Worked Example 8. Fig. 2.9 shows the same tree diagram.



This is P(second marble is Y, given that first marble is G).

This is P(second marble is Y, given that first marble is Y).

Fig. 2.9

We observe:

- P(second marble is Y, given that first marble is G) = $\frac{3}{10}$ (see ○ in Fig. 2.9);
- P(second marble is Y, given that first marble is Y) = $\frac{3}{10}$ (see ○ in Fig. 2.9).

In other words, the probability of obtaining a yellow marble in the second draw is the same, regardless of whether the first draw is a green marble or a yellow marble.

Similarly, the probability of obtaining a green marble in the second draw is the same, regardless of whether the first draw is a green marble or a yellow marble.

Therefore, the first event of drawing a green marble or a yellow marble does *not* affect the second event of drawing a yellow marble.

We say that the second event is independent of the first event.

In general, two events are **independent** if the chance of one of them occurring does not affect the chance of the other event occurring.

Attention

Independent events are *not* the same as mutually exclusive events.

Referring to Worked Example 9 again, let A be the event that the first marble drawn is green and B be the event that the second marble drawn is yellow.

Since there are 7 green and 3 yellow marbles, P(A) = P(first marble is G) = $\frac{7}{10}$.

From part **(iv)**, P(B) = P(second marble is Y) = $\frac{3}{10}$.

From part **(iii)**, P(A and B) = P(first marble is G and second marble is Y)

$$= \frac{7}{10} \times \frac{3}{10}.$$

Therefore, we observe that P(A and B) = P(A) × P(B) in this case.

In general, the **Multiplication Law of Probability** states that:

> If A and B are **independent events**,
> $P(A \cap B)$ or **$P(A \text{ and } B) = P(A) \times P(B)$**.

Attention

P(A and B) = P(GY) = $\frac{7}{10} \times \frac{3}{10}$.
It is different from P(one marble is G and the other is Y), which is P(GY) + P(YG)
$= \left(\frac{7}{10} \times \frac{3}{10}\right) + \left(\frac{3}{10} \times \frac{7}{10}\right)$
(see Worked Example 9**(v)**).

Information

In higher level mathematics, we will learn a more general Multiplicative Law of Probability that applies to any two events, regardless of whether they are independent or dependent.

Worked Example 10

Calculating probability of combined events that are independent

There are 25 boys and 15 girls in a class. 12 of the boys and 5 of the girls wear spectacles. A class monitor and a class monitress are selected at random from the 25 boys and the 15 girls respectively. What is the probability that both the class monitor and class monitress wear spectacles?

Solution

P(class monitor wears spectacles) = $\frac{12}{25}$

P(class monitress wears spectacles) = $\frac{5}{15}$

$= \frac{1}{3}$

P(class monitor and class monitress both wear spectacles)

$= \frac{12}{25} \times \frac{1}{3}$ selections of monitor and monitress are independent

$= \frac{4}{25}$

Practise Now 10

Similar and Further Questions
Exercise 2C
Questions 3, 4, 12, 13, 19

Workers from a company work in either the 'Administrative' department or the 'Technical' department. There are 18 men and 12 women in the company. 12 men and 4 women are from the 'Technical' department. A chairman and a chairwoman are selected at random from the 18 men and the 12 women respectively. Find the probability that

(i) both the chairman and chairwoman are from the 'Technical' department,

(ii) the chairman is from the 'Administrative' department and the chairwoman is from the 'Technical' department.

Thinking Time

For two events A and B, what do $P(A \cup B)$ and $P(A \cap B)$ mean? Use Venn diagram(s) to illustrate your answers.

C. Dependent events

In Worked Example 9, the experiment involves replacing the item after the first draw.
What happens if the item is not replaced? How will this affect the outcome of the second draw?

Investigation Dependent events

There are 7 green marbles and 3 yellow marbles in a bag. The marbles are identical except for their colour. Two marbles are drawn at random from the bag (i.e. *without replacement*).

1. Copy and complete the probabilities on the tree diagram in Fig. 2.10.

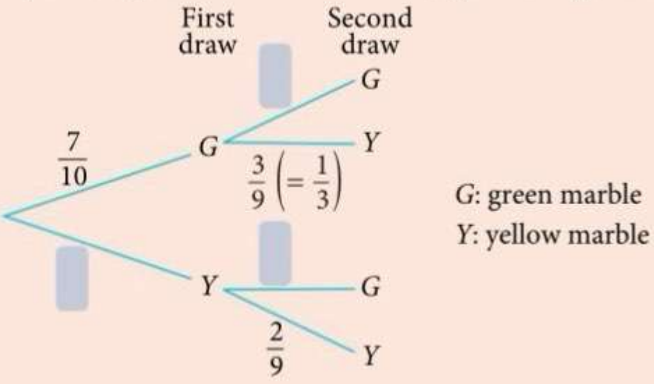


Fig. 2.10

2. Find the probability that
   (i) the second marble drawn is yellow, given that the first marble drawn is green,
   (ii) the second marble drawn is yellow, given that the first marble drawn is yellow.
3. Are the probabilities in Questions 2**(i)** and **(ii)** equal? Does the probability of drawing a yellow marble in the second draw *depend* on the outcome in the first draw? Explain.
4. Find the probability that the second marble drawn is yellow. Is this probability equal to the probabilities in Questions 2**(i)** and **(ii)**?
5. Let A be the event that the first marble drawn is green and B be the event that the second marble drawn is yellow.
   (i) Is event B independent or dependent on event A? Explain.
   (ii) Does the Multiplication Law of Probability, $P(A \text{ and } B) = P(A) \times P(B)$ apply in this case?

Attention

Drawing two marbles from the bag is equivalent to drawing the first marble and then drawing the second marble *without replacing* the first marble in the bag.

Information

For Question 4, notice that P(second marble is Y) = P(first marble is Y), which was first observed in Worked Example 9 for independent events. This is an interesting result that is true regardless of whether the two events are independent or dependent. It is beyond the scope of this Book to prove this result, but if you are interested, you can try to prove it for the general case of h green marbles and k yellow marbles for two draws that are independent, and then for two draws that are dependent. In fact, this result is true for n draws, where n is an integer greater than 1.

From the above Investigation, we observe that the Multiplication Law of Probability, $P(A \text{ and } B) = P(A) \times P(B)$, does not apply if A and B are **dependent** events.
In general:

If A and B are **dependent events**,
$P(A \cap B)$ or $P(A \text{ and } B) \neq P(A) \times P(B)$.

Attention

When finding P(A and B) in Question 5**(ii)** in the above Investigation, we still *multiply* the probabilities across the two respective branches, i.e.
P(A and B)
= P(first marble is G and second marble is Y)
$= \frac{7}{10} \times \frac{3}{9}$,
where P(A) = P(first marble is G) $= \frac{7}{10}$ and P(second marble is Y, given first marble is G) $= \frac{3}{9}$.
This was first explained in the text just before Fig. 2.8 on page 49.

Worked Example 11

Calculating probability of combined events that are dependent

Out of 33 students in a class, 21 study Geography and 12 study History. None of the students study both subjects. Two students are selected at random from the class.

(i) Draw a tree diagram to show the probabilities of the possible outcomes.

(ii) Albert said that the probability that both students study History is $\frac{16}{121}$. Explain what he has done incorrectly.

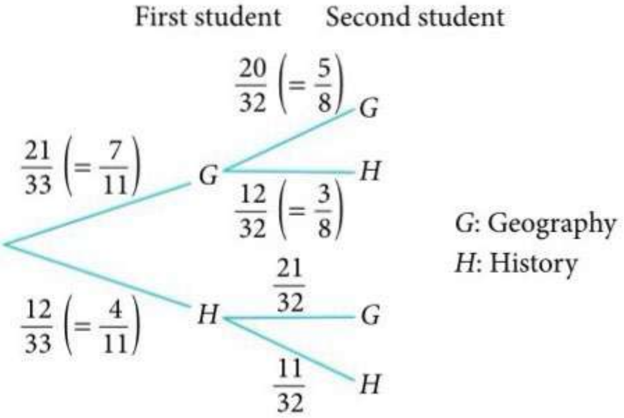
(iii) Find the probability that

  (a) the first student studies History and the second student studies Geography,

  (b) one student studies History while the other student studies Geography.

Solution

(i)



(ii) Albert calculated the probability as $\frac{12}{33} \times \frac{12}{33} = \frac{4}{11} \times \frac{4}{11} = \frac{16}{121}$. He did not take into account that there are only 11 History students and 32 students left after the first student was selected.

(iii) (a)
$$\begin{aligned} \text{P}(HG) &= \frac{4}{11} \times \frac{21}{32} \\ &= \frac{21}{88} \end{aligned}$$

Attention

(iii) (a) P(HG) means P(first student studies History and second student studies Geography).

(b) P(one student studies History and the other Geography)

$$\begin{aligned} &= \text{P}(HG) + \text{P}(GH) \\ &= \left(\frac{4}{11} \times \frac{21}{32}\right) + \left(\frac{7}{11} \times \frac{3}{8}\right) \\ &= \frac{21}{88} + \frac{21}{88} \\ &= \frac{21}{44} \end{aligned}$$

Practise Now 11

Similar and Further Questions
Exercise 2C
Questions 5–9, 14–18, 20, 21

1. A Science teacher needs two students to assist him with a Science demonstration. Two students are selected at random from his class of 16 boys and 12 girls.

(i) Draw a tree diagram to show the probabilities of the possible outcomes.

(ii) Bernard said that the probability that both students are girls is $\frac{9}{49}$. Explain what he has done incorrectly.

(iii) Find the probability that

  (a) the first student is a boy and the second student is a girl,

  (b) one student is a boy while the other student is a girl,

  (c) at least one of the students is a girl.

2. A bag contains 8 red balls, 7 blue balls and 1 white ball. Two balls are drawn fr bag at random, one after another, without replacement.
   - **(i)** Draw a tree diagram to show the probabilities of the possible outcomes.
   - **(ii)** Cheryl said that the probability that one ball is red and the other ball is blue is $\frac{7}{30}$. Explain what she has done incorrectly.
   - **(iii)** Find the probability that
     - **(a)** the first ball is red and the second ball is blue,
     - **(b)** one ball is red while the other ball is blue,
     - **(c)** the two balls are of the same colour.

Reflection

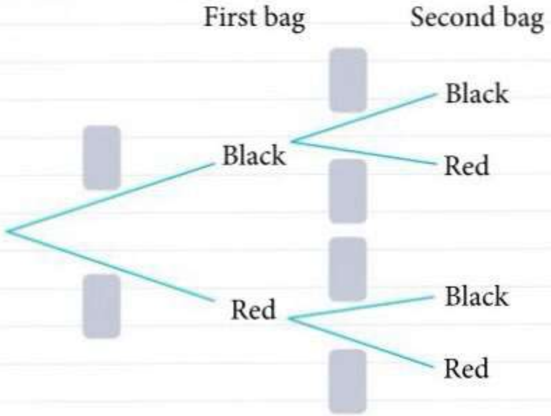
1. How do I decide which type of tree diagram to draw to represent the sample space of a probability experiment?
2. How do I decide whether two events are independent or dependent?
3. How do I decide when to use the Multiplication Law of Probability?
4. What have I learnt in this section or chapter that I am still unclear of?

Advanced Intermediate Basic

Exercise 2C

1. Raju has two bags, each containing 5 black marbles and 4 red marbles. He takes one marble at random from each bag.

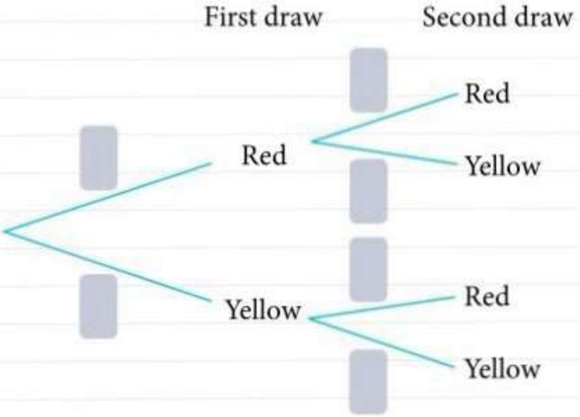
(i) Copy and complete the tree diagram.



(ii) Find the probability that Raju draws
- **(a)** a black marble from the first bag,
- **(b)** a red marble from the second bag, given that he draws a black marble from the first bag,
- **(c)** a black marble from the first bag and a red marble from the second bag,
- **(d)** a red marble from the second bag.

2. A bag contains 6 red balls and 4 yellow balls. A ball is chosen at random and then put back into the bag. The process is carried out twice.

(i) Copy and complete the tree diagram.



(ii) Find the probability of choosing
- **(a)** two red balls,
- **(b)** one ball of each colour,
- **(c)** a yellow ball on the second draw.

Exercise 2C

3. Albert takes either Bus A or Bus B to school every day. Buses A and B either arrive punctually or late. The probabilities of Bus A and Bus B arriving punctually are $\frac{2}{3}$ and $\frac{7}{8}$ respectively.

Find the probability that

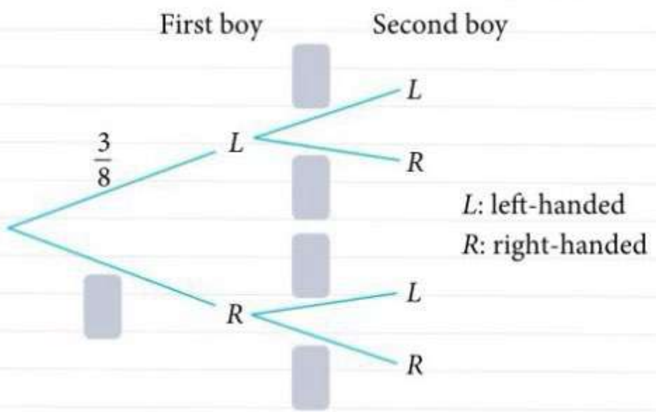
(i) both buses are punctual,
(ii) Bus A is late while Bus B is punctual,
(iii) exactly one of the buses is late.

4. Li Ting has two laptops, laptop X and laptop Y. In any one year, the probability of laptop X breaking down is 0.1 and the probability of laptop Y breaking down is 0.35. In any one year, what is the probability that

(i) both laptops break down,
(ii) laptop X breaks down but laptop Y does not,
(iii) exactly one of the laptops breaks down?

5. In a group of 8 boys, 3 are left-handed. The remaining 5 boys are right-handed. 2 boys are chosen at random from the same group of 8 boys.

(i) Copy and complete the tree diagram.



(ii) By using the tree diagram, find the probability that
- **(a)** the first boy chosen is right-handed and the second boy chosen is left-handed,
- **(b)** one boy is right-handed and the other is left-handed,
- **(c)** both boys chosen are left-handed.

6. A bag contains 6 green cards and 4 blue cards. After mixing the cards thoroughly, Yasir takes two cards at random from the bag, one after another.

(i) Draw a tree diagram to show the probabilities of the possible outcomes.
(ii) Yasir said that the probability that both cards are blue is $\frac{4}{25}$. Explain what he has done incorrectly.
(iii) Calculate the probability that Yasir takes out
- **(a)** two green cards,
- **(b)** one card of each colour,
- **(c)** at least one blue card.

7. Ten cards are marked with the letters 'P', 'R', 'O', 'P', 'O', 'R', 'T', 'I', 'O' and 'N' respectively. These cards are placed in a box. Two cards are drawn at random, without replacement.

(i) Draw a tree diagram to show the probabilities of the possible outcomes.
(ii) Joyce said that the probability that both cards bear the letter 'O' is $\frac{9}{100}$. Explain what she has done incorrectly.
(iii) Calculate the probability that
- **(a)** the two cards bear the letters 'P' and 'O' in that order,
- **(b)** the two cards bear the letters 'P' and 'O' in any order,
- **(c)** the two cards bear the same letter.

8. A class has 30 girls and 15 boys. Two representatives are to be selected at random from the class.

(i) Draw a tree diagram to show the probabilities of the possible outcomes.
(ii) Albert said that the probability that a boy and a girl are selected as representatives is $\frac{5}{22}$. Explain what he has done incorrectly.
(iii) Find the probability that
- **(a)** the first representative is a girl,
- **(b)** the second representative is a girl, given that the first representative is a boy,
- **(c)** the first representative is a boy and the second representative is a girl,

Exercise 2C

(d) a boy and a girl are selected as representatives,

(e) the second representative is a girl.

9. Five balls numbered 1, 2, 5, 8 and 9 are put in a bag.

(i) One ball is selected at random from the bag. Write down the probability that it is numbered '8'.

(ii) On another occasion, two balls are selected at random from the bag. By drawing a tree diagram, find the probability that

(a) the number on each ball is even,

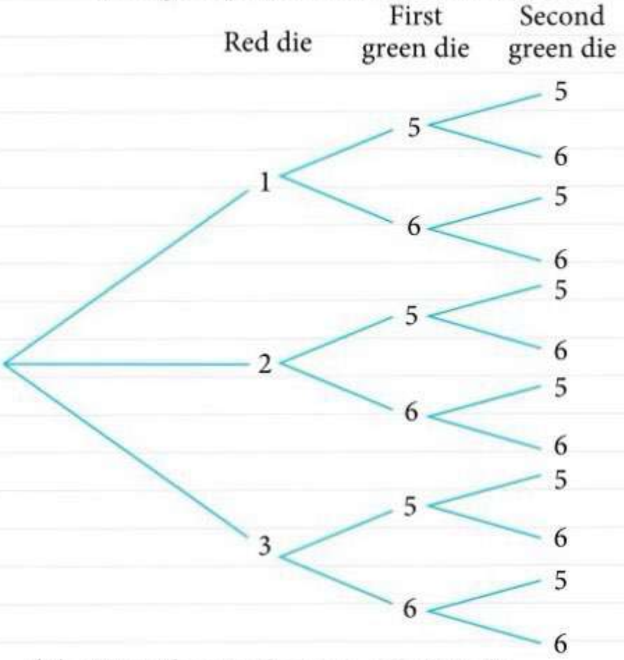
(b) the sum of the numbers on the balls is more than 10,

(c) the number on each ball is not a prime number,

(d) only one ball bears an odd number.

10. A red die has the number 1 on one face, the number 2 on two faces and the number 3 on three faces. Two green dice each has the number 6 on one face and the number 5 on five faces. All the dice are fair. The three dice are rolled together.

(i) Copy and complete the tree diagram by writing the probabilities on the 'branches'.



(ii) Using the tree diagram, calculate the probability of obtaining

(a) 2 on the red die, 5 on the first green die and 6 on the second green die,

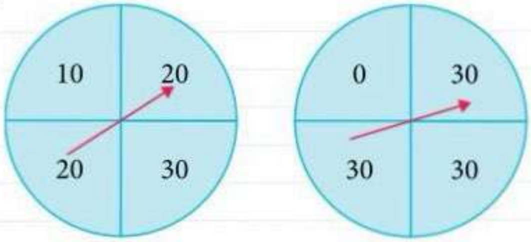
(b) 3 on the red die and 6 on each of the two green dice,

(c) exactly two sixes,

(d) a sum of 12,

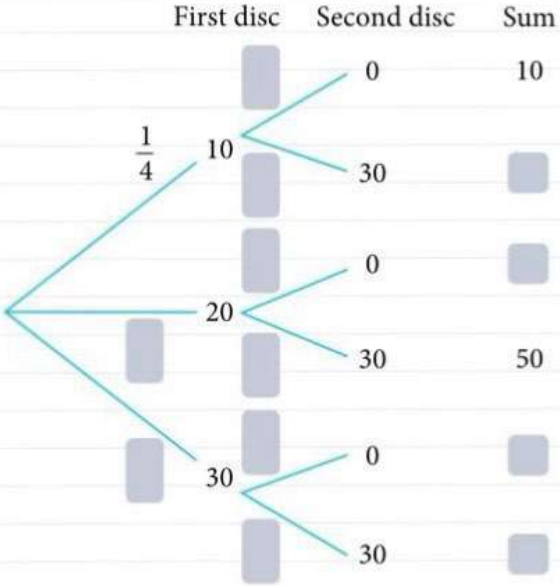
(e) a sum which is divisible by 3.

11. The diagram below shows two discs, each with four equal sectors. Each disc has a spinning pointer which, when spun, is equally likely to come to rest in any of the four equal sectors.
In a game, the player spins each pointer once. His score is the sum of the numbers shown by the pointers.



First disc Second disc

(i) Copy and complete the tree diagram shown.



(ii) With the help of the diagram, calculate the probability that

(a) the first number obtained is less than or equal to the second number obtained,

(b) the second number obtained is zero.

Exercise 2C

(iii) If the player's score is between 10 and 50, he receives \$2. If his score is more than 40, he receives \$5. Otherwise, he receives nothing. What is the probability that he receives
- **(a)** \$2,
- **(b)** \$5,
- **(c)** \$2 or \$5,
- **(d)** nothing?

12. The table below shows the number of male and female employees working in the front office, middle office and back office of an investment bank.

Department Gender	Front office	Middle office	Back office
Male	22	24	15
Female	18	16	25

Three representatives, one from each department, are selected at random to attend a seminar. What is the probability that
- **(i)** all three representatives are females,
- **(ii)** the representative from the front office is a male while the others are females,
- **(iii)** exactly one of the representatives is a male?

13. Bernard has three pairs of headphones – A, B and C. In a year, the probabilities of Headphones A, B and C malfunctioning are 0.03, 0.12 and 0.3 respectively. Find the probability that, in a year,
- **(i)** all three pairs of headphones malfunction,
- **(ii)** all three pairs of headphones function properly,
- **(iii)** at least one pair of headphones malfunctions,
- **(iv)** exactly two pairs of headphones malfunction.

14. In a wardrobe, there are 16 shirts, of which 8 are black, 6 are white and 2 are blue. The shirts are identical except for their colour.
- **(i)** If two shirts are taken out of the wardrobe at random, find the probability that
  - **(a)** both are black,
  - **(b)** one shirt is black and the other is white,
  - **(c)** the two shirts are of the same colour.
- **(ii)** If a third shirt is taken out from the wardrobe at random, calculate the probability that all three shirts are black.

15. Box A contains 7 blue balls and 5 yellow balls. Box B contains 3 blue balls and 7 yellow balls. One ball is removed at random from Box A and placed into Box B. After thoroughly mixing the balls, a ball is drawn at random from Box B and placed back into Box A.
- **(i)** Draw a tree diagram to show the probabilities of the possible outcomes.
- **(ii)** Find the probability that at the end of the experiment, Box A has
  - **(a)** more yellow balls than blue balls,
  - **(b)** exactly 7 blue and 5 yellow balls,
  - **(c)** twice as many blue balls as yellow balls.

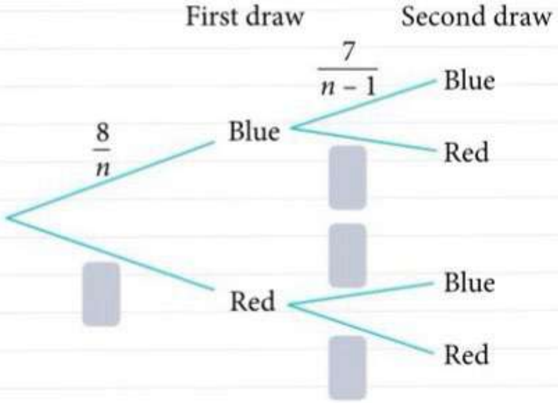
16. Class A has 18 boys and 17 girls and Class B has 14 boys and 22 girls. A student from Class A is selected at random and transferred to Class B. A teacher then selects a student at random from the extended Class B. Find the probability that the student selected is
- **(i)** the student who was initially from Class A,
- **(ii)** a boy.

17. A bag contains 10 red balls, 9 blue balls and 7 yellow balls. Three balls are drawn in succession, at random, without replacement. By drawing a tree diagram or otherwise, find the probability of obtaining
- **(i)** a red and two blue balls in that order,
- **(ii)** a red, a yellow and a blue ball in that order,
- **(iii)** three balls of different colours.

Exercise 2C

18. A box contains n marbles. 8 of the marbles are blue and the rest are red. Ken takes two marbles from the box, at random, without replacement.

(i) Complete the tree diagram.



(ii) The probability that Ken takes two blue marbles is $\frac{4}{13}$. Write down an equation to represent this information and show that it simplifies to $n^2 - n - 182 = 0$.

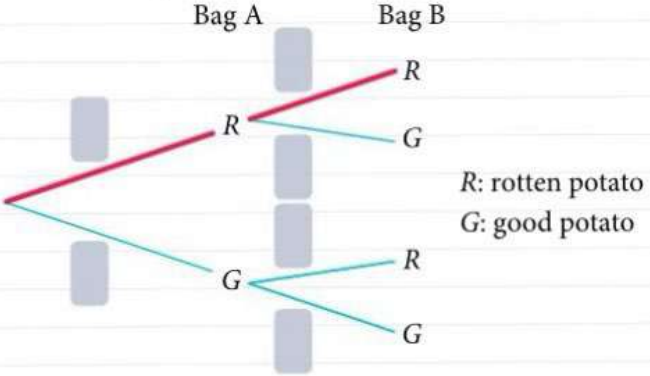
(iii) Solve the equation $n^2 - n - 182 = 0$.

(iv) Explain why one of the solutions in part **(iii)** must be rejected.

(v) Find, as a fraction in its simplest form, the probability that Ken takes one blue marble and one red marble.

19. Bag A contains 20 potatoes, 4 of which are rotten. Bag B contains 12 potatoes, 3 of which are rotten. David selects one potato at random from each bag.

(i) Complete the tree diagram below to show the possible outcomes of David's selections.



(ii) David wants to find out the probability of selecting two rotten potatoes. He multiplies the probabilities along the 'RR' branch (highlighted in red) and he says that he is using the 'Multiplication Law of Probability'. Do you agree with what David says? Explain your answer clearly.

20. A box contains 7 electrical components. The box was dropped in transit and one of the components became defective, but not visibly. The components are taken out from the box at random and tested until the defective component is obtained. What is the probability that the defective component is the third component tested?

21. A game is such that a fair die is rolled repeatedly until a '6' is obtained. Find the probability that

(i) **(a)** the game ends on the third roll,
(b) the game ends on the fourth roll,
(c) the game ends by the fourth roll.

(ii) Suppose now that the game is such that the same die is rolled repeatedly until two '6's are obtained.
Find the probability that
(a) the game ends on the third roll,
(b) the game ends on the third roll and the sum of the scores is odd.

Looking Back

In this chapter, we continue our study of probability to find the probability of combined events. We have learnt about the idea of independent and dependent events and used different **diagrams**, such as sample space diagrams and tree diagrams, to represent the sample space of multi-stage probability experiments. These diagrams are useful in helping us make sense of the problem so that we can apply the appropriate rules to work out the probability. The ideas we have learnt in this chapter provide the basis for us to analyse the possible outcomes of games, sporting events, and other real-world situations. We have also learnt how we can apply simple ideas of probability to **model** and estimate the population of pangolins in Singapore. In the future, you might learn more about how the idea of probability can be used to model population changes of an animal species.

Summary

1. In a probability experiment, the probability, P(E), of an event E occurring is given by:

$$P(E) = \frac{\text{measure of favourable outcomes for event } E}{\text{measure of all possible outcomes for sample space } S}$$

2. For an event E, $P(E') = 1 - P(E)$.

3. For a probability experiment with equally likely outcomes, we can represent the sample space in various ways, depending on the number of stages of the experiment.

Number of stages of probability experiment	Example(s)	Representation of sample space
One-stage	Tossing 1 coin	List of outcomes in set notation
Two-stage	Tossing 2 coins; tossing a coin 2 times	**Sample space diagram** or tree diagram
Three-stage	Tossing 3 coins; tossing a coin 3 times	**Tree diagram**

- What is a key feature of a sample space diagram and of a tree diagram?

4. For a probability experiment with outcomes that are not equally likely to occur, we can represent the sample space using a different type of tree diagram, i.e. one with probabilities written on its branches.

- Can you represent the sample space of a probability experiment with equally likely outcomes using a tree diagram with probabilities written on its branches?

5. The **Addition Law of Probability** states that if A and B are **mutually exclusive events**, then $P(A \cup B)$ or $P(A \text{ or } B) = P(A) + P(B)$.

- Give an example of a set of mutually exclusive events.

6. The **Multiplication Law of Probability** states that if A and B are **independent events**, then $P(A \cap B)$ or $P(A \text{ and } B) = P(A) \times P(B)$.

- Give an example of a set of independent events.

CHAPTER 3

Statistical Data Analysis

In Books 2 and 3, you took the first steps in learning about the basics of statistics, which is the study of the collection, interpretation, and use of numerical data. The use of data is critical for us to gain insights into the world we live in and make key decisions about how we live.

For instance, do you know how much plastic waste is generated every year in Pakistan? How much of that is recycled? How much waste do you generate each week? Such data is useful in understanding the magnitude and extent of waste management issues and in helping us make decisions to reduce wastage in our daily lives.

What is a suitable way to describe these datasets? While we have learnt about measures of central tendency in Book 3, these measures of central tendency alone may not be enough to describe a dataset completely.

In this chapter, we will learn about measures of spread and examine how the various **measures** can be incorporated to provide a more complete picture about our datasets.

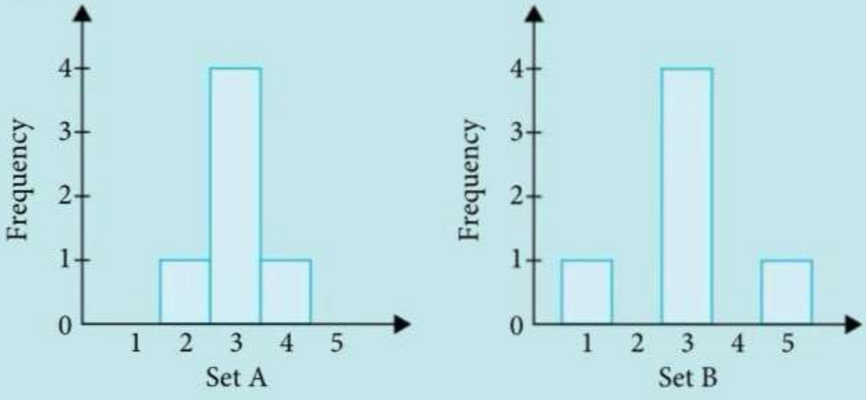
Learning Outcomes

What will we learn in this chapter?

- What cumulative frequency tables, cumulative frequency curves and scatter diagrams are
- How to estimate the median, quartiles and percentiles from cumulative frequency curves
- How to calculate quartiles of a set of discrete data
- How to estimate the spread of a set of data using range and interquartile range
- How to compare two sets of data using the median and the interquartile range, or the mean and the range
- Why statistical data analysis is useful in real life

Introductory Problem

Fig. 3.1(**a**) and (**b**) show the histograms for Set A and Set B respectively, both with size $n = 6$, mean = 3, median = 3 and mode = 3. Although the three averages (mean, median and mode) are all equal to 3 for Set A and Set B, the two distributions are different.



(a) **(b)**

Fig. 3.1

1. Draw another two histograms to represent two different distributions with size $n = 6$, mean = 3, median = 3 and mode = 3.
2. Are the three averages (mean, median and mode) always adequate for comparing two sets of data? Explain.

In the Introductory Problem, we observe that two sets of data can have the same averages (mean, median and mode), but the distributions can still be different. Therefore, there is a need to describe the distributions using other measures, such as **quartiles**, **interquartile range** and **range**.
But first, let us learn how to present a set of data (called **dataset** or data set) by constructing a **table of cumulative frequencies**.

3.1 Cumulative frequency table and curve

A. Cumulative frequency table

Class Discussion Constructing table of cumulative frequencies

Table 3.1(**a**) shows the frequency table for the time 40 students spent using the computer on a particular day.
It is called a *'less than or equal' frequency table* because the class intervals are of the form $a < t \leqslant b$ (except for the first class interval). Why do you think the first class interval is an exception?
Table 3.1(**b**) shows the corresponding table of cumulative frequencies. It is called a *'less than or equal'* **cumulative frequency table** because it is constructed from a 'less than or equal' frequency table.
To find the cumulative frequency of a particular hour k, we must add up the frequencies which are less than or equal to k, i.e. $t \leqslant k$. For example, the cumulative frequency of 4 hours, i.e. $t \leqslant 4$, is $3 + 5 = 8$.

Attention

The first class interval can also be $0 < t \leqslant 2$ if all the students used the computer.

Number of hours 40 students spent using the computer

Number of hours, t	Frequency
$0 \leqslant t \leqslant 2$	3
$2 < t \leqslant 4$	5
$4 < t \leqslant 6$	16
$6 < t \leqslant 8$	12
$8 < t \leqslant 10$	4

(a)

Number of hours, t	Cumulative frequency
$t \leqslant 2$	3
$t \leqslant 4$	$3 + 5 = 8$

(b)

Table 3.1

1. Using the information from Table 3.1(**a**), copy and complete Table 3.1(**b**).
2. Using your answers in Table 3.1(**b**), find the number of students who used the computer for
   - **(i)** 6 hours or less,
   - **(ii)** more than 8 hours,
   - **(iii)** more than 4 hours but not more than 10 hours.
3. What does the last entry under 'Cumulative frequency' of Table 3.1(**b**) represent? Explain.
4. Can you use Table 3.1(**a**) or (**b**) to find the number of students who used the computer for
   - **(i)** less than 6 hours,
   - **(ii)** at least 8 hours,
   - **(iii)** at most 7 hours?

   Explain.
5. Suppose the frequency table in Table 3.2 is given instead. We call this frequency table a *'less than' frequency table* because the class intervals are of the form $a \leqslant t < b$. Can you find the answers to Question 4 now?

Problem-solving Tip

(i) We do not know the number of students who used the computer for exactly 6 hours.

Number of hours 40 students spent on the computer

Number of hours, t	Frequency
$0 \leqslant t < 2$	3
$2 \leqslant t < 4$	5
$4 \leqslant t < 6$	14
$6 \leqslant t < 8$	13
$8 \leqslant t < 10$	5

Table 3.2

From the above Class Discussion, we learn that the cumulative frequency of a particular value can be obtained by *adding up* the frequencies which are *less than or equal to*, or *less than*, that value, depending on whether we are given a 'less than or equal' frequency table or a 'less than' frequency table respectively.

Big Idea

Diagrams

A cumulative frequency table is a statistical diagram used to display the cumulative frequencies of a dataset. It allows us to gather information such as the number of students whose score is below a certain mark or the number of insects shorter than a certain length, as shown in Practise Now 1A.

Practise Now 1A

Similar and Further Questions
Exercise 3A
Questions 1–3, 7, 11, 12

The lengths of 40 insects of a certain species were measured to the nearest millimetre. The frequency distribution is given in the table below.

Length (x mm)	Frequency
$25 \leqslant x < 30$	1
$30 \leqslant x < 35$	3
$35 \leqslant x < 40$	6
$40 \leqslant x < 45$	12
$45 \leqslant x < 50$	10
$50 \leqslant x < 55$	6
$55 \leqslant x < 60$	2

(i) Construct a cumulative frequency table for the given data.

(ii) Using the cumulative frequency table which you have constructed, find the number of insects which are
  (a) less than 50 mm in length,
  (b) at least 45 mm in length,
  (c) at least 35 mm but less than 50 mm in length.

(iii) Explain why it is not possible to state the number of insects which are 50 mm or less in length.

(iv) If an insect is chosen at random from the 40 insects, find the probability that its length will be less than 50 mm.

B. Cumulative frequency curve

In Questions 4 and 5 of the previous Class Discussion, we have learnt that it is not possible to find the cumulative frequency of a data value that lies inside a class interval, e.g. the data value $t = 7$ lies in the class interval $6 < t \leqslant 8$ or $6 \leqslant t < 8$.
But how can we *estimate* the cumulative frequency?
To do that, we have to draw a **cumulative frequency curve**. In fact, we can estimate other statistical measures, such as the median, quartiles and percentiles from the cumulative frequency curve.
In Worked Example 1, we will make use of the cumulative frequencies from Table 3.1(**b**) to learn how to draw a cumulative frequency curve.

Worked Example 1

Drawing and interpreting cumulative frequency curve

The table below shows the cumulative frequencies for the number of hours (t) 40 students spent using the computer, on a particular day.

Number of hours (t)	$t \leqslant 2$	$t \leqslant 4$	$t \leqslant 6$	$t \leqslant 8$	$t \leqslant 10$
Cumulative frequency	3	8	24	36	40

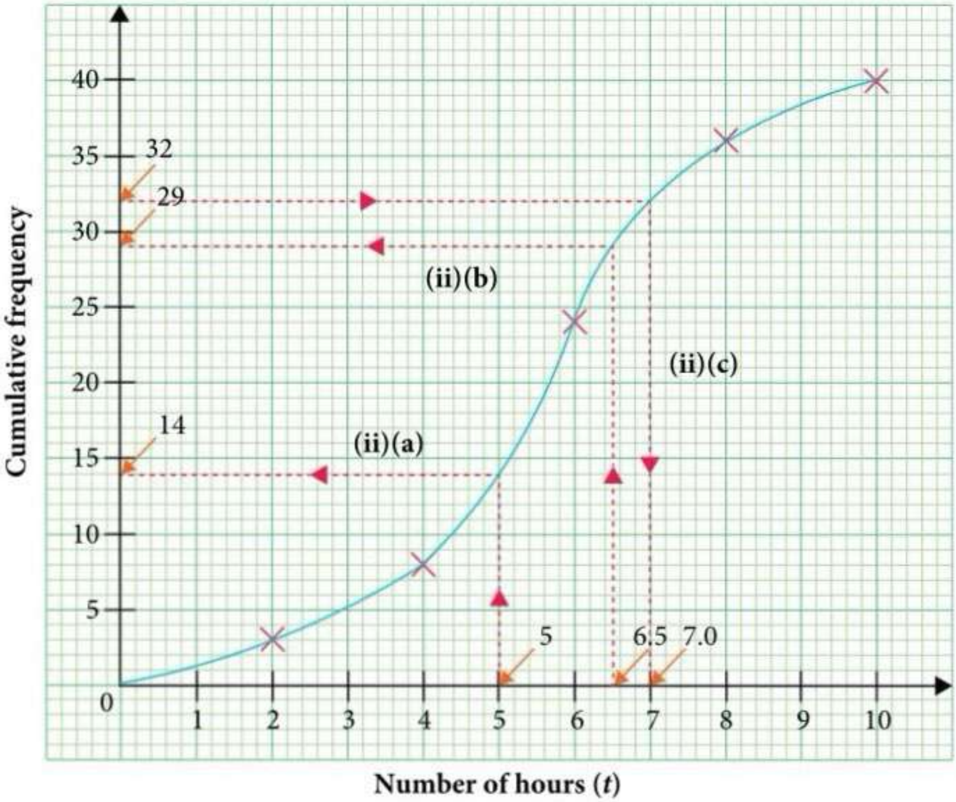
(i) Using a scale of 1 cm to represent 1 hour on the horizontal axis and 1 cm to represent 5 students on the vertical axis, draw a cumulative frequency curve for the data given in the table.

(ii) Using the cumulative frequency curve, estimate
  (a) the number of students who used the computer for 5 hours or less,
  (b) the relative frequency of students who used the computer for more than 6.5 hours,
  (c) the value of t, such that 80% of the students used the computer for at most t hours.

(iii) Explain why it is not possible to state the number of students who used the co u for less than 5 hours.

Solution

(i) **Cumulative frequency curve for the number of hours spent using the computer**



(ii) **(a)** From the curve, the number of students who used the computer for 5 hours or less is 14.

(b) From the curve, the number of students who used the computer for 6.5 hours or less is 29.

Then 40 – 29 = 11 students used the computer for more than 6.5 hours.

∴ relative frequency of students who used the computer for more than 6.5 hours is $\frac{11}{40}$

(c) 80% of the students means $\frac{80}{100} \times 40 = 32$, i.e.

32 students used the computer for at most t hours.

From the curve, $t = 7.0$.

(iii) Since the curve is drawn based on a 'less than or equal' cumulative frequency table, the reading on the curve corresponding to $t = 5$ will also include students who used the computer for exactly 5 hours. Therefore, it is not possible to state the number of students who used the computer for less than 5 hours.

Problem-solving Tip

To plot a cumulative frequency curve on graph paper:

Step 1: Draw and label the horizontal axis 'Number of hours (t)'.

Step 2: Draw and label the vertical axis 'Cumulative frequency'.

Step 3: Plot the points (2, 3), (4, 8), (6, 24), (8, 36) and (10, 40).

Step 4: Join all the points with a smooth curve.

Problem-solving Tip

(ii) **(a)** Draw a vertical dotted line from $t = 5$ on the horizontal axis until it intersects the curve. Then draw a horizontal line from the point of intersection on the curve to the vertical axis. The reading of '14' indicates that 14 students used the computer for 5 hours or less. Why is this an *estimate*?

(c) The answer can only be accurate up to half of a small square grid.

Recall

(ii) (b) In Book 2, we learnt that relative frequency = $\frac{\text{number of occurrences}}{\text{total number of trials}}$.

Here, there are 11 students who used the computer for more than 6.5 hours, out of a total of 40 students.

Practise Now 1B

Similar and Further Questions
Exercise 3A
Questions 4–6, 8–10, 13–15

The table below shows the amount of milk (in litres) produced by each of the 70 cows in a dairy farm, on a particular day.

Amount of milk (*x* litres)	Number of cows
$0 \leqslant x < 4$	7
$4 \leqslant x < 6$	11
$6 \leqslant x < 8$	17
$8 \leqslant x < 10$	20
$10 \leqslant x < 12$	10
$12 \leqslant x < 14$	5

(i) Copy and complete the following cumulative frequency table for the data given.

Amount of milk (*x* litres)	Number of cows
$x < 4$	7
$x < 6$	18
$x < 8$	
$x < 10$	
$x < 12$	
$x < 14$	

(ii) Using a scale of 1 cm to represent 1 litre on the horizontal axis and 1 cm to represent 5 cows on the vertical axis, draw a cumulative frequency curve for the data given.

(iii) Using the curve in part **(ii)**, estimate

- **(a)** the number of cows that produced less than 9.4 litres of milk,
- **(b)** the fraction of cows that produced at least 7.4 litres of milk,
- **(c)** the value of *x*, if 70% of the cows produced at least *x* litres of milk.

(iv) If two cows are chosen at random from the 70 cows, find the probability that the cows each produced less than 9.4 litres of milk.

Reflection

1. What information can I obtain from a 'less than' cumulative frequency table that I cannot obtain from a 'less than or equal' cumulative frequency table?
2. Given a cumulative frequency table, how do I estimate the cumulative frequency of a data value that lies inside a class interval?
3. What have I learnt in this section that I am still unclear of?

Exercise 3A

1. 120 students took a Mathematics examination and their results are shown in the table below.

Marks (m)	Number of students
$0 < m \leqslant 10$	3
$10 < m \leqslant 20$	12
$20 < m \leqslant 30$	9
$30 < m \leqslant 40$	11
$40 < m \leqslant 50$	17
$50 < m \leqslant 60$	19
$60 < m \leqslant 70$	20
$70 < m \leqslant 80$	14
$80 < m \leqslant 90$	10
$90 < m \leqslant 100$	5

(i) Copy and complete the following cumulative frequency table.

Marks (m)	Number of students
$m \leqslant 10$	3
$m \leqslant 20$	
$m \leqslant 30$	
$m \leqslant 40$	
$m \leqslant 50$	
$m \leqslant 60$	
$m \leqslant 70$	
$m \leqslant 80$	
$m \leqslant 90$	
$m \leqslant 100$	

(ii) Using the table in part **(i)**, find the number of students who

(a) scored less than or equal to 30 marks,

(b) scored more than 80 marks,

(c) scored more than 40 marks but not more than 90 marks.

(iii) Explain why it is not possible to state the number of students who scored less than 30 marks.

(iv) If a student is chosen at random from the 120 students, find the relative frequency that he or she has scored more than 80 marks.

2. The table below shows the daily total rainfall, in mm, recorded in a particular month.

Daily total rainfall (x mm)	Number of days
$0 \leqslant x < 10$	21
$10 \leqslant x < 20$	3
$20 \leqslant x < 30$	2
$30 \leqslant x < 40$	1
$40 \leqslant x < 50$	2
$50 \leqslant x < 60$	1
$60 \leqslant x < 70$	1

(i) Construct a cumulative frequency table for the given data.

(ii) Using the cumulative frequency table which you have constructed, find the number of days when the rainfall was

(a) less than 30 mm,

(b) at least 40 mm,

(c) at least 20 mm but less than 50 mm.

(iii) Explain why it is not possible to state the number of days when the rainfall was at most 60 mm.

(iv) If a day is chosen at random from the month, find the relative frequency that the day has at least a total rainfall of 40 mm.

Exercise 3A

3. In a toy factory, quality checks were performed on 65 batches of toys to check for defects. The frequency distribution is given in the table below.

Number of defects (x)	Frequency
$0 \leqslant x \leqslant 4$	21
$4 < x \leqslant 8$	18
$8 < x \leqslant 12$	14
$12 < x \leqslant 16$	6
$16 < x \leqslant 20$	3
$20 < x \leqslant 24$	2
$24 < x \leqslant 28$	1

(i) Construct a cumulative frequency table for the given data.

(ii) The batches were graded as follows:

Grade 1: more than 20 defects,

Grade 2: more than 12 and less than or equal to 20 defects,

Grade 3: more than 4 and less than or equal to 12 defects.

Using the cumulative frequency table which you have constructed, find the number of batches graded as

(a) Grade 1,

(b) Grade 2,

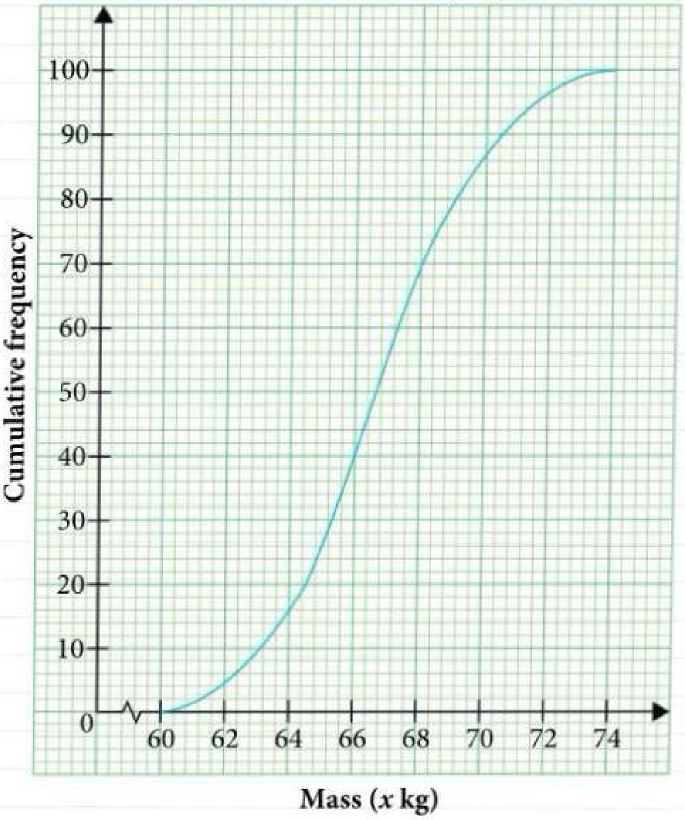
(c) Grade 3.

(iii) Explain why it is not possible to state the number of batches which have at least 16 defects.

(iv) If a batch is chosen at random from the 65 batches, find the relative frequency that the batch has more than 8 defects.

4. The masses, in kg, of 100 people were measured. The cumulative frequency curve shows the mass, x kg, and the number of people with masses less than or equal to x kg.

Cumulative frequency curve for the masses of people



(i) Use the curve to estimate

(a) the number of people whose masses are less than or equal to 65 kg,

(b) the number of people whose masses are more than 68.6 kg,

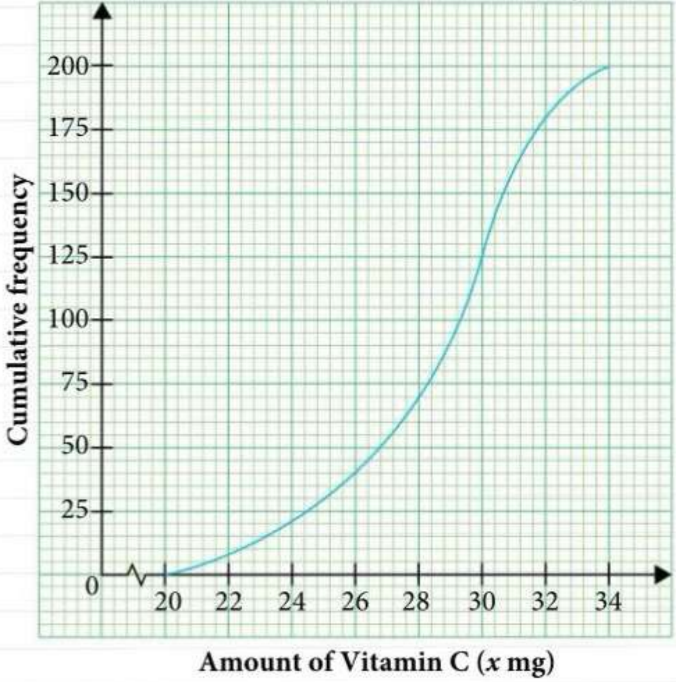
(c) the percentage of the total number of people whose masses are more than 64.4 kg.

(ii) If a person is chosen at random from the 100 people, find the relative frequency that the person has a mass of more than 68.6 kg.

Exercise 3A

5. The Vitamin C contents of 200 oranges were measured. The cumulative frequency curve below shows the Vitamin C content, x mg, and the number of oranges having Vitamin C content less than x mg.

Cumulative frequency curve for the Vitamin C contents of oranges



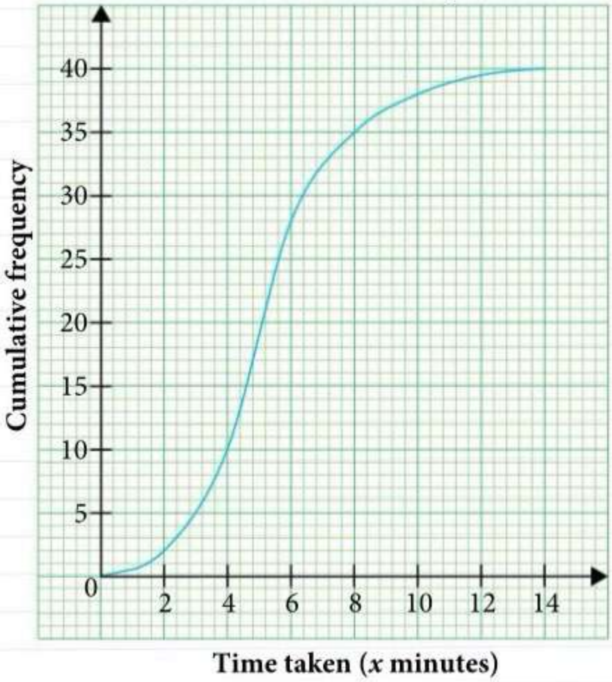
(i) Use the curve to estimate

- **(a)** the number of oranges that contain less than 32 mg of Vitamin C,
- **(b)** the fraction of oranges that contain 26 mg or more of Vitamin C,
- **(c)** the value of p, given that 40% of the oranges contain at least p mg of Vitamin C.

(ii) If an orange is chosen at random from the 200 oranges, find the relative frequency that the orange contains at least 28 mg of Vitamin C.

6. The time taken for 40 students to solve an online puzzle was recorded. The cumulative frequency curve below shows the time taken, x minutes, and the number of students who required less than x minutes to solve the puzzle.

Cumulative frequency curve for the time taken by students to solve the puzzle



(i) Use the curve to estimate

- **(a)** the number of students who took at least 4 minutes to solve the puzzle,
- **(b)** the percentage of students who took less than 6 minutes to solve the puzzle,
- **(c)** the value of q, given that $\frac{1}{4}$ of the students took less than q minutes to solve the puzzle.

(ii) If a student is chosen at random from the 40 students, find the relative frequency that the student took at least 10 minutes to solve the puzzle.

Exercise 3A

7. The table below shows the cumulative frequencies for the annual income of 200 households in a certain district.

Annual income ($$x$, in thousands)	Cumulative frequency
$x = 0$	0
$x < 20$	42
$x < 40$	80
$x < 60$	102
$x < 80$	122
$x < 110$	188
$x < 120$	200

(i) Construct a frequency table for the given data.

(ii) On a sheet of graph paper, draw a histogram to represent the frequency distribution.

(iii) If a household is selected at random from this district, what is the interval of annual incomes the household is most likely to earn?

(iv) If two households are selected at random from this district, find the relative frequency that the annual income of the households will be at least $20 000 but less than $60 000.

8. The table below shows the ages of 80 workers in a company.

Age (x years)	Number of workers
$20 \leqslant x < 25$	9
$25 \leqslant x < 30$	1
$30 \leqslant x < 35$	6
$35 \leqslant x < 40$	24
$40 \leqslant x < 45$	21
$45 \leqslant x < 50$	12
$50 \leqslant x < 55$	7

(i) Copy and complete the following cumulative frequency table for the data given.

Age (x years)	Number of workers
$x < 25$	9
$x < 30$	10
$x < 35$	
$x < 40$	
$x < 45$	
$x < 50$	
$x < 55$	

(ii) Using a scale of 1 cm to represent 5 years on the horizontal axis and 1 cm to represent 10 workers on the vertical axis, draw a cumulative frequency curve for the data given.

(iii) Using the curve in part **(ii)**, estimate

(a) the number of workers who are younger than 33 years old,

(b) the fraction of workers who are at least 46 years old,

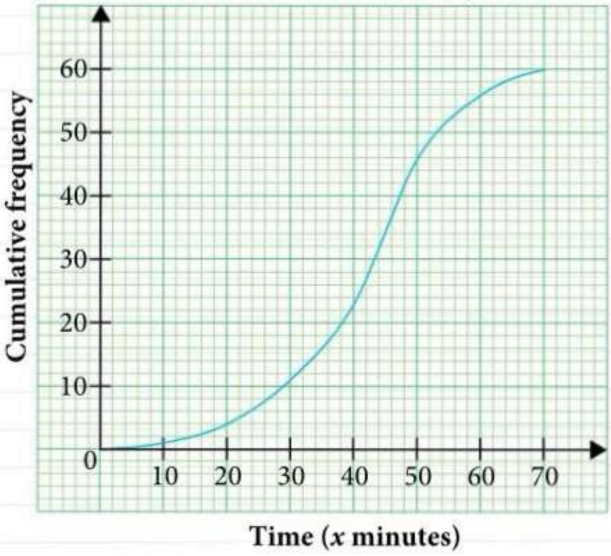
(c) the value of p, if 75% of the workers are at least p years old.

(iv) If two workers are chosen at random from the 80 workers, find the probability that the workers are younger than 38 years old.

Exercise 3A

9. The average amounts of time that 60 adults exercise daily are recorded. The cumulative frequency curve below shows the amount of time, x minutes, and the number of adults who exercise at most x minutes daily.

Cumulative frequency curve for the average amount of time adults exercise daily



(i) Use the curve to estimate

- **(a)** the number of adults who exercise more than 38 minutes daily,
- **(b)** the percentage of adults who exercise at most 35 minutes daily,
- **(c)** the value of t, if $\frac{1}{3}$ of the adults exercise more than t minutes daily.

(ii) A newspaper article claims that one in three adults exercises at most 20 minutes daily. Comment on whether the above data supports this claim.

(iii) If two adults are selected at random from the 60 adults, find the probability that both adults exercise at most 66 minutes daily.

(iv) Copy and complete the following cumulative frequency table for the data given in the cumulative frequency curve.

Time (x minutes)	Number of adults
$x \leqslant 0$	
$x \leqslant 10$	
$x \leqslant 20$	
$x \leqslant 70$	60

(v) **(a)** Copy and complete the grouped frequency table for the data given in the cumulative frequency table.

Time (x minutes)	Frequency
$0 < x \leqslant 10$	
$10 < x \leqslant 20$	
$20 < x \leqslant 30$	
$30 < x \leqslant 40$	
$40 < x \leqslant 50$	
$50 < x \leqslant 60$	
$60 < x \leqslant 70$	
$70 < x \leqslant 80$	

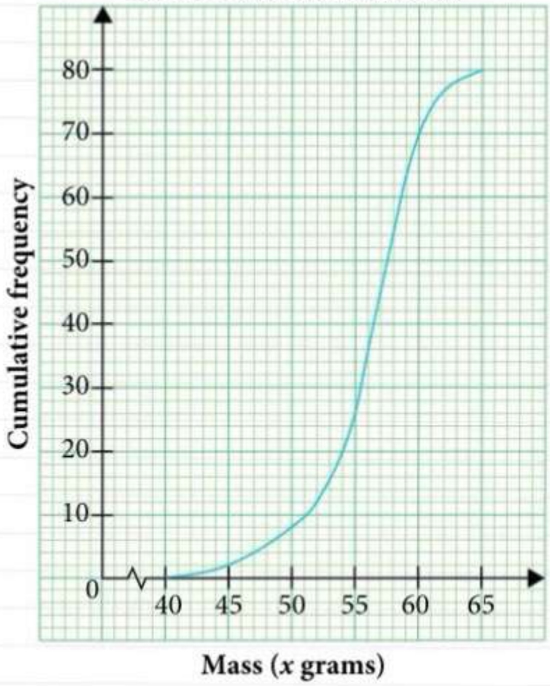
(b) Using the table, find an estimate of the mean amount of time that adults exercise daily.

(c) Explain why the mean is an estimate.

Exercise 3A

10. The masses of 80 tomatoes produced at a nursery are recorded. The cumulative frequency curve below shows the mass, x grams, and the number of tomatoes which are less than or equal to x grams.

Cumulative frequency curve for the masses of tomatoes



(i) **(a)** Tomatoes with masses more than 56 g are rated Grade A tomatoes. Estimate the percentage of Grade A tomatoes.

(b) Tomatoes that are y g or less are rated Grade C. Estimate the value of y if 15% of the tomatoes are rated Grade C.

(c) Estimate the number of Grade B tomatoes, which are between Grades A and C.

(d) The owner of the nursery claims that 60% of the tomatoes produced in the nursery are rated Grade A. Comment on whether your answer in part **(i)(a)** supports his claim.

(e) If two tomatoes are chosen at random from the 80 tomatoes, find the probability that the two tomatoes are rated either Grade A or Grade B.

(ii) **(a)** Copy and complete the grouped frequency table for the masses of the tomatoes.

Mass (x g)	Frequency
$40 < x \leqslant 45$	
$45 < x \leqslant 50$	
$50 < x \leqslant 55$	
$55 < x \leqslant 60$	
$60 < x \leqslant 65$	

(b) Using the table, find an estimate of the mean mass of a tomato produced at the nursery.

(c) Explain why the mean is an estimate.

11. 230 men took part in a physical fitness test and were required to do pull-ups. The number of pull-ups done by the men is shown in the frequency table below.

Number of pull-ups (x)	Frequency
$0 \leqslant x < 6$	69
$6 \leqslant x < 8$	63
$8 \leqslant x < 10$	28
$10 \leqslant x < 12$	24
$12 \leqslant x < 16$	19
$16 \leqslant x < 20$	14
$20 \leqslant x < 25$	13

Exercise 3A

(i) Copy and complete the following cumulative frequency table.

Number of pull-ups (x)	Frequency
$x \geqslant 0$	
$x \geqslant 6$	
$x \geqslant 8$	
$x \geqslant 10$	
$x \geqslant 12$	
$x \geqslant 16$	
$x \geqslant 20$	13

(ii) Explain why it is not possible to state the number of men who did more than 8 pull-ups.

(iii) The men have to do at least 12 pull-ups to achieve the Gold Award, 8 to 11 pull-ups to achieve the Silver Award and 6 to 7 pull-ups to achieve the Bronze Award. Using the table in part **(i)**, find the number of men who achieved the

- **(a)** Gold Award,
- **(b)** Silver Award,
- **(c)** Bronze Award.

(iv) If a man is chosen at random from the 230 men, find the relative frequency that he has achieved the Gold Award.

12. The table below shows the average monthly water consumption of 50 households.

Average monthly water consumption (x m^3)	Cumulative frequency
$x \geqslant 0$	50
$x \geqslant 3$	47
$x \geqslant 10$	40
$x \geqslant 15$	25
$x \geqslant 20$	12
$x \geqslant 25$	6
$x \geqslant 31$	3
$x \geqslant 35$	1
$x \geqslant 40$	0

(i) Construct a frequency table for the given data.

(ii) On a sheet of graph paper, draw a histogram to represent the frequency distribution.

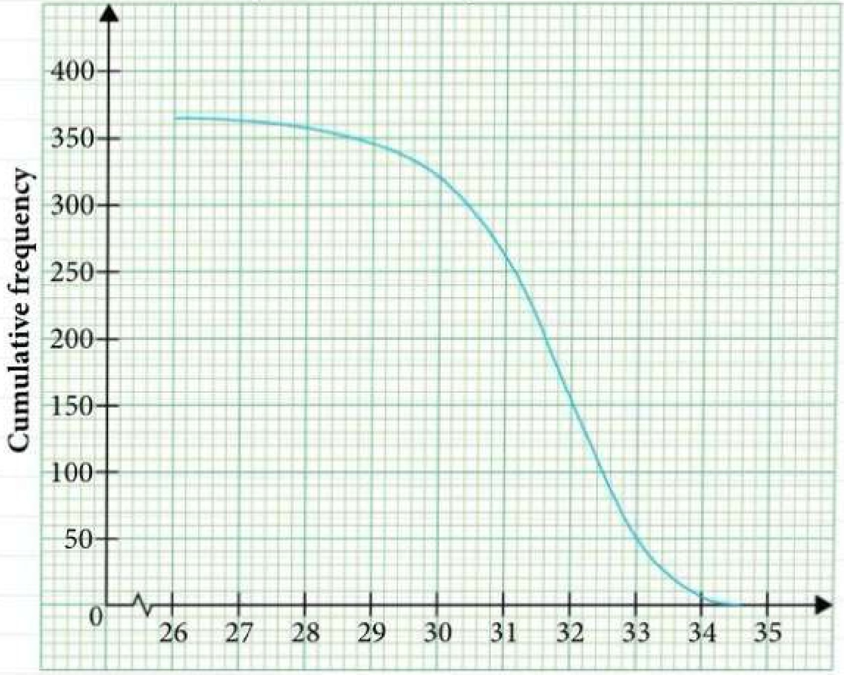
(iii) If a household is selected at random from the 50 households, what is the most likely interval of average monthly water consumption of the household?

(iv) If two households are selected at random from the 50 households, find the relative frequency that the average monthly water consumption of the households will be at least 20 m^3 but less than 35 m^3.

Exercise 3A

13. The daily maximum temperature at a town in a particular year was recorded. The cumulative frequency curve below shows the daily maximum temperature, x °C, and the number of days when the daily maximum temperature exceeded x °C.

Cumulative frequency curve for the daily maximum temperature recorded



Daily maximum temperature (x °C)

(i) Use the curve to estimate

  (a) the number of days when the daily maximum temperature exceeded 30 °C,

  (b) the fraction of the days that had a daily maximum temperature of at most 32 °C,

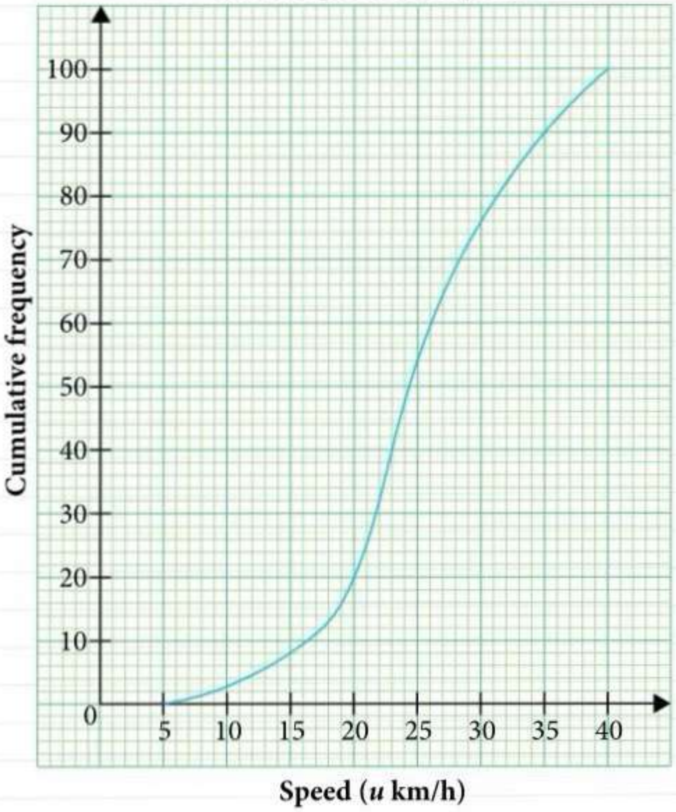
  (c) the value of k, given that 165 days had a daily maximum temperature of k °C or less.

(ii) If a day is chosen at random from that year, find the probability that the day had a daily maximum temperature of at most 32.4 °C.

Exercise 3A

14. The speeds of 100 bicycles on a cycling lane at a particular park are recorded. The cumulative frequency curve below shows the speed u km/h, and the number of bicycles which travelled at a speed less than or equal to u km/h.

Cumulative frequency curve for the speeds of bicycles



(i) Use the curve to estimate

- **(a)** the number of bicycles that travelled at a speed of at most 18 km/h,
- **(b)** the fraction of bicycles that travelled at a speed greater than 25 km/h,
- **(c)** the value of v, if 40% of the bicycles travelled at a speed less than or equal to v km/h.

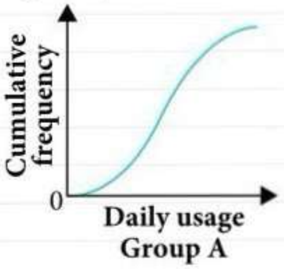
(ii) The speed limit of a bicycle on a cycling lane is 25 km/h. A newspaper article states that 50% of cyclists do not keep to the speed limit. Comment on whether the data in the question supports this claim.

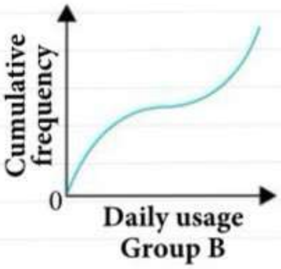
(iii) If two bicycles are selected at random from the 100 bicycles, find the probability that both bicycles exceeded the speed limit.

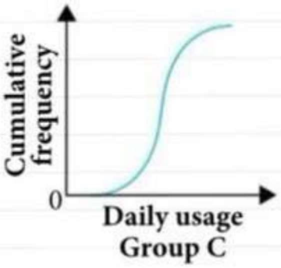
(iv) Copy and complete the following cumulative frequency table for the data given in the cumulative frequency curve.

Speed (u km/h)	Number of bicycles
$u \leqslant 5$	0
$u \leqslant 10$	
$u \leqslant 15$	
$u \leqslant 40$	

15. The cumulative frequency curves for the daily mobile data usage of three groups of participants in a survey are shown below. Each group has 100 participants.







Out of the three groups, explain which of them is likely to have participants who use

(i) the least mobile data,

(ii) the most mobile data.

Explain your answers.

3.2 Median, quartiles, percentiles, range and interquartile range

In the previous section, we have learnt how to estimate the cumulative frequencies from a cumulative frequency curve. What else can we estimate from a cumulative frequency curve?
In fact, we can estimate the median and other useful statistical measures such as quartiles and the interquartile range from a cumulative frequency curve.
Let us first recap how to find the median of a set of discrete data and then learn how to find the quartiles, before returning to finding these statistics from a cumulative frequency curve which is used to display continuous data.

A. Quartiles, interquartile range and range for discrete data

In Book 3, we learnt how to find the **median** of a set of *discrete* data.
The median is a measure of the average and is the 'middle value' when the data are arranged in an ascending (or descending) order.

Discrete data refers to a set of data which only takes on distinct values. For example, a dataset showing the number of phone calls received in a day can only take on distinct values such as 1, 5, 12, etc., but not 1.5, $4\frac{2}{3}$, etc.

Consider the following set of discrete data arranged in ascending order:

Dataset A: 2, 5, 6, 7, 8, 9, 14, 16, 20, 21, 30.

The **total frequency n** (i.e. the total number of data values or the *size of the dataset*) is 11. The position of the median is $\frac{n+1}{2} = \frac{11+1}{2} = 6$.

Therefore, the median is 9 as shown in Fig. 3.2.

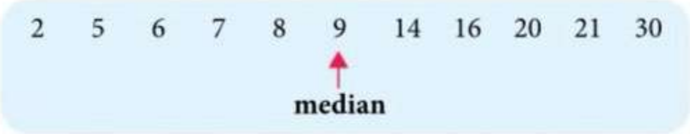


Fig. 3.2

From Fig. 3.3, we see that the median 9 divides the dataset in two equal halves, with 5 values on each side of the median. We consider the 5 values on the left of the median. The middle value of these 5 values is 6 and it is called the **lower quartile** or the **first quartile Q_1**.

The first quartile can be considered as the 'first-quarter value', where *about* one quarter (or 25%) of the data is less than (or less than or equal to) this value.

Since the median is the middle value or 'second-quarter value', the **median** is also called the **second quartile Q_2**, where *about* half (or 50%) of the data is less than (or less than or equal to) this value.

Attention

There are no standard methods used to find the lower and the upper quartiles of discrete data. Microsoft Excel uses another more complicated method. The method here is a simpler method that is suitable for a small set of discrete data. However, there is only one method to find the median of a set of discrete data.

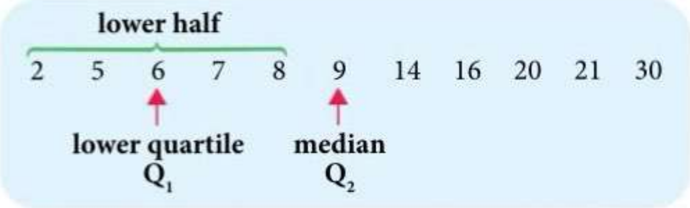


Fig. 3.3

Similarly, in Fig. 3.4, we consider the 5 values on the right of the median. The middle value of the 5 values is 20, and it is called the **upper quartile** or the **third quartile Q_3**, where *about* three quarters (or 75%) of the data is less than (or less than or equal to) this value.

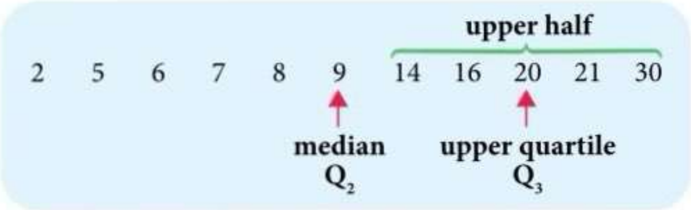


Fig. 3.4

From Fig. 3.3 and 3.4, we see that the **quartiles** obtained by the above method divide the dataset, which is arranged in ascending order, into 4 roughly equal parts.

Consider another set of discrete data arranged in ascending order:

Dataset B: 2, 5, 7, 7, 8, 9, 10, 10, 11, 21, 30.

From Table 3.3, we observe that the total frequency, the smallest value, the largest value and the median are the same for both sets of data. Only the lower and upper quartiles are different. But what can the lower and upper quartiles tell us?

	Dataset A	Dataset B
Smallest value	2	2
Lower quartile	6	7
Median	9	9
Upper quartile	20	11
Largest value	30	30
Total frequency	11	11

Table 3.3

Big Idea

Measures

The **lower or first quartile Q_1** is a measure of the *lower 25%* of the dataset because about 25% of the data values are less than (or less than or equal to) Q_1.
The **median** (or the second quartile Q_2) is a measure of the *centre* of the dataset because about 50% of the data values are less than (or less than or equal to), or more than (or more than or equal to) Q_2.
The **upper or third quartile Q_3** is a measure of the lower 75% (or *upper 25%*) of the dataset because about 75% of the data values are less than (or less than or equal to) Q_3, and about 25% of the dataset are more than (or more than or equal to) Q_3.
These measures are useful for comparison and analysis of datasets.

Attention

We can only say that in general, *about*, and not exactly, 50% of the data is less than (or less than or equal to) the median.
Similarly, we can only say that in general, *about* 25% of the data is less than (or less than or equal to) the lower quartile; and *about* 75% of the data is less than (or less than or equal to) the upper quartile.
In Dataset A, the median is 9, $\frac{5}{11}$ (or about 45%) of the data is less than the median, and $\frac{6}{11}$ (or about 55%) of the data is less than or equal to the median.
Consider this dataset, where $n = 12$:
2, 5, 6, 7, 8, 9, 11, 14, 16, 20, 21, 30
The median is 10 and $\frac{6}{12}$ (or 50%) of the data is less than (or less than or equal to) the median.
Consider yet another dataset, where n is still 12:
2, 5, 6, 10, 10, 10, 10, 14, 16, 20, 21, 30
The median is still 10, but $\frac{3}{12}$ (or 25%) of the data is less than the median, whereas $\frac{7}{12}$ (or about 58%) of the data is less than or equal to the median.

Fig. 3.5 **(a)** and **(b)** show histograms representing the data values in Dataset A and in Dataset B respectively.

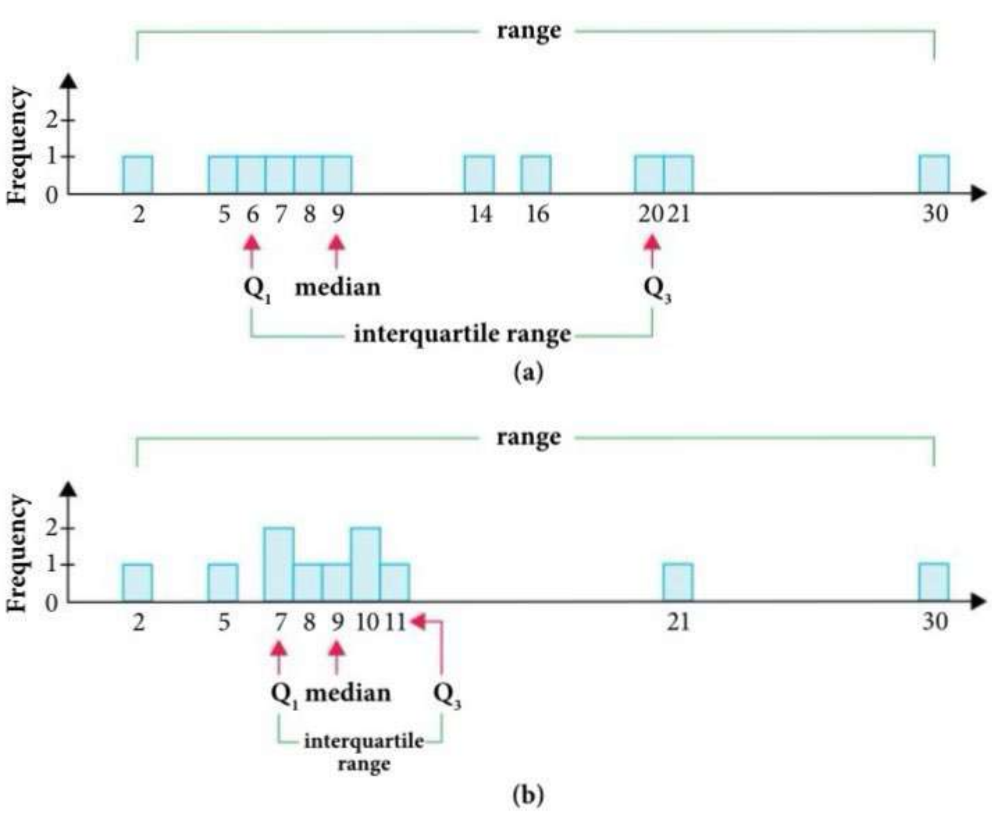


Fig. 3.5

Big Idea

Diagrams

Do you notice that the histograms in Fig. 3.5 help us visualise the spread of the data values more clearly than the numbers in Fig. 3.4 and Table 3.3?

From Fig. 3.5**(a)** and **(b)**, we notice that the data values are spread out differently even though the total frequency, the smallest value, the largest value and the median are the same for both datasets.

The **range** and the **interquartile range** are indicated. These can be used to measure the **spread** of a dataset.

Dataset A

Range = largest value − smallest value
= 30 − 2
= 28

Interquartile range = $Q_3 - Q_1$
= 20 − 6
= 14

Dataset B

Range = largest value − smallest value
= 30 − 2
= 28

Interquartile range = $Q_3 - Q_1$
= 11 − 7
= 4

Since both datasets have the same largest and smallest values, the range is the same (range = 28). However, from Fig. 3.5(**a**) and (**b**), we can see that the data values in Dataset A are more spread out from 6 to 20 (interquartile range = 14), while the data values in Dataset B are clustered around 7 to 11 (interquartile range = 4). In other words, a smaller interquartile range means that the middle 50% of the data values are less spread out around the median and are more consistent, compared to a larger interquartile range. Hence, we can compare the spread using the interquartile range.

The **range** of a dataset only depends on its smallest and largest data values, and does not take into account all the other data values. The presence of extreme values will also affect the range.

On the other hand, the **interquartile range** of a dataset describes the distribution of the *middle 50%* of the data values around the *median* and is not affected by extreme values. Thus, it is more representative of the distribution of the dataset.

Range = largest value − smallest value
Interquartile range = $Q_3 - Q_1$

Big Idea

Measures

Both the range and interquartile range (IQR) measure the *spread* of a dataset. We use either of them depending on what we want to find out. The IQR is usually a better measure of spread as it considers how the middle 50% of the data are distributed and is not affected by extreme values, unlike the range. Extreme values are called *outliers*.

Worked Example 2

Finding quartiles, interquartile range and range for discrete data

The data below shows the number of books 8 students read.

10, 12, 12, 13, 9, 17, 11, 14

Find

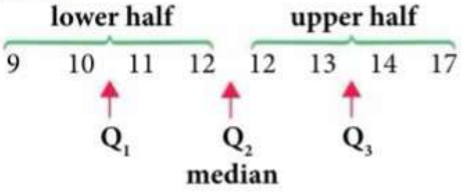
(i) Q_1, Q_2 and Q_3,

(ii) the interquartile range,

(iii) the range.

Solution

(i)



Problem-solving Tip

(i) Arrange the data in ascending order. What if the data are arranged in descending order? What will it affect?

$Q_2 = \frac{12 + 12}{2} = 12$ When n is even, the median is the mean of the two middle values.

$Q_1 = \frac{10 + 11}{2} = 10.5$ When the number of data in the lower half is even, Q_1 is the mean of the two middle values.

$Q_3 = \frac{13 + 14}{2} = 13.5$ When the number of data in the upper half is even, Q_3 is the mean of the two middle values.

(ii) Interquartile range $= Q_3 - Q_1$
$= 13.5 - 10.5$
$= 3$

(iii) Range = largest value − smallest value
$= 17 - 9$
$= 8$

Practise Now 2

Similar and Further Questions
Exercise 3B
Questions 1, 2, 9, 14

1. The following set of data shows the number of sit-ups done in 1 minute by 10 students during a physical fitness test.

12, 22, 36, 10, 14, 45, 59, 44, 38, 25

Find

(i) Q_1, Q_2 and Q_3,
(ii) the interquartile range,
(iii) the range.

2. Another physical fitness test is conducted one month later. However, only 9 of the students took the test as one of the students was sick. The following dataset shows the number of sit-ups done in 1 minute by these 9 students.

23, 54, 15, 32, 16, 26, 47, 9, 35

Find

(i) the median, the lower quartile and the upper quartile,
(ii) the interquartile range and the range.

Thinking Time

In Book 3, we learnt that the position of the median or the second quartile is $\frac{n+1}{2}$ or $\frac{1}{2}(n+1)$, where n is the total frequency.

Is it true that the position of the first quartile is $\frac{n+1}{4}$ or $\frac{1}{4}(n+1)$?

Is it true that the position of the third quartile is $\frac{3}{4}(n+1)$?

Calculate $\frac{1}{4}(n+1)$ and $\frac{3}{4}(n+1)$ for the data values in Dataset A on page 76, in Worked Example 2 and in the two questions of Practise Now 2, where $n = 11, 8, 10$ and 9 respectively.

When is the position of the first quartile $\frac{1}{4}(n+1)$ and when is the position of the third quartile $\frac{3}{4}(n+1)$?

B. Quartiles, interquartile range and range for continuous data

After learning how to find quartiles, range and interquartile range for discrete data, let us look at the cumulative frequency curve from Worked Example 1 on page 64, which is reproduced in Fig. 3.6.

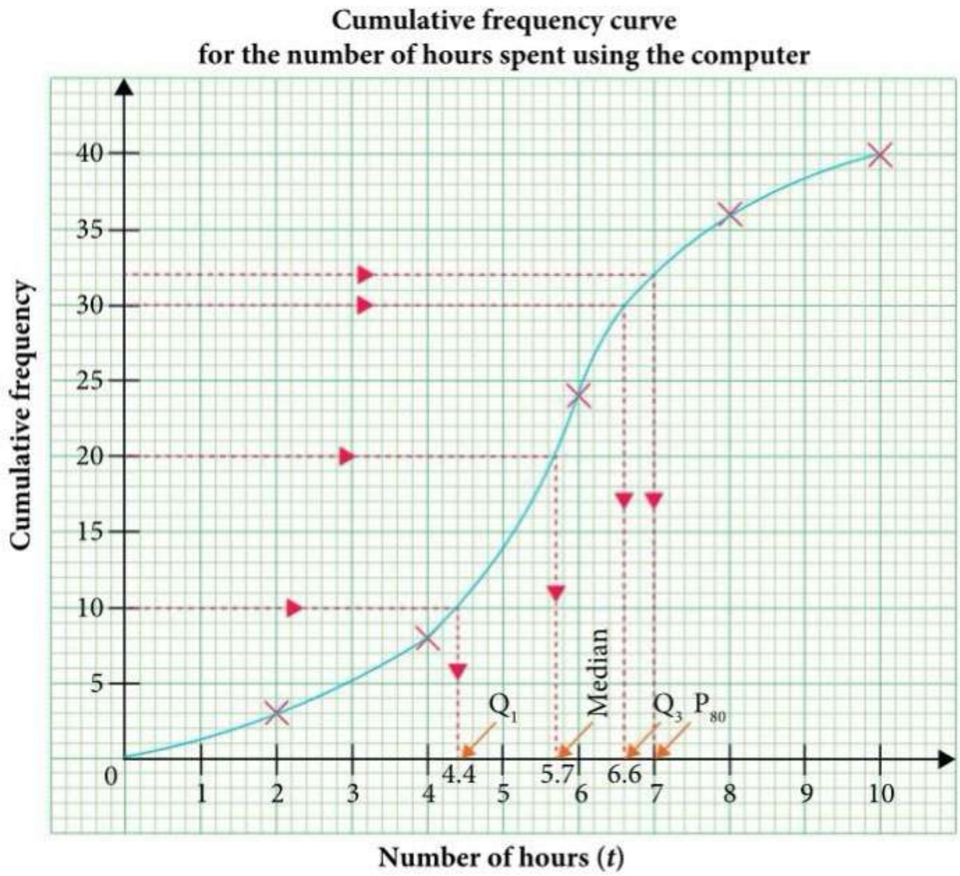
The cumulative frequency curve displays **continuous data** since the time spent using the computer can take on any value within a range of numbers.

Although the number of students (cumulative frequency) is supposed to be discrete, we will treat the cumulative frequency as continuous since it is displayed on the vertical axis from 0 to 40. This will affect how we find the position of the median.

For **discrete** data with 40 values, the median position is $\frac{n+1}{2} = \frac{40+1}{2} = 20.5$, i.e. the median is the mean of the 20th and the 21st data values that have been arranged in ascending order. It makes sense because there are 40 values from the 1st value to the 40th value and the middle is 20.5.

But for **continuous** data from 0 to 40 (not from 1 to 40), the position of the median is $\frac{n}{2} = \frac{40}{2} = 20$. It makes sense because the middle of the vertical axis from 0 to 40 in Fig. 3.6 is 20, not 20.5.

To *estimate* the median from a cumulative frequency curve, we find the position of the median using $\frac{n}{2}$ (which is 20 in this case) and then look for the data value corresponding to the cumulative frequency of 20 in Fig. 3.6 (which is 5.7 hours in this case). Therefore, an estimate of the median Q_2 is 5.7 hours. Why is this an estimate?



Cumulative frequency curve for the number of hours spent using the computer

Fig. 3.6

Similarly, to estimate the **lower quartile** from a cumulative frequency curve, we find the position of the lower quartile using $\frac{n}{4} = \frac{40}{4} = 10$. Is 10 one quarter of the vertical axis from 0 to 40 in Fig. 3.6?

Then we look for the data value corresponding to the cumulative frequency of 10 in Fig. 3.6 (which is 4.4 hours in this case). Therefore, an estimate of the lower quartile Q_1 is 4.4 hours.

Similarly, to estimate the **upper quartile** from a cumulative frequency curve, we find the position of the upper quartile using $\frac{3}{4}n = \frac{3}{4} \times 40 = 30$. Is 30 three quarters of the vertical axis from 0 to 40 in Fig. 3.6?

Then we look for the data value corresponding to the cumulative frequency of 30 in Fig. 3.6 (which is 6.6 hours in this case). Therefore, an estimate of the upper quartile Q_3 is 6.6 hours.

Therefore, an estimate of the **interquartile range** is $Q_3 - Q_1 = 6.6 - 4.4 = 2.2$ hours.

An estimate of the **range** is the difference between the *t*-coordinate of the largest endpoint and the *t*-coordinate of smallest endpoint of the cumulative frequency curve, i.e. 10 – 0 = 10 hours.

Attention

Unlike discrete data, the positions of the **first and third quartiles** for continuous data are always $\frac{n}{4}$ and $\frac{3}{4}n$ respectively.

Since the three quartiles are obtained from the cumulative frequency curve, they are only *estimates*.

The range is also an estimate because we cannot conclude the actual smallest and largest data values from the cumulative frequency curve.

C. Percentiles for continuous data

For the cumulative frequency curve, we can also find a measure called **percentiles**, which are values that divide the dataset into 100 equal parts.

For example, in Fig. 3.6, 80% of the distribution (i.e. $\frac{80}{100} \times 40 = 32$ students) used the computer for 7 hours or less.

We say that an estimate of the **80th percentile (or P_{80})** is 7 hours.

Since the median value of 5.7 hours means that 50% of the distribution, i.e. 20 students, used the computer for 5.7 hours or less, the **median** is the same as the **50th percentile, P_{50}**.

Similarly, $Q_1 = P_{25}$ and $Q_3 = P_{75}$.

Big Idea

Diagrams

Just like a cumulative frequency table, a cumulative frequency curve is a statistical diagram used to display the cumulative frequencies of a dataset.

However, a cumulative frequency curve allows us to estimate the cumulative frequencies for data values that lie within a class interval, and other useful statistical measures such as the median, lower and upper quartiles, the interquartile range and percentiles.

Worked Example 3

Estimating range from frequency table

The table shows the weekly work durations, in hours, of a group of working adults.

Work duration (x hours)	Number of adults
$30 < x \leqslant 35$	9
$35 < x \leqslant 40$	8
$40 < x \leqslant 45$	3
$45 < x \leqslant 50$	10
$50 < x \leqslant 55$	6

Estimate the range of the weekly work durations.

***Solution**

Work duration (x hours)	Frequency (f)	Mid-value (x)
$30 < x \leqslant 35$	9	32.5
$35 < x \leqslant 40$	8	37.5
$40 < x \leqslant 45$	3	42.5
$45 < x \leqslant 50$	10	47.5
$50 < x \leqslant 55$	6	52.5

Estimated range $= 52.5 - 32.5$

$= 20$ hours

Practise Now 3

Similar and Further Questions
Exercise 3B
Question 3

50 people were surveyed on the total time they spend exercising weekly and the results are shown in the table below.

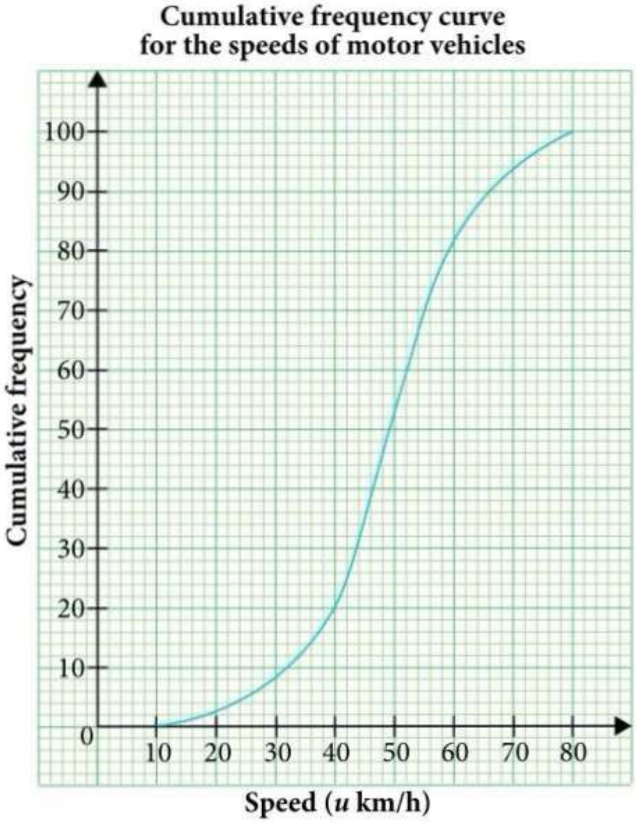
Total weekly exercise time (x minutes)	Number of people
$30 < x \leqslant 50$	6
$50 < x \leqslant 70$	11
$70 < x \leqslant 90$	9
$90 < x \leqslant 110$	6
$110 < x \leqslant 130$	7
$130 < x \leqslant 150$	11

Estimate the range of the total weekly exercise times.

Worked Example 4

Estimating quartiles, interquartile range and percentiles from cumulative frequ nc curve

The cumulative frequency curve represents the instantaneous speeds, u km/h, of 100 motor vehicles taken at a particular point on a street travelling at speeds less than or equal to u km/h.



Estimate

(i) the median, the lower and the upper quartiles,

(ii) the interquartile range,

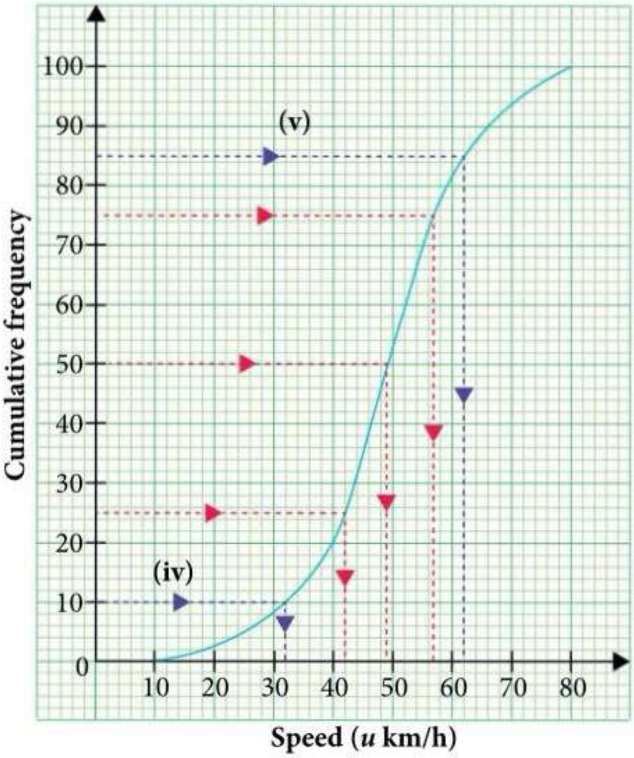
(iii) the range,

(iv) the 10th percentile,

(v) the value of v, if 85% of motor vehicles have speeds less than or equal to v km/h.

Solution

(i)



Cumulative frequency curve for the speeds of motor vehicles

Total frequency $n = 100$

So $\frac{n}{2} = 50$, $\frac{n}{4} = 25$ and $\frac{3n}{4} = 75$.

From the graph, median speed = 49 km/h,
lower quartile = 42 km/h,
upper quartile = 57 km/h.

(ii) Interquartile range = 57 – 42
= 15 km/h

(iii) Range = largest value – smallest value
= 80 – 10
= 70 km/h

(iv) 10% of total frequency $= \frac{10}{100} \times 100$
$= 10$

From the graph, the 10th percentile = 32 km/h.

(v) 85% of drivers $= \frac{85}{100} \times 100 = 85$ drivers have speeds less than or equal to v km/h.

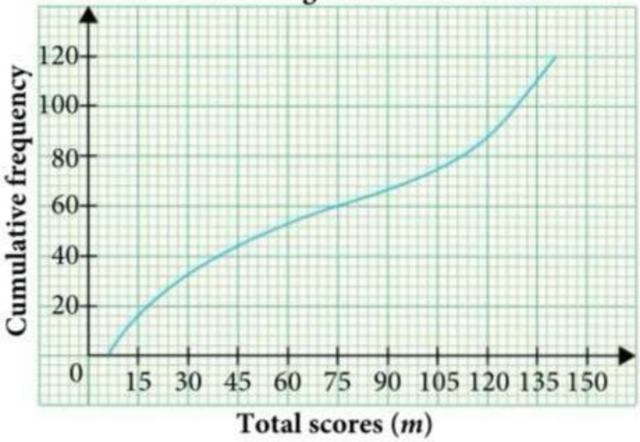
From the graph, $v = 62$.

Practise Now 4

Similar and Further Questions
Exercise 3B
Questions 4, 5, 10, 15

120 students went bowling. The cumulative frequency curve shows the total scores (m) and the number of students scoring less than m points.

Cumulative frequency curve
for the bowling scores of students



From the graph, estimate

(i) the median, the lower quartile and the upper quartile,

(ii) the interquartile range and the range,

(iii) the 10th and the 80th percentiles,

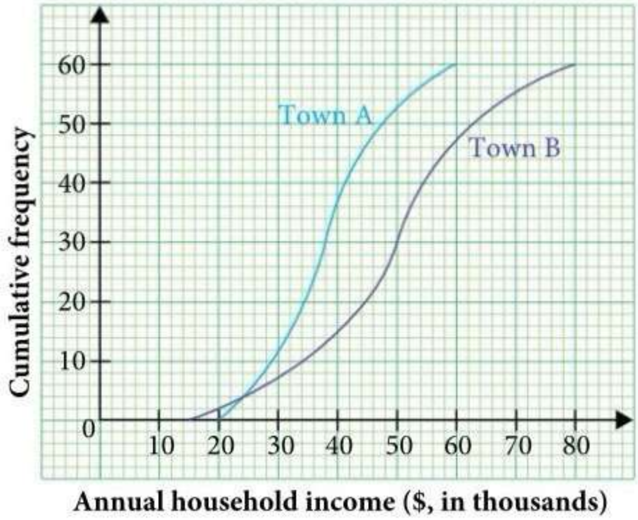
(iv) the value of x, if 60% of the students scored at least x points.

Worked Example 5

Comparing and analysing two cumulative frequency curves

The diagram below shows the cumulative frequency curves for the annual incomes (in thousands of dollars) of 60 households in two towns, A and B.

Cumulative frequency curves
for the annual household incomes of households



(i) For Town A, estimate
  (a) the median income level,
  (b) the interquartile range.

(ii) For Town B, estimate
  (a) the median income level,
  (b) the interquartile range.

(iii) A newspaper article states that 'households in Town B generally have higher in nc vel than households in Town A'. Comment on whether the data from the two towns suppor this claim.

(iv) Which town is more likely to have an income inequality problem? Justify your answer.

Attention

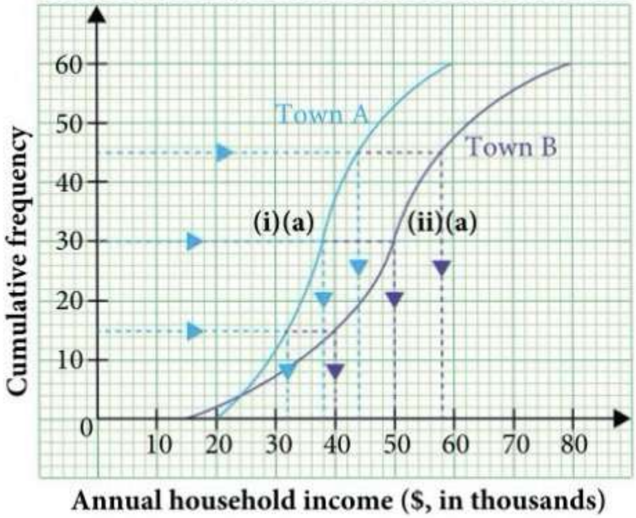
Income inequality refers to the unequal distribution of income between different groups of people. A simplified indicator can be obtained by comparing the top 25% and bottom 25% incomes of the population.

Solution

(i) For each set of data, total frequency $n = 60$.

So $\frac{n}{2} = 30$, $\frac{n}{4} = 15$ and $\frac{3n}{4} = 45$.

Cumulative frequency curves for the annual household incomes of households



(a) From the graph, median income level of Town A = \$38 000

(b) From the graph, lower quartile = \$32 000

upper quartile = \$44 000

Interquartile range for Town A = \$44 000 − \$32 000

= \$12 000

(ii) **(a)** From the graph, median income level of Town B = \$50 000

(b) From the graph, lower quartile = \$40 000

upper quartile = \$58 000

Interquartile range for Town B = \$58 000 − \$40 000

= \$18 000

(iii) From parts **(i)(a)** and **(ii)(a)**, the median annual income level of Town B is \$50 000, which is higher than the median annual income level of Town A at \$38 000.

Therefore, the data from the two towns support this claim.

Problem-solving Tip

(iii) Although most of the curve for Town A lies above the curve for Town B, it does not mean that households in Town A generally have higher income levels than households in Town B. This is because the horizontal axis represents the income levels, instead of the vertical axis. Hence, the curve that is further to the right (in this case, Town B) indicates higher income levels. This can also be seen by comparing the median income levels.

(iv) Town B is more likely to have an income inequality problem. The interquartile range of \$18 000 for Town B is much greater than the interquartile range of \$12 000 for Town A.

Alternatively, we can also see from the graph that the middle part of the curve for Town A (between the cumulative frequency of 15 to 45 households) has a steeper gradient (or is narrower) than that of Town B. This shows that the income levels of the middle 50% of Town A are less spread out than those of Town B, and thus the interquartile range for Town A is smaller.

Problem-solving Tip

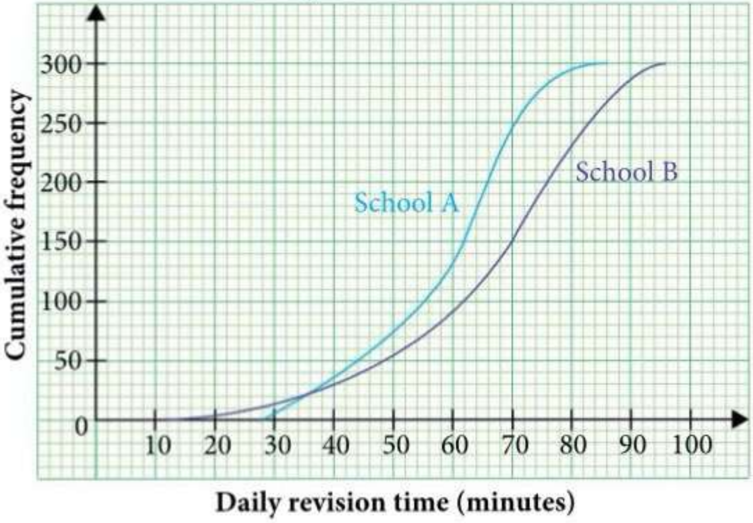
(iv) The interquartile range indicates the difference (or gap) in income levels between the bottom 25% of the households and the top 25% of the households. Hence, a greater interquartile range suggests a higher likelihood of an income inequality problem.

Practise Now 5

Similar and Further Questions
Exercise 3B
Questions 6–8, 11–13

300 students, each from School A and School B, participated in a survey. The cumulative frequency curves below show the daily revision time, in minutes, of the students.

Cumulative frequency curve
for the daily revision time of students



(i) For School A, estimate
  (a) the median, **(b)** the interquartile range.

(ii) For School B, estimate the
  (a) the median, **(b)** the interquartile range.

(iii) State, with a reason, if students from School A or School B, spend more time revising daily.

(iv) In which school is the daily revision time more consistent? Justify your answer.

Reflection

1. What is the difference between calculating the lower and upper quartiles from a set of discrete data and from a set of continuous data displayed on a cumulative frequency curve?
2. What do the interquartile range and the range measure? Which one is a better measure and why?
3. If the interquartile range of a dataset is large, what does it tell us about the middle 50% of the distribution?
4. What have I learnt in this section that I am still unclear of?

Exercise 3B

1. Find the range, lower quartile, median, upper quartile and interquartile range for the following sets of data.
   (a) 7, 6, 4, 8, 2, 5, 10
   (b) 63, 80, 54, 70, 51, 72, 64, 66
   (c) 14, 18, 22, 10, 27, 32, 40, 16, 9
   (d) 10.4, 8.5, 13.1, 11.8, 6.7, 22.4, 4.9, 2.7, 15.1, 16.7

2. The following dataset shows the number of distinctions scored by 10 classes for a particular examination. Each class has 40 students.

   0, 1, 6, 9, 24, 0, 27, 6, 9, 29

   (i) For the given data, find the first, second and third quartiles.
   (ii) Find the interquartile range and the range.

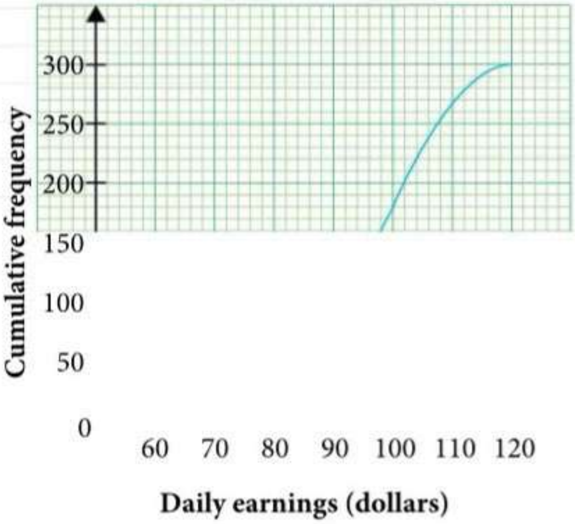
3. The table shows the heights, in centimetres, of 40 children.

Height (x hours)	Number of adults
$100 \leqslant x \leqslant 105$	4
$105 < x \leqslant 110$	13
$110 < x \leqslant 115$	10
$115 < x \leqslant 120$	8
$120 < x \leqslant 125$	4
$125 < x \leqslant 130$	1

Estimate the range of the heights.

4. The graph shows the cumulative frequency curve for the daily earnings of 300 employees in a company.

Cumulative frequency curve for the daily earnings of 300 employees

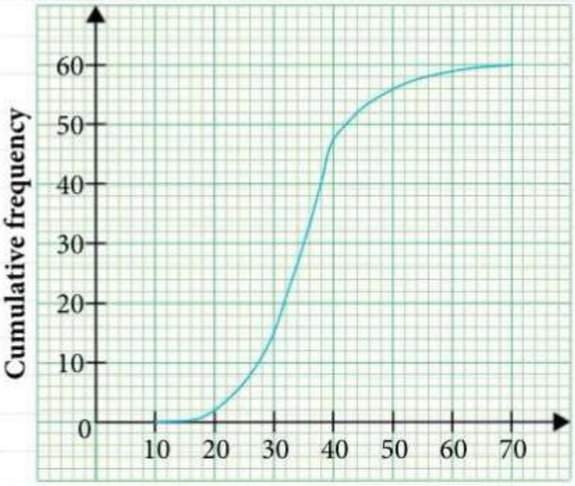


Use the graph to estimate
(i) the median, the lower and upper quartiles,
(ii) the interquartile range and the range,
(iii) the 20th and the 90th percentiles.

Exercise 3B

5. The following diagram shows the cumulative frequency curve for the heights, h cm, of 60 plants that have heights less than or equal to h cm.

Cumulative frequency curve for the heights of plants



Heights (h cm)

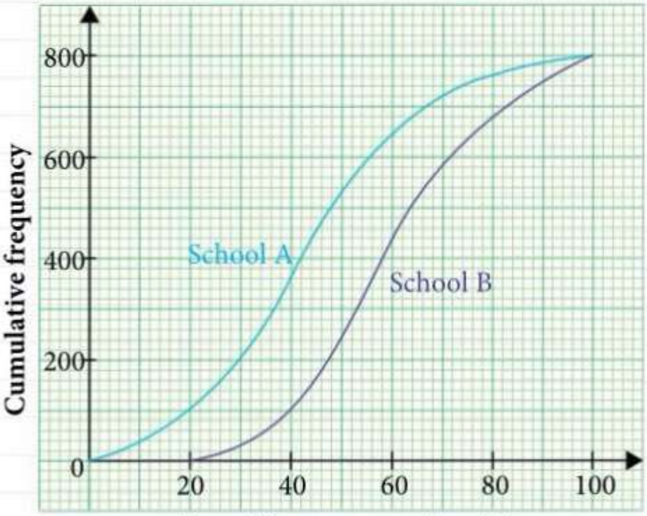
(i) Use the curve to estimate

- **(a)** the median height,
- **(b)** the first and third quartiles,
- **(c)** the 30th and 55th percentiles,
- **(d)** the number of plants that are taller than 50 cm.

(ii) One of the plants is chosen at random. Find the probability that the plant has a height within the interquartile range.

6. The graph shows the cumulative frequency curves of the daily travelling expenses of 800 students in two schools, A and B.

Cumulative frequency curves for the daily travelling expenses of students



Travelling expenses (cents)

(i) Use the graph to estimate the median travelling expenses of the students from

(a) School A, **(b)** School B.

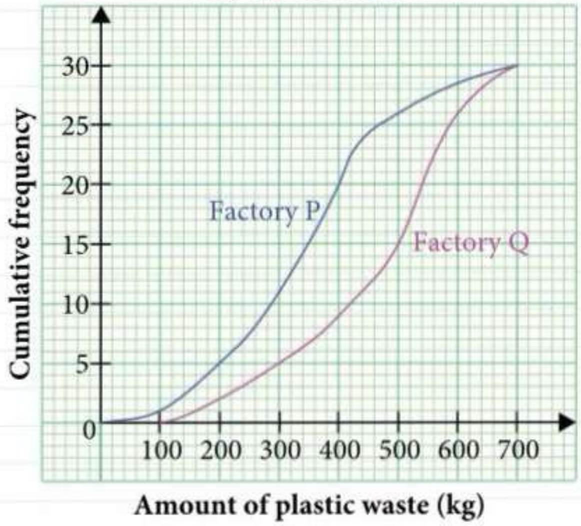
(ii) Estimate the interquartile range of the travelling expenses of the students from

(a) School A, **(b)** School B.

(iii) State, with a reason, whether the students from School A or School B spend more on daily travelling expenses.

Exercise 3B

7. The graph shows the cumulative frequency curves of the amount of plastic waste, in kg, produced in 30 days by two factories, P and Q.

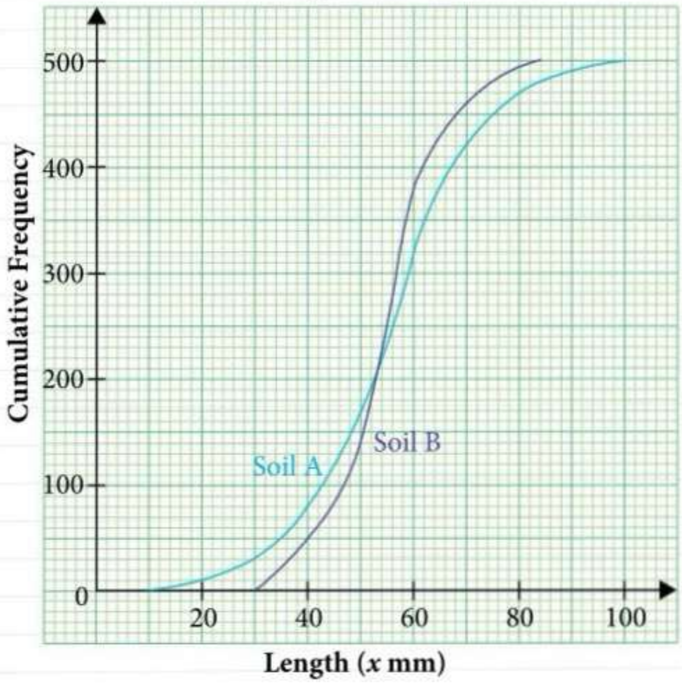

Cumulative frequency curve for the amount of plastic waste produced by two factories

(i) Use the graph to estimate the median amount of plastic waste produced by
   (a) Factory P, (b) Factory Q.

(ii) Find an estimate of the interquartile range of the amount of plastic waste produced by
   (a) Factory P, (b) Factory Q.

(iii) Which factory, P or Q, produced more plastic waste? Explain.

8. 500 earthworms were collected from a sample of Soil A and 500 earthworms from Soil B, and their lengths were measured. The cumulative frequency curve below shows the length, x mm, and the number of earthworms which have lengths less than or equal to x mm.

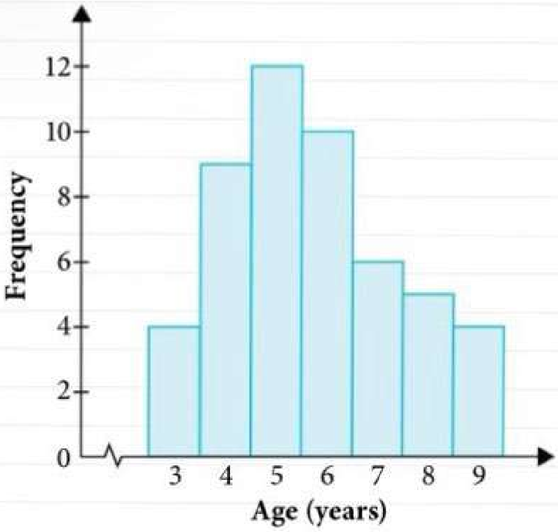

Cumulative frequency curve for the lengths of earthworms

(i) Which soil was the shortest earthworm, among the 1000 earthworms, found in?

(ii) Earthworms which grew more than 60 mm are said to be 'satisfactory'. Which soil had more 'satisfactory' earthworms? Explain.

Exercise 3B

9. The histogram represents the ages, in years, of 50 children.



Find

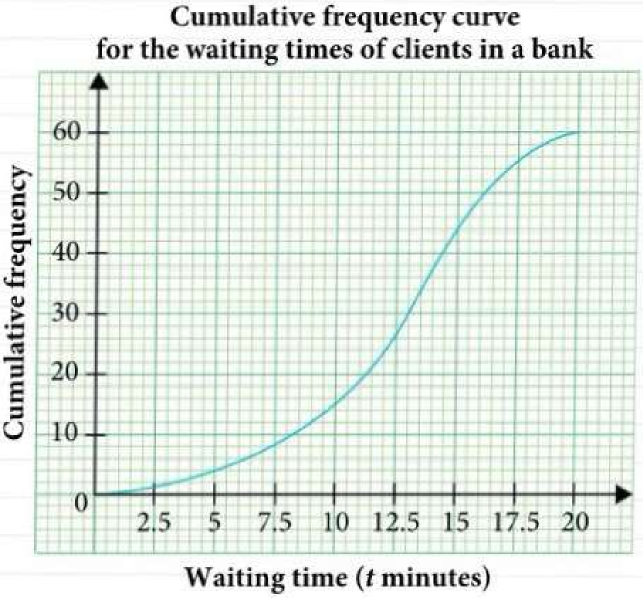
(i) the range,

(ii) the median age,

(iii) the lower and upper quartiles,

(iv) the interquartile range.

10. The waiting times (in minutes) of 60 clients at a bank, on a particular day were measured. The cumulative frequency curve shows the waiting times, *t* minutes, and the number of clients with waiting times less than or equal to *t* minutes.



(i) **(a)** Estimate the lower quartile, median and upper quartile of the waiting times in the bank.

(b) Estimate the interquartile range.

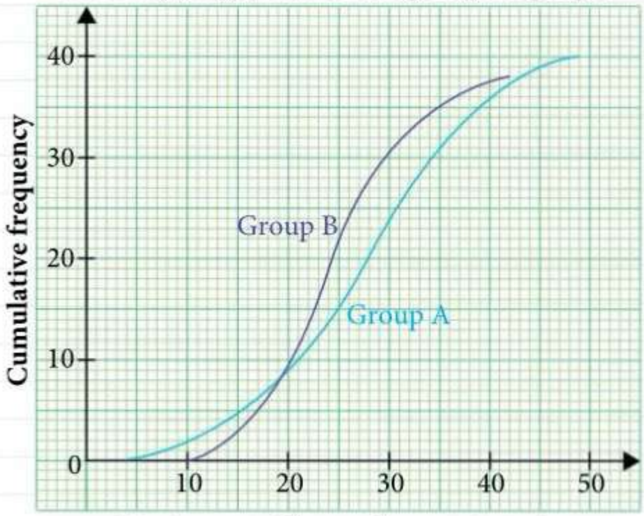
(ii) Find an estimate of the percentage of clients who waited for more than 15 minutes at the bank.

(iii) For the same 60 clients, a second cumulative frequency curve is plotted to show the waiting times (t), and the number of clients with waiting times more than t minutes.
What does the t-coordinate of the point of intersection of the two cumulative curves represent? Explain your answer clearly.

Exercise 3B

11. The cumulative frequency curves below show the average typing speeds, x words per minute, of two groups of people, A and B, and the number of people whose typing speeds are less than or equal to x words per minute.

Cumulative frequency curves for the typing speeds of two groups of people



Typing speed (x words per minute)

(i) Estimate the lower quartile, median, and upper quartile of Group A.

(ii) How many people are there in Group B?

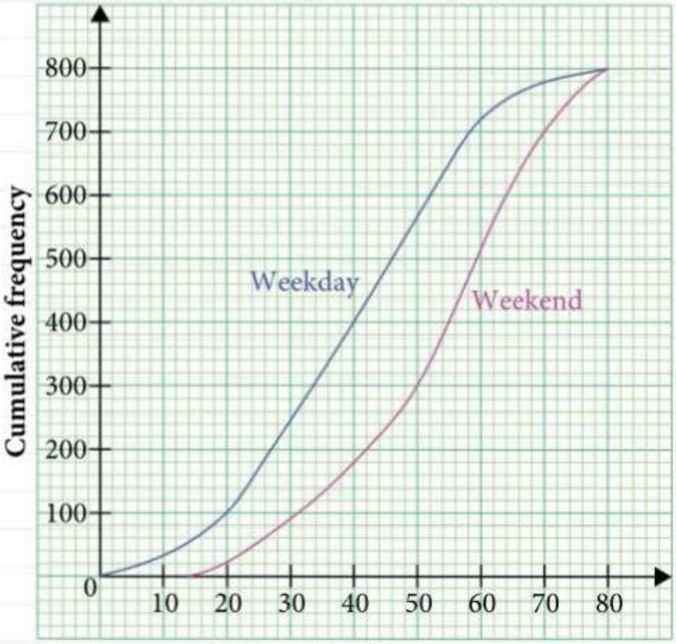
(iii) Estimate the interquartile range of Group B.

(iv) Estimate the percentage of people in Group B who can type faster than 38 words per minute.

(v) Do you agree with the statement that 'The people in Group A generally type faster and have a more consistent typing speed.'? Justify your answer.

12. 800 people were surveyed on their average waiting times at a polyclinic, on a weekday and on a weekend respectively. The cumulative frequency curves show the waiting times, x minutes, and the number of people who waited for less than x minutes.

Cumulative frequency curve for the waiting times at a polyclinic



Waiting time (x minutes)

(a) Estimate the interquartile range of the waiting times on a weekday.

(b) Estimate the number of people who waited for at least 64 minutes on a weekend.

(c) David visited the polyclinic on a weekday and waited for 36 minutes. Estimate his waiting time if he were to visit on a weekend.

(d) **(i)** Are the waiting times shorter on a weekday, or on a weekend? Justify your answer.

(ii) Do you agree with the statement that 'The waiting times on a weekend are generally longer and more consistent.'? Justify your answer.

Exercise 3B

13. The table below shows the Pollutant Standards Index (PSI) of City X and City Y, measured in the same period of 10 days. A higher PSI reading indicates poorer air quality.

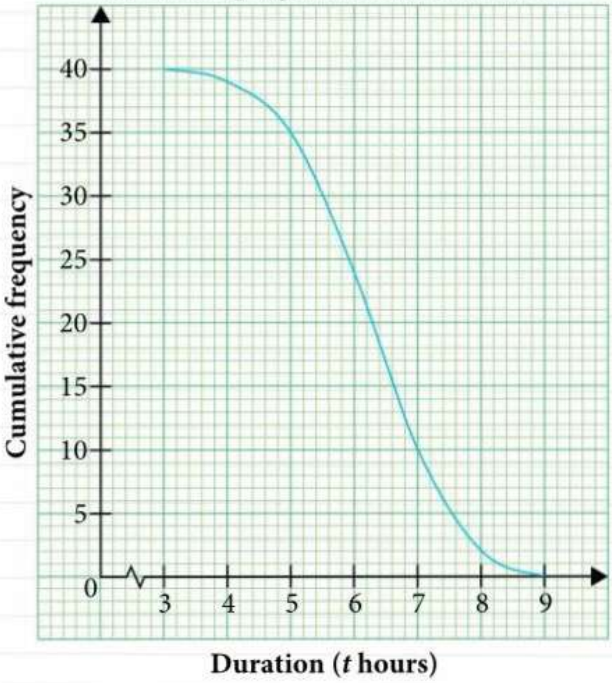
City X					City Y				
80	65	21	81	16	103	79	99	121	200
23	37	50	53	100	308	114	171	198	235

(i) For the PSI data of City X, find
  (a) the range,
  (b) the median,
  (c) interquartile range.
(ii) For the PSI data of City Y, find
  (a) the range,
  (b) the median,
  (c) the interquartile range.
(iii) Which city's data show a greater spread?
(iv) Compare and comment on the air quality of the two cities. Give two reasons to support your answer.

14. Imran has written down 7 integers.
The mode of these integers is 19, the median is 10 and the lower quartile is 8.
The smallest integer is 3 and the largest integer is twice the upper quartile.
List the integers in set notation in ascending order.

15. The duration, in hours, that 40 students slept for on a Wednesday night were recorded. The cumulative frequency curve shows the duration, t hours, and the number of students who slept for at least t hours.

Cumulative frequency curve for the sleeping duration of students



(i) **(a)** Estimate the lower quartile, upper quartile, 40th and 80th percentiles of the duration that the students slept for.
  (b) Estimate the interquartile range.
(ii) Find an estimate of the fraction of students who slept for less than 6 hours.
(iii) If two students are selected at random from the 40 students, find the probability that both students had a duration of sleep that was between 6 and 7.2 hours.
(iv) A student claims that only 10% of the class had a duration of sleep that was between 6 and 7.2 hours. Comment on whether the above data supports this claim.

3.3 Further comparison of data

In the previous two sections, the average of choice is the *median* because only the median (and not the mean) can be estimated from a cumulative frequency curve. As such, the spread of choice is the **interquartile range**, which is a measure of how the middle 50% of the data are spread around the median.

In Book 3, we learnt that the median is an appropriate measure of the average of the data if there are extreme values (or outliers) because the median is not affected by outliers. Consequently, the interquartile range is an appropriate measure of the spread of a distribution when there are outliers.

However, if a dataset does not contain outliers, the *mean* is an appropriate measure of the average of the **data**, and the *range* provides a good measure of how the data is spread out because they take into account all the values of the dataset.

In Book 3, we learnt that the mean of a set of data is $\overline{x} = \frac{\Sigma fx}{\Sigma f}$, where f is the frequency of each data value x.

If $f = 1$ for each data value x, then $\overline{x} = \frac{\Sigma x}{n}$, where $n = \Sigma f$ is the total frequency or the size of the dataset.

We have already learnt how to compare data using the median and the interquartile range in the previous section. In this section, we will learn how to compare data involving other averages and the range of a dataset.

Worked Example 6

Comparing data with mean and range

(i) The data for the temperature in City A is shown below. Find the mean and range of the temperatures.

Time	Temperature in City A, °C
0000	25
0400	24
0800	26
1200	33
1600	31
2000	29

(ii) The mean and range of the temperatures in another City B are summarised in the table below.

Temperature in City B, °C	
Mean	27
Range	12

Provide two comparisons between the temperatures in City A and City B.

*Solution

(i)

x
25
24
26
33
31
29
$\Sigma x = 168$

$$\begin{aligned}\text{Mean } \overline{x} &= \frac{\Sigma x}{n} \\ &= \frac{168}{6} \\ &= 28\end{aligned}$$

$$\begin{aligned}\text{Range} &= \text{largest value} - \text{smallest value} \\ &= 33 - 24 \\ &= 9\end{aligned}$$

(ii) The mean temperatures for both cities are about the same, indicating that *on average*, the temperatures in both cities are about the same.
However, City B has a greater range, which indicates that there is a greater spread of the temperatures in City B.

Practise Now 6

Similar and Further Questions
Exercise 3C
Questions 1–4, 7

(i) The table below shows the number of grammatical errors made by Yasir in eight English essays submitted this semester. Find the mean and range of the number of errors made. Show your working clearly.

Essay number	1	2	3	4	5	6	7	8
Number of errors	6	9	15	26	10	14	21	3

(ii) The mean and range of the number of grammatical errors made by another student, Bernard, in the eight English essays are summarised in the table below.

Number of grammatical errors made by Bernard	
Mean	15
Range	27

Provide two comparisons between the numbers of grammatical errors made by Yasir and Bernard.

Problem-solving Tip

How do you interpret the table? Are the essay numbers the data values and the number of errors the frequency? This would mean that for Essay number 2, the data value 2 occurs 9 times. Is this how you would interpret the table?
Or are the number of errors the data values? In other words, this table is *not* a frequency table.

Worked Example 7

Comparing data with mean, median, mode and range

At a grocery store, oranges are sold in packs of five. The store manager claims that each orange should have a mass of about 150 g. Sara, not believing the claim, buys fifteen oranges and measures their masses. The masses of the oranges are shown below.

151, 142, 150, 149, 154, 143, 140, 143, 141, 152, 151, 147, 141, 141, 148

(i) Calculate the mean, median, mode and range of the masses of oranges.
(ii) Comment on the claim made by the store manager.

Solution

(i) Rearranging the data in ascending order:

140, 141, 141, 141, 142, 143, 143, 147, 148, 149, 150, 151, 151, 152, 154

$$\begin{aligned}\text{Mean mass} &= \frac{\text{sum of data values}}{\text{number of data values}}\\ &= \frac{2193}{15}\\ &= 146.2\text{ g}\end{aligned}$$

Total number of data values, $n = 15$

$$\begin{aligned}\text{Position of median} &= \frac{n+1}{2}\\ &= \frac{15+1}{2}\\ &= 8\end{aligned}$$

Median mass = 147 g

Modal mass = 141 g

$$\begin{aligned}\text{Range} &= \text{largest value} - \text{smallest value}\\ &= 154 - 140\\ &= 14\text{ g}\end{aligned}$$

(ii) Since the mean, median and modal values are all below 150 g, the claim made by the store manager is not true. In addition, the range, which is 14 g, indicates that there is a great spread in the masses of the oranges.

Practise Now 7

Similar and Further Questions
Exercise 3C
Questions 5, 6

Cheryl bought two boxes of pencils, A and B. Each box has 10 pencils. She measured the lengths of the pencils, in cm, in box A, as shown below.

17.1, 18.4, 17.1, 18.9, 18.3, 18.4, 17.1, 17.7, 17.9, 18.1

(i) Find the mean, median, mode and range of the lengths of pencils in box A.

(ii) The lengths of pencils in box B are summarised below.

Mean length = 18.2 cm
Median length = 18.1 cm
Modal length = 18 cm
Range = 0.9 cm

Compare the lengths of pencils from both boxes.

Reflection

1. How do we decide whether to use the interquartile range or the range to measure the spread of a dataset or distribution?
2. What have I learnt in this section or chapter that I am still unclear of?

Exercise 3C

1. The number of days in a month, in which working adults from Group A and Group B cook dinner, is shown below.

Group A: 4, 6, 6, 7, 8, 10, 11, 12
Group B: 0, 1, 1, 2, 3, 14, 17, 25

(i) Calculate the mean and range for Group A and Group B. Show your working clearly.

(ii) Provide two comparisons between the results of Group A and Group B.

2. Shaha makes breakfast every morning. The time taken for her to make breakfast in a particular school week is shown in the table below.

Day	Time taken to make breakfast (minutes)
Monday	4
Tuesday	13
Wednesday	6
Thursday	11
Friday	13
Saturday	21
Sunday	26

(i) Calculate the mean and range of the time taken by Shaha to make breakfast during the school week.

The time Shaha took to make breakfast in a week during the school holidays is shown in the table below.

Day	Time taken to make breakfast (minutes)
Monday	22
Tuesday	14
Wednesday	7
Thursday	12
Friday	27
Saturday	5
Sunday	14

(ii) Calculate the mean and range of the time Shaha took to make breakfast in the week during the school holidays.

(iii) Provide two comparisons between the time taken by Shaha to make breakfast during the school week and in the week during the school holidays.

3. Two trains, A and B, are scheduled to arrive at a station at certain time. The number of minutes the trains arrived after the scheduled time are recorded in the table below.

Number of minutes	Number of days for Train A	Number of days for Train B
2	3	4
3	2	3
4	5	9
5	12	9
6	10	7
7	6	5
8	1	3
9	1	0

(i) For each train, calculate the mean and range of the data.

(ii) Which train consistently arrives after the scheduled time? Justify your answer.

(iii) Which train is more punctual on the whole? Justify your answer.

Exercise 3C

4. The waiting times, in minutes, for 60 patients at two hospitals are given in the tables below.

Hospital A

Time (t minutes)	Number of patients
$20 < t \leqslant 22$	5
$22 < t \leqslant 24$	11
$24 < t \leqslant 26$	27
$26 < t \leqslant 28$	13
$28 < t \leqslant 30$	4

Hospital B

Mean	Range
25	11.2

(i) For Hospital A, find an estimate of the
  (a) mean waiting time,
  (b) range of the waiting time.
(ii) Provide two comparisons between the waiting times at the two hospitals.

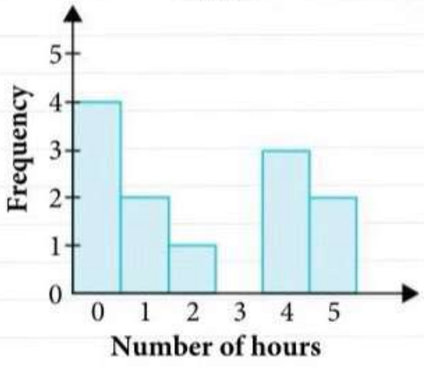
5. The table shows the daily temperatures in two cities in the Sahara over a period of 50 days.

Temperature (x °C)	Number of days	
	City A	City B
$35 \leqslant x < 40$	0	2
$40 \leqslant x < 45$	4	14
$45 \leqslant x < 50$	12	16
$50 \leqslant x < 55$	23	10
$55 \leqslant x < 60$	8	5
$60 \leqslant x < 65$	3	3

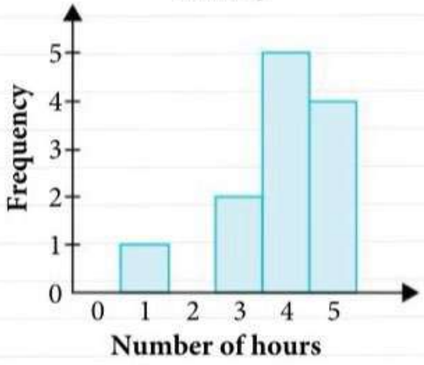
(i) For each city, find
  (a) the estimated mean temperature,
  (b) the estimated range,
  (c) the class interval where the median lies,
  (d) the modal class.
(ii) Which city is warmer on the whole? Justify your answer.
(iii) Which city experiences a greater variability in its daily temperatures? Justify your answer.

6. The histograms represent the average number of hours that 12 students from each of two classes, P and Q, spend reading each week.

Class P



Class Q



(i) For each class, calculate the mean, median, mode and range of the data.
(ii) Which class has reading times with a smaller spread? Justify your answer.
(iii) Which class spends more time reading? Justify your answer.

Exercise 3C

7. The tables below show the masses of 100 students from School R and 100 students from School S. (All masses are corrected to the nearest 5 kg.)

School R

Mass (x kg)	45	50	55	60	65	70
Number of students	5	36	28	22	7	2

School S

Mass (y kg)	40	45	50	55	60	65	70	75	80
Number of students	7	21	24	6	3	26	8	1	4

The National Statistics Division requires the combined statistics (mean and range) of both schools.

(i) Can we use $\frac{\overline{x}+\overline{y}}{2}$ to find the combined mean? Justify your answer.

(ii) Can we add the range of the masses for both schools to find the combined range? Explain your answer.

(iii) Find an estimate of the combined mean and range of all 200 students.

3.4 Scatter diagrams

A **scatter diagram** is commonly used to illustrate the results of a statistical survey or enquiry by comparing the two sets of data. Values of the two sets of data are recorded in pairs and these are plotted on a graph just like plotting a pair of x-y coordinates. If these pairs of coordinate points tend to lie on a straight line, we can conclude that there is a close relationship or **correlation** between these two sets of data.

The scores obtained by 12 pupils in Paper 1 and Paper 2 of a mathematics examination are shown in the table below.

Paper 1	55	45	20	34	53	58	38	40	15	29	18	27
Paper 2	47	42	17	26	48	51	31	34	12	23	16	24

Fig. 3.7 shows the scatter diagram when the set of data is plotted. We usually plot the independent variable on the horizontal axis and the dependent variable on the vertical axis. In the case above where there are no clear independent and dependent variables, we plot the first variable on the horizontal axis and the second variable on the vertical axis.

Information

A scatter diagram is also known as a scatter plot or a scatter graph.

Attention

An advantage of a scatter diagram is that it can be easily drawn to represent large quantities of data between two variables that are being measured. A disadvantage is that it does not give the exact extent of correlation.

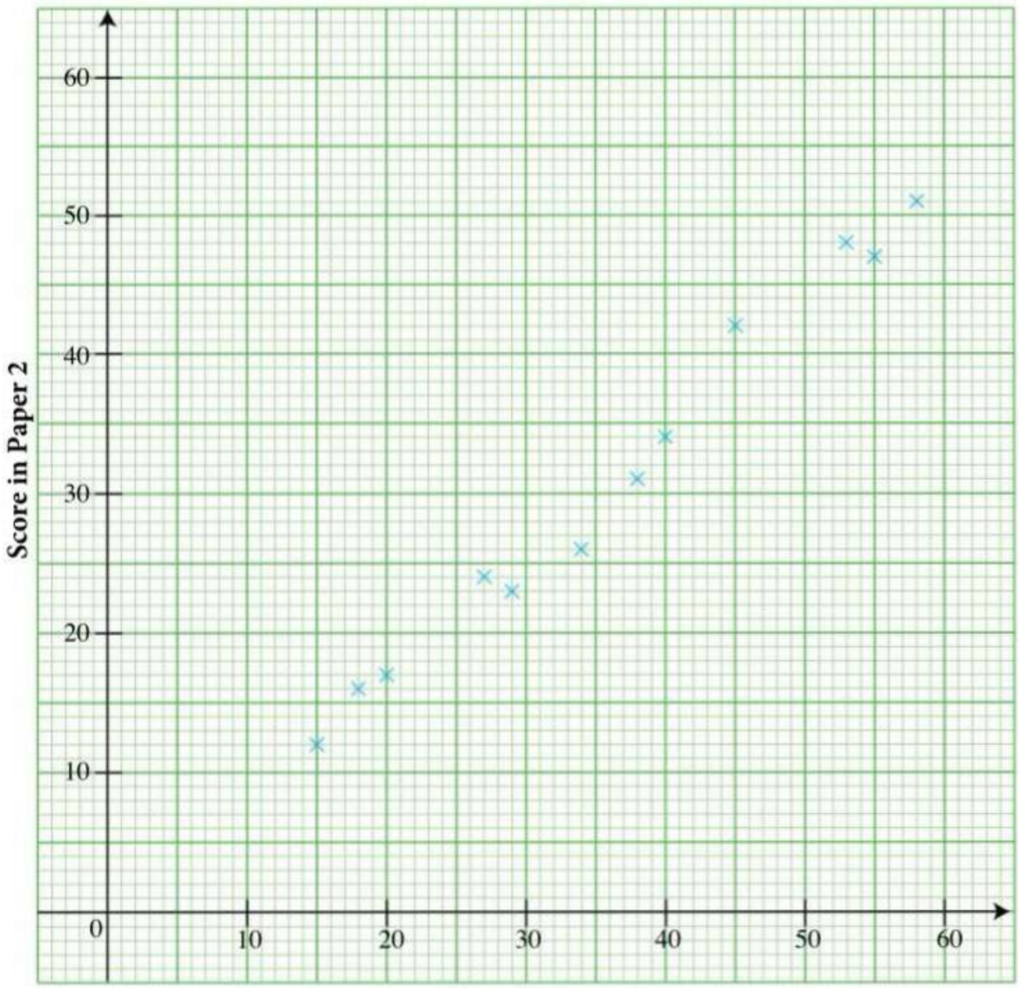


Fig. 3.7

Fig. 3.7 suggests that there is a close relationship between the scores for Paper 1 and Paper 2. The higher the score for Paper 1, the higher it is for Paper 2. This is called a **positive correlation**. We may also use the term "strong", "moderate" or "weak" to give a clearer indication of the correlation.

We can also show the general trend by drawing a line that will pass through most of the points or as close to most of the points as possible. This line is called the **line of best fit** as shown in Fig. 3.8.

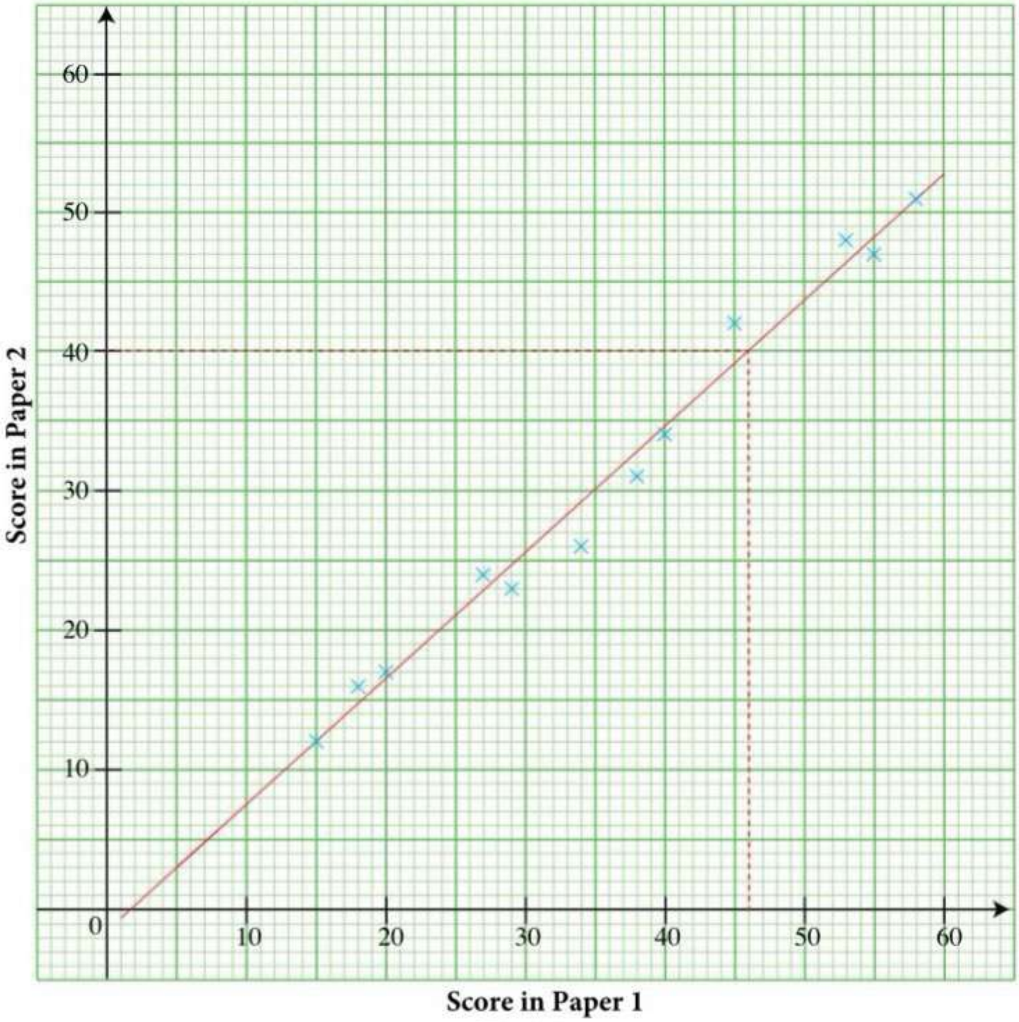


Fig. 3.8

Attention

When data points are more scattered, the correlation between the two variables weakens.

As most of the marks lie very close to the line of best fit, we say that there is **strong positive correlation** between Paper 1 and Paper 2. We can use the line of best fit to predict the performance of a pupil who was absent from sitting Paper 2. For example, if a pupil scored 46 marks in Paper 1 but was absent for Paper 2, we can use the graph to determine the likely score he will obtain for Paper 2. This can be done by locating the corresponding score for Paper 2 from the line of best fit when the score for Paper 1 is 46, as shown in Fig. 3.8.

Thinking Time

In Book 3, we learnt that the equation of a straight line is $y = mx + c$, where the constant m is the gradient of the line and the constant c is the y-intercept. Can we apply this to find the equation of the line of best fit?

From the above Thinking Time, we can conclude that we can obtain the equation of the line of best fit in the same way as we do for any linear graph.

Worked Example 8

Scatter diagram with positive correlation

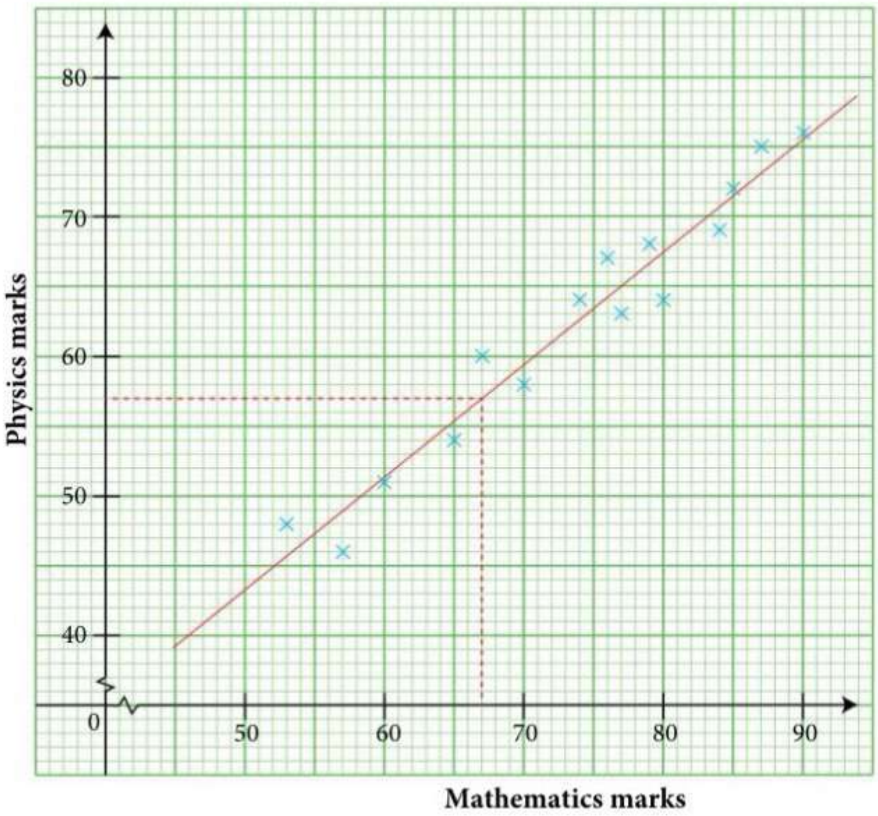
The table below shows the marks scored by 15 pupils in a Mathematics and Physics examination

Mathematics	85	67	65	84	53	80	70	87	79	74	90	60	76	57	77
Physics	72	60	54	69	48	64	58	75	68	64	76	51	67	46	63

(i) Draw a scatter diagram for the above data.
(ii) What type of correlation is shown in the scatter diagram?
(iii) Draw a line of best fit on the scatter diagram.
(iv) Use your line of best fit to estimate the number of marks that a pupil who scored 57 in Physics is likely to score in Mathematics.
(v) Would it be reliable to use your line of best fit to estimate the number of marks that a pupil who scored 45 in Mathematics is likely to score in Physics? Explain your answer clearly.

Solution

(i)



(ii) At first glance these values suggest that there is close connection between the Mathematics and Physics marks. The higher the Mathematics marks, the higher the Physics marks. It is a **strong positive correlation**.

(iii) The line of best fit is drawn with approximately half of the plots lying above the line and the other half lying below the line of best fit.

(iv) Using the line of best fit on the scatter diagram, a student who scored 57 in Physics is likely to score 67 in Mathematics.

Problem-solving Tip

The jagged lines near the origin are used to indicate a broken scale. They are used when values close to 0 are not required. In this case, we start with 50 marks on the horizontal axis and 40 marks on the vertical axis.

(v) It would be unreliable since the score of 45 lies outside of the range as no pupil had scored less than 50 marks for Mathematics.

Attention

Some points to take note of when drawing the line of best fit:
- The line drawn may pass through all the points, some of the points or none of them.
- The data points have to be evenly distributed on either sides of the line of best fit.
- Use a transparent ruler to draw the line of best fit so as to gauge the best position to fit the line.

Practise Now 8

Similar and Further Questions
Exercise 3D
Questions 1(b), (c), 2, 4

The fuel consumption of a small truck measured against the total mass of the truck and the mass of goods it carries is shown in the table below.

Mass (tonnes)	3.00	3.04	2.26	2.70	3.10	2.50	2.80	2.56	2.84	2.30	2.90	2.32	2.44
Fuel consumption (l / 100 km)	14.8	15.2	11.2	13.3	15.1	12.1	13.5	12.7	13.8	11.2	14.5	11.7	11.9

(i) Draw a scatter diagram for the above data.
(ii) What type of correlation is shown in the scatter diagram?
(iii) Draw a line of best fit on the scatter diagram.
(iv) Use your line of best fit to estimate the number of litres of petrol that a truck with a total mass of 2.62 tonnes will need to travel 100 km.
(v) Would it be reliable to use your line of best fit to estimate the number of litres of petrol that a truck with a total mass of 4.8 tonnes will need to travel 100 km?
Explain your answer clearly.

Worked Example 9

Scatter diagram with negative correlation

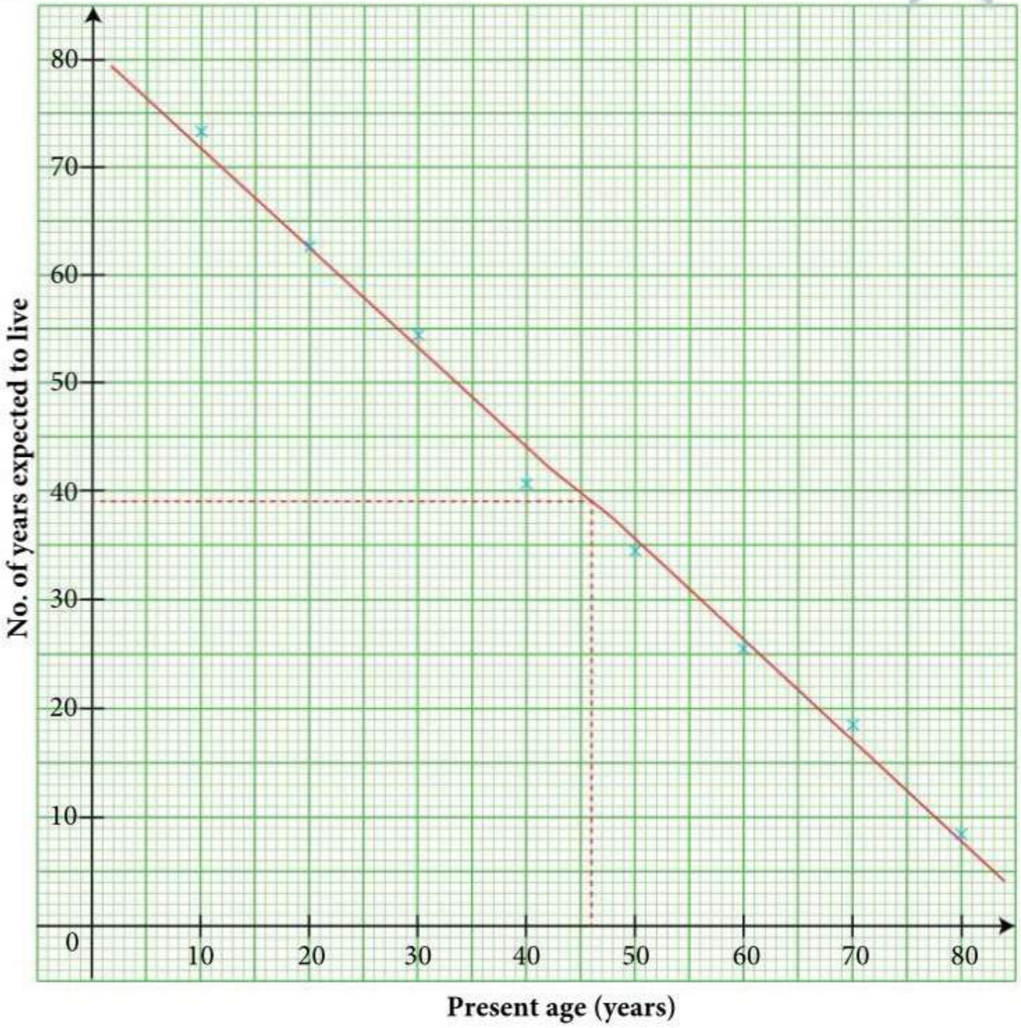
The table below shows the number of years that people of various ages in an Asian country are expected to live out their remaining years in life.

Present age (years)	10	20	30	40	50	60	70	80
Additional number of years expected to live	73.2	62.8	54.3	40.7	34.5	25.1	18.7	8.3

(i) Draw a scatter diagram for the above data.
(ii) What type of correlation is shown in the scatter diagram?
(iii) Draw a line of best fit on the scatter diagram.
(iv) Use your line to estimate the additional number of years that a person aged 46 is expected to live.
(v) Would it be reliable to use your line of best fit to estimate the additional number of years that the oldest person in the country aged 105 is expected to live? Explain your answer.

Solution

(i)



(ii) The data shows strong negative correlation.

(iii) The line of best fit is drawn passing through as many points as possible and as close as possible to all the other points.

(iv) Using the line of best fit on the scatter diagram, a person aged 46 is expected to live for another 39 years.

(v) No, it would be highly unreliable as it lies outside the range as there is no data for people with a present age of more than 100.

Practise Now 9

Similar and Further Questions
Exercise 3D
Questions 1(a), (d), 3, 5

The table below shows the time taken (in seconds) by 15 pupils aged between 11 and 16 years to run the 100 m race.

Age (years)	13.3	15.3	13.6	15.5	14.0	14.3	15.3	12.8	12.2	14.8	11.7	14.9
Time (s)	13.9	12.3	13.2	11.6	13.1	12.3	12.3	13.9	14.6	13.1	14.7	12.1

Age (years)	12.5	13.2	11.1
Time (s)	14.5	13.3	15.2

(i) Draw a scatter diagram for the above data.

(ii) What type of correlation is shown in the scatter diagram?

(iii) Draw a line of best fit on the scatter diagram.

(iv) Use your line to estimate the time that a pupil aged 14.5 years is expected to comple th 100 m race.

(v) Can we use the above data to predict the time that a pupil aged 17 years old will take to complete the 100 m race? Explain your answer.

Class Discussion

Scatter diagram with no correlation

Work in pairs.

The table below shows the height of a group of pupils and the number of marks they scored in an English examination.

Height (cm)	143	144	145	149	155	157	162	165	169	173	173	177	180	184	187	188
English (marks)	62	85	45	67	55	40	52	78	47	38	89	64	45	71	53	42

1. Draw a scatter diagram for the above data.
2. Describe the correlation shown in the scatter diagram.

From the above Class Discussion, we observe that the scatter diagram for two variables with no correlation will have plots with no visible upward or downward trend.

Thinking Time

What sort of correlation (if any) would you expect between each of the following? Explain your assumptions.

1. The number of hours one spends doing mathematical problems and the number of marks in the examination.
2. The number of cars on the road and the number of traffic accidents.
3. The weight of a person and the house number that they live in.
4. The amount of money one has in the bank and the amount of interest he will earn.
6. The price of a shirt and the colour of its buttons.
6. The size of a cupcake and the number of cupcakes you can eat.
7. The height of a man and the number of fingers that he has.
8. The speed of a sprinter and the number of races he wins.

Journal Writing

If you are required to find the correlation between two quantities, write down the steps that you would need to prepare. Do you have pre-conceived results that you expect to observe? How do you intend to set about finding data to support your investigation? Explain some of the difficulties that you encounter in your investigation and some steps to avoid if you were to carry out another investigation.

Reflection

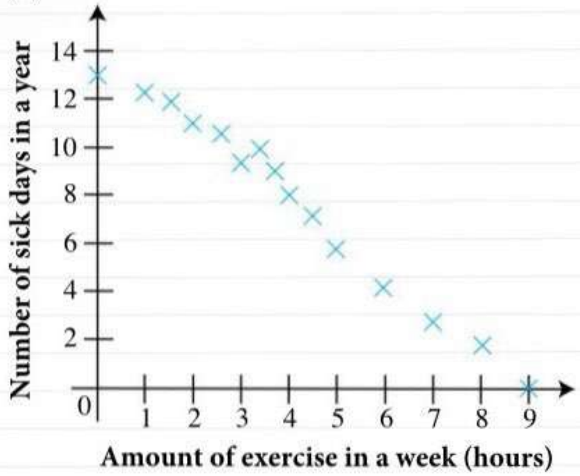
1. How do I draw the line of best fit through a set of data points on a scatter diagram?
2. How do I tell if there is a strong correlation between two variables from a scatter diagram?
3. What have I learnt in this section or chapter that I am still unclear of?

Advanced Intermediate Basic

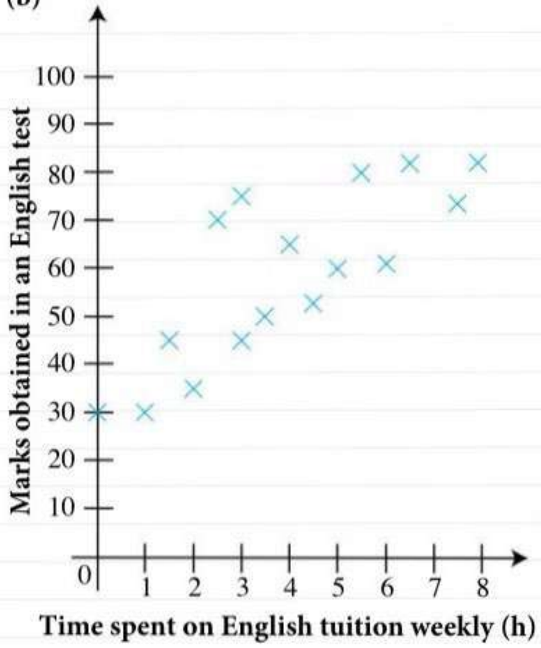
Exercise 3D

1. State the type of correlation for each of the scatter diagrams below.

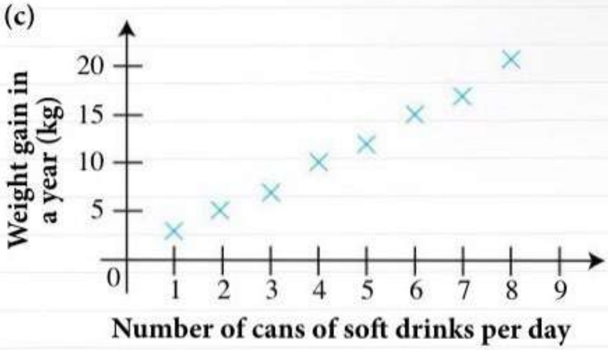
(a)



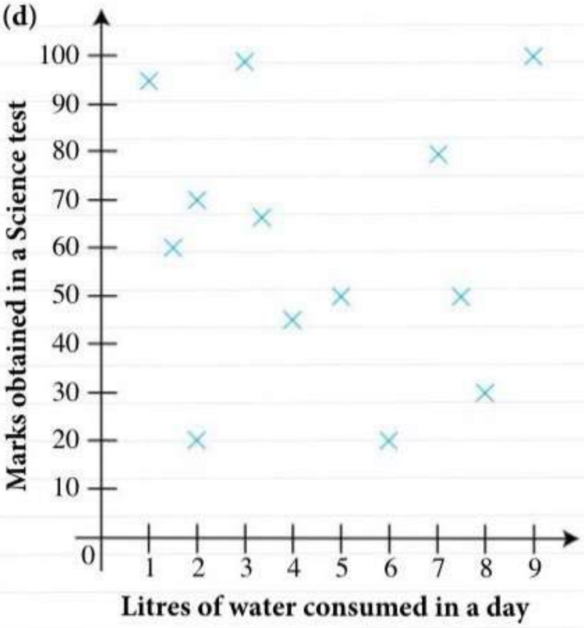
(b)



(c)

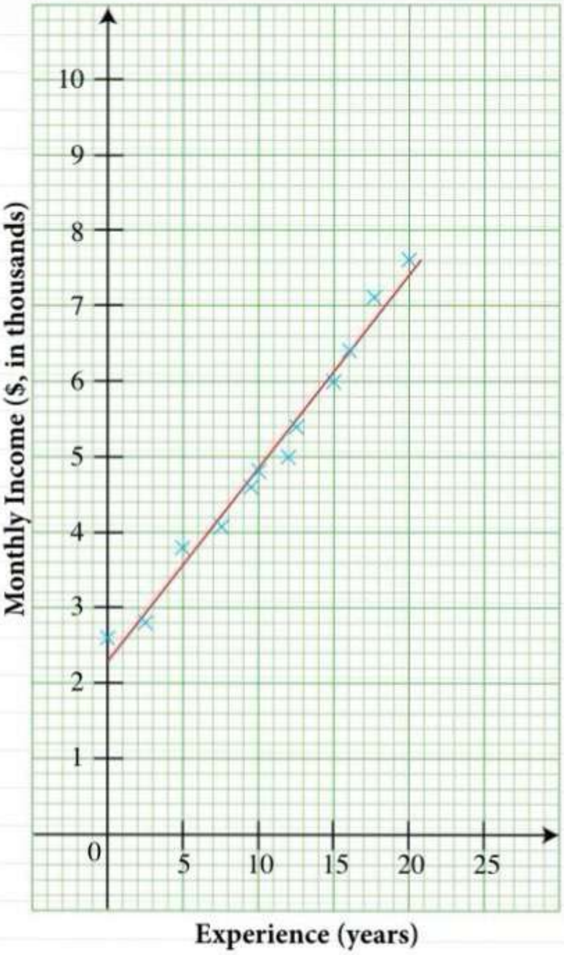


(d)



Exercise 3D

2. The scatter diagram shows the monthly income earned by individuals with varying years of experience. The line of best fit for the scatter diagram has been drawn.



Using the line of best fit, estimate the monthly income earned by an individual with 18 years of experience.

3. The table below shows the mass of a group of pupils and their respective intelligence quotient (IQ) score.

Mass (kg)	66	62	56	75	50	43	90
IQ score	111	95	113	94	107	92	91
Mass (kg)	56	47	88	67	82	85	72
IQ score	91	102	101	90	106	115	104

(i) Draw a scatter diagram for the above data.

(ii) What type of correlation is shown in the scatter diagram?

4. The following table shows the age and their systolic blood pressure of 14 women.

Age	45	51	75	33	52	55	68
BP (mm Hg)	123	127	151	107	135	131	152
Age	44	55	48	62	35	65	42
BP (mm Hg)	125	134	130	138	113	145	117

(i) Draw a scatter diagram to show the age and blood pressure of the 14 women.

(ii) Describe the type of correlation between the age and the blood pressure of the women in the study.

(iii) Draw a line of best fit for this set of data.

(iv) What could the blood pressure of a 58-year-old lady be?

(v) David is 86 years old. Would it be reliable to use the line of best fit to predict his blood pressure? Explain your answer clearly.

Exercise 3D

5. The table below shows the number of cigarettes smoked per day and the life expectancy of a group of smokers.

Number of cigarettes smoked	42	13	45	23	40	48	16	38	56	27	8	35	5	33
Life expectancy (years)	56	75	53	71	51	50	73	61	44	64	77	55	86	63

(i) Draw a scatter diagram to show the number of cigarettes smoked per day and the life expectancy of this group of smokers.

(ii) Describe the type of correlation between the number of cigarettes smoked per day and the life expectancy of this group of smokers.

(iii) Draw a line of best fit for this set of data.

(iv) A man smoked 25 cigarettes per day, what could his life expectancy be?

(v) Would it be reliable to use the line of best fit to predict the life expectancy of a non-smoker? Explain your answer clearly.

Looking Back

In this chapter, we have learnt most of the basic ideas of statistics that provide the means for us to collect, interpret and use data to make decisions. Statistical literacy – the ability to read, understand and reason with data to make decisions – is a critical 21st century competency. Many of the ideas we have learnt in this chapter allow us to read and understand datasets that are summarised with **measures** such as mean and range. Beyond reading and understanding datasets, we should also know how to represent data with appropriate **diagrams** so that we can communicate the information embedded in the dataset clearly. Most importantly we should be able to make decisions after carrying out a reasoned and meaningful interpretation of the data. We are currently in the fourth industrial revolution, where data is the new currency. There is little doubt that statistics will continue to play an important role in the future and we have just taken the first small step towards understanding this important field of mathematics.

Summary

1. A **cumulative frequency table** is a statistical diagram used to display the cumulative frequencies of a dataset.
   - Explain how the cumulative frequencies can be obtained from a frequency table.

2. The cumulative distribution can be displayed graphically by a **cumulative frequency curve**, which can be used to estimate corresponding data values.

3. A cumulative frequency curve can also be used to *estimate* the **quartiles** and **percentiles** of a distribution.
   (i) The **lower or first quartile Q_1** is a measure of the *lower 25%* of the dataset.
   (ii) The **median** (or the **second quartile Q_2**) is a measure of the *centre* of the dataset.
   (iii) The **upper or third quartile Q_3** is a measure of the lower 75% (or *upper 25%*) of the dataset.

4. **Range** = largest value – smallest value
   Interquartile range = $Q_3 - Q_1$
   The interquartile range measures the spread of the *middle 50%* of the data about the *median*. It is a better measure of spread than range because it is not affected by extreme values (or outliers).
   - Find the interquartile range and range of the following set of data.

     2, 5, 1, 7, 4, 3, 2

5. Data in a **scatter diagram** are represented by plotting pairs of coordinates on a Cartesian plane. A **line of best fit** is often drawn on the scatter diagram, passing through the middle of all the data points, to determine the strength of **correlation**.

CHAPTER 4

Vectors

The picture shows a signpost in Cape Town, South Africa. Each panel provides two pieces of information: the direction of a city from the signpost, and its distance from the signpost. From the picture, can you tell how far Singapore is from the signpost? Based on your geographical knowledge, in what general direction is Pakistan from South Africa?

Just like the case of signposts, it is sometimes important to tell the size and direction of a quantity. For example, we may wish to state the speed of a car which is moving in a certain direction; or we may want to think about how we can cross a flowing river in the most efficient way. In all these cases, we need to deal with quantities with **measures** of magnitude and direction, or what mathematicians term as vectors.

In this chapter, we will expand our understanding of quantities to include the idea of vectors. We will learn how we can represent them using vector **notations** and vector **diagrams**, so that we can work with vectors to **model** and solve real-world problems.

Learning Outcomes

What will we learn in this chapter?

- What vectors are
- How to use vector notations and to represent vectors as directed line segments and in column vector form
- How to add and subtract vectors, and to multiply a vector by a scalar
- How to express a vector in terms of two non-zero and non-parallel coplanar vectors
- How to express a vector in terms of position vectors
- Why vectors have useful applications in mathematics and real-world contexts

Introductory Problem

Cheryl walked 100 metres towards the north from point *P*, as shown in Fig. 4.1.
David also walked 100 metres from point *P*, but towards the east.

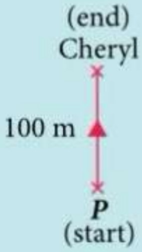


Fig. 4.1

1. On Fig. 4.1, draw the route taken by David.
2. Although both Cheryl and David walked the same distance of 100 metres each, did they end up at the same point? Explain.
3. In the real world, distance is not enough to describe motion. What else do you need?

A **scalar** quantity only has a *magnitude*. For example, **distance** is a scalar. In the above scenario, the distances travelled by both Cheryl and David are the same: 100 metres (magnitude) each.

A **vector** quantity has both a *magnitude* and a *direction*. For example, **displacement** is a vector. In the scenario presented above, the displacement of Cheryl from *P* is 100 metres in the north direction.

4. Describe the displacement of David from *P*.

Another real-world example of a scalar is **speed** (e.g. 50 km/h) while another real-world example of a vector is **velocity** (e.g. 50 km/h southwards).

5. Give a few more real-life examples of scalars and vectors.

From the Introductory Problem, we realise that scalars alone may not be enough to describe some quantities. Therefore, in this chapter, we will learn about vectors, and how to represent and use them to solve problems.

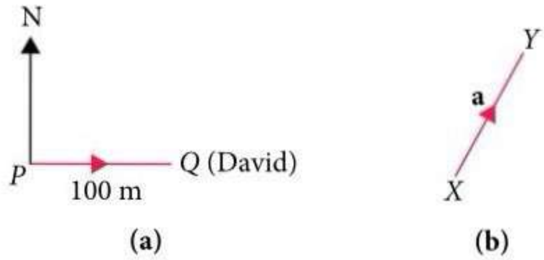
4.1 Vectors in two dimensions

A. Representation of vector as directed line segment

A **non-zero vector** can be represented by a **directed line segment**, where the direction of the line segment is that of the vector, and the length of the line segment represents the magnitude of the vector. Fig. 4.2 shows some examples of vectors.

Attention

We will discuss the zero vector in Section 4.2B.



(a) (b)

Fig. 4.2

In Fig. 4.2(**a**), the displacement vector is denoted by $\overrightarrow{\mathbf{PQ}}$, where P is the *starting or initial point*, and Q is the *ending or terminal point*. The **magnitude** of $\overrightarrow{PQ}$ is denoted by $|\overrightarrow{\mathbf{PQ}}|$. In this case, $|\overrightarrow{PQ}| = 100$ m.

In Fig. 4.2(**b**), the vector $\overrightarrow{XY}$ is denoted by a bold letter **a**. Its magnitude is denoted by $|\mathbf{a}|$. Since it is tedious to write in bold by hand, we write the vector as $\underset{\sim}{a}$ and its magnitude as $|\underset{\sim}{a}|$.

Big Idea

Notations

Since a vector has both a magnitude and a direction, we use an arrow to indicate the direction from the starting to the ending point. By convention, we write the starting point first, e.g. a vector from P to Q would be written as $\overrightarrow{PQ}$.

We represent the magnitude of $\overrightarrow{PQ}$ using the notation $|\overrightarrow{PQ}|$.

We can also use the notation **a** to represent a vector. Take note that **a** is not a variable (like a in algebra), so **a** is typed in bold but not in italics. Understanding these conventions makes it easier for mathematical ideas to be expressed and communicated.

B. Representation of vector on Cartesian plane as column vector

Fig. 4.3 shows two vectors lying on a Cartesian plane.

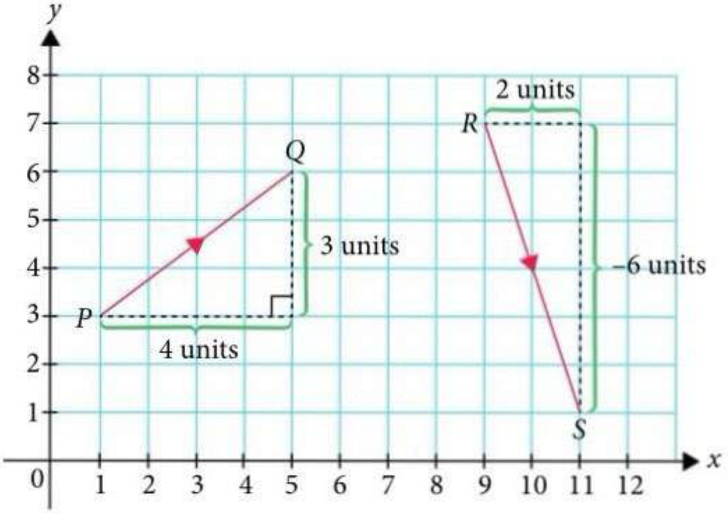


Fig. 4.3

To move from P to Q, we can move 4 units in the positive x-direction from P and then 3 units in the positive y-direction.

So another way to describe the vector $\overrightarrow{PQ}$ is to use a **column vector**, namely, $\overrightarrow{PQ} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$, where the first entry 4 in the column vector represents the number of units in the positive x-direction and the second entry 3 represents the number of units in the positive y-direction.

4 and 3 are called the **components** of the vector $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$, where 4 is the x-component and 3 is the y-component.

Similarly, $\overrightarrow{RS} = \begin{pmatrix} 2 \\ -6 \end{pmatrix}$. The y-component is negative because S is 6 units in the *negative* y-direction from R.

Is $\begin{pmatrix} 2 \\ -6 \end{pmatrix}$ equal to $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$? Explain.

To find the magnitude of $\overrightarrow{PQ}$, we can use Pythagoras' Theorem to find the length of the line segment PQ, i.e.

$$\left|\overrightarrow{PQ}\right| = \sqrt{4^2 + 3^2} = 5 \text{ units.}$$

What is the magnitude of $\overrightarrow{RS}$?

In general,

the magnitude of a column vector $\mathbf{a} = \begin{pmatrix} x \\ y \end{pmatrix}$ is given by $|\mathbf{a}| = \sqrt{x^2 + y^2}$.

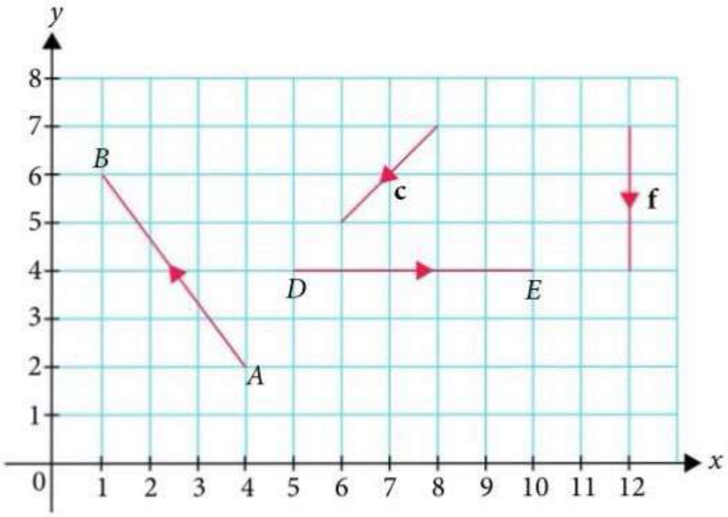
Consider the *horizontal vectors* $\mathbf{b} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} -3 \\ 0 \end{pmatrix}$, and the *vertical vectors* $\mathbf{d} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$ and $\mathbf{e} = \begin{pmatrix} 0 \\ -4 \end{pmatrix}$.

What are the values of $|\mathbf{b}|$ and $|\mathbf{c}|$?
What are the values of $|\mathbf{d}|$ and $|\mathbf{e}|$?

Practise Now 1A

Similar and Further Questions
Exercise 4A
Questions 1(a)–(e), 5, 10

Express each of the vectors in the diagram as a column vector and find its magnitude.



C. Equal vectors

Class Discussion

Equal vectors

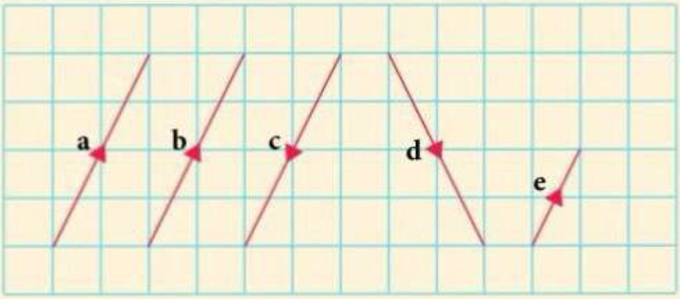


Fig. 4.4

1. Compare vectors **a** and **b**.
    - **(i)** What do you notice about their magnitudes? How can you confirm your answer?
    - **(ii)** What do you notice about their directions? How can you confirm your answer?
    - **(iii)** What do you notice about their x- and y-components?
2. Compare vectors **a** and **c**.
    - **(i)** What do you notice about their magnitudes?
    - **(ii)** What do you notice about their directions?
    - **(iii)** What do you notice about their x- and y-components?
3. Compare vectors **a** and **d**. How are the two vectors similar or different?
4. Compare vectors **a** and **e**. How are the two vectors similar or different?

From the above Class Discussion, we observe that:

- Two vectors **a** and **b** are **equal** (i.e. **a** = **b**) if and only if they have the same magnitude and the same direction.

  In column vector forms, $\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} r \\ s \end{pmatrix}$ if and only if $p = r$ and $q = s$.
- If two vectors **a** and **c** have the same magnitude but opposite directions, vector **c** is called the **negative vector** of vector **a** (and vice versa) and we write **a** = –**c** (or **c** = –**a**).
- Two vectors are **parallel** if they have the same or opposite directions.

Attention

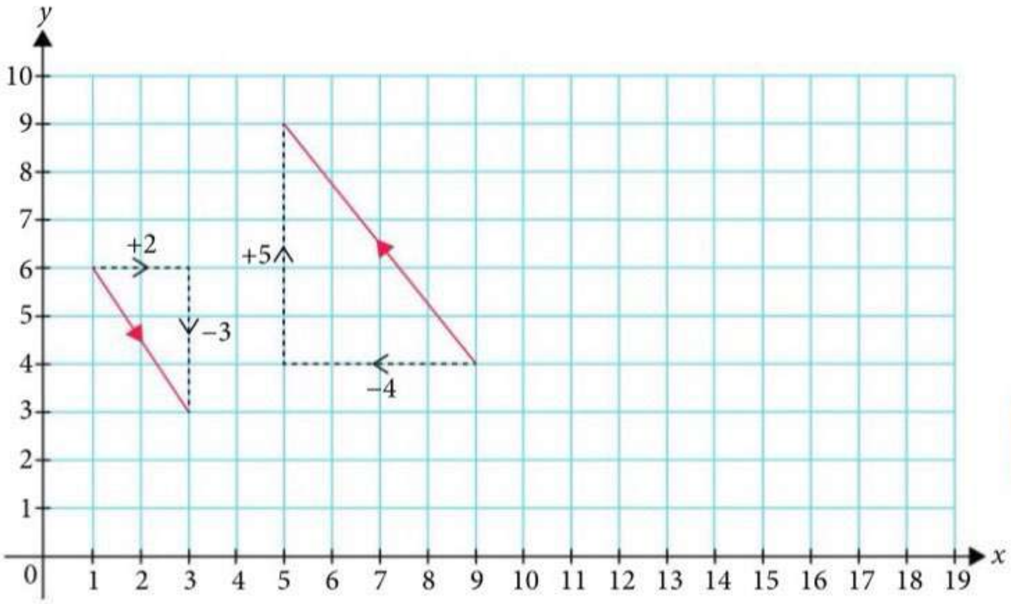
'Opposite directions' is not the same as 'different directions'. In the above Class Discussion, vectors **a** and **c** have *opposite directions* but **a** and **d** have *different directions*.

Worked Example 1

Drawing vector on Cartesian plane

On the 1 unit by 1 unit square grid below, draw the column vectors $\begin{pmatrix}2\\-3\end{pmatrix}$ and $\begin{pmatrix}-4\\5\end{pmatrix}$.

***Solution**



Problem-solving Tip

Choose any starting point.

For $\begin{pmatrix}2\\-3\end{pmatrix}$, move 2 units in the positive x-direction from the starting point and then 3 units in the negative y-direction to the ending point. Join the starting and ending points with a directed line segment. Since the y-component of the column vector is negative, the starting point should be higher than the ending point.

Information

It does not matter if your starting point is different from your classmates because all the vectors with the same column vector form are equal.

Practise Now 1B

Similar and Further Questions
Exercise 4A
Questions 2(a)–(e), 6(a)–(f), 11(a), (b)

Draw the following column vectors on the square grid provided in Worked Example 1.

(a) $\begin{pmatrix}4\\-6\end{pmatrix}$

(b) $\begin{pmatrix}-2\\3\end{pmatrix}$

(c) $\begin{pmatrix}4.5\\0\end{pmatrix}$

(d) Negative of $\begin{pmatrix}0\\1\end{pmatrix}$

Thinking Time

(a) Consider the vector $\begin{pmatrix}2\\-3\end{pmatrix}$ in Worked Example 1 and the vector $\begin{pmatrix}4\\-6\end{pmatrix}$ in Practise Now 1B(**a**).

- **(i)** Are the two vectors equal?
- **(ii)** Compare their magnitudes and directions and describe the relationship between the two vectors.

(b) Consider the vector $\begin{pmatrix}2\\-3\end{pmatrix}$ in Worked Example 1 and the vector $\begin{pmatrix}-2\\3\end{pmatrix}$ in Practise Now 1B(**b**).

- **(i)** Are the two vectors equal?
- **(ii)** Compare their magnitudes and directions and describe the relationship between the two vectors.

Worked Example 2

Finding magnitude and direction of vector

Two column vectors **a** and **b** are such that $\mathbf{a} = \begin{pmatrix} x-2 \\ 3-y \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 4-x \\ y-6 \end{pmatrix}$.

(i) If **a** = **b**,

- **(a)** find the value of x and of y,
- **(b)** write down the negative of **a** as a column vector,
- **(c)** show that $|\mathbf{a}| = |\mathbf{b}| = \sqrt{\frac{13}{4}}$ units.

(ii) If $|\mathbf{a}| = |\mathbf{b}|$,

- **(a)** express y in terms of x,
- **(b)** explain why **a** might not be equal to **b**.

Solution

(i) (a) $\begin{pmatrix} x-2 \\ 3-y \end{pmatrix} = \begin{pmatrix} 4-x \\ y-6 \end{pmatrix}$

$$\therefore x-2 = 4-x \quad \text{and} \quad 3-y = y-6$$

$$2x = 6 \qquad\qquad 2y = 9$$

$$x = 3 \qquad\qquad y = 4\tfrac{1}{2}$$

$$\therefore x = 3 \text{ and } y = 4\tfrac{1}{2}$$

(b) $\mathbf{a} = \begin{pmatrix} x-2 \\ 3-y \end{pmatrix}$

$$= \begin{pmatrix} 3-2 \\ 3-4\frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ -1\frac{1}{2} \end{pmatrix}$$

Negative of **a** = −**a**

$$= -\begin{pmatrix} 1 \\ -1\frac{1}{2} \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ 1\frac{1}{2} \end{pmatrix}$$

Big Idea

Equivalence

A **vector equation**, in which two 2 × 1 column vectors are equal, e.g.

$$\begin{pmatrix} x-2 \\ 3-y \end{pmatrix} = \begin{pmatrix} 4-x \\ y-6 \end{pmatrix},$$

is equivalent to a **pair of simultaneous equations** in two variables, i.e.

$$x-2 = 4-x,$$
$$3-y = y-6,$$

where the x-components of both vectors are equal, and the y-components of both vectors are also equal.

Attention

The x-component and y-component of a vector can be expressed as a mixed number, an improper fraction, or a decimal, e.g.

$$\mathbf{a} = \begin{pmatrix} 1 \\ -1\frac{1}{2} \end{pmatrix} \text{ or } \begin{pmatrix} 1 \\ -\frac{3}{2} \end{pmatrix}$$

$$\text{or } \begin{pmatrix} 1 \\ -1.5 \end{pmatrix}.$$

(c)
$$
\begin{aligned}
|\mathbf{a}| &= \sqrt{1^2 + \left(-1\frac{1}{2}\right)^2} \\
&= \sqrt{1^2 + \left(-\frac{3}{2}\right)^2} \\
&= \sqrt{1 + \frac{9}{4}} \\
&= \sqrt{\frac{13}{4}} \text{ units}
\end{aligned}
$$

$$
\text{Since } \mathbf{b} = \mathbf{a} = \begin{pmatrix} 1 \\ -1\frac{1}{2} \end{pmatrix}, \text{ then } |\mathbf{b}| = \sqrt{1^2 + \left(-1\frac{1}{2}\right)^2} = \sqrt{\frac{13}{4}} \text{ units}
$$

$$
\therefore |\mathbf{a}| = |\mathbf{b}| = \sqrt{\frac{13}{4}} \text{ units (shown)}
$$

(ii) (a)
$$
\begin{aligned}
|\mathbf{a}| &= |\mathbf{b}| \\
\sqrt{(x-2)^2 + (3-y)^2} &= \sqrt{(4-x)^2 + (y-6)^2} \\
x^2 - 4x + 4 + 9 - 6y + y^2 &= 16 - 8x + x^2 + y^2 - 12y + 36 \qquad \text{square both sides} \\
\cancel{x^2} - 4x + 4 + 9 - 6y + \cancel{y^2} &= 16 - 8x + \cancel{x^2} + \cancel{y^2} - 12y + 36 \\
-4x + 13 - 6y &= -8x - 12y + 52 \\
6y &= -4x + 39 \\
y &= \frac{39 - 4x}{6}
\end{aligned}
$$

(b) In $y = \frac{39 - 4x}{6}$, x and y have no fixed values.

For example, if $x = 0$, $y = \frac{39 - 0}{6} = 6\frac{1}{2}$.

$$
\text{Then } \mathbf{a} = \begin{pmatrix} x-2 \\ 3-y \end{pmatrix} = \begin{pmatrix} 0-2 \\ 3-6\frac{1}{2} \end{pmatrix} = \begin{pmatrix} -2 \\ -3\frac{1}{2} \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 4-x \\ y-6 \end{pmatrix} = \begin{pmatrix} 4-0 \\ 6\frac{1}{2}-6 \end{pmatrix} = \begin{pmatrix} 4 \\ \frac{1}{2} \end{pmatrix}
$$

$$
|\mathbf{a}| = \sqrt{(-2)^2 + \left(-3\frac{1}{2}\right)^2} = \sqrt{4 + \frac{49}{4}} = \sqrt{16\frac{1}{4}} \text{ units} \quad \text{and} \quad |\mathbf{b}| = \sqrt{4^2 + \left(\frac{1}{2}\right)^2} = \sqrt{16 + \frac{1}{4}} = \sqrt{16\frac{1}{4}} \text{ units}
$$

$\therefore |\mathbf{a}| = |\mathbf{b}|$ but $\mathbf{a} \neq \mathbf{b}$ in this case.

Attention

If $x = 3$, then $y = \frac{39 - 4 \times 3}{6} = 4\frac{1}{2}$, which is the value of x and of y in **(i)(a)**. For this special case, $\mathbf{a} = \mathbf{b}$. For other values of x and y, $\mathbf{a} \neq \mathbf{b}$.

Practise Now 2

Similar and Further Questions
Exercise 4A
Questions 3, 7, 8, 12

Two column vectors **a** and **b** are such that $\mathbf{a} = \begin{pmatrix} x+2 \\ 4-y \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 10-x \\ y-5 \end{pmatrix}$.

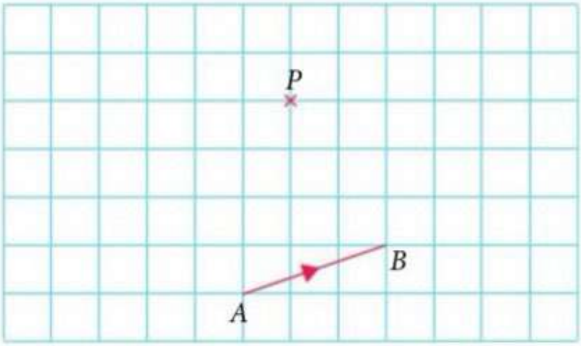
(i) If $\mathbf{a} = \mathbf{b}$,
  (a) find the value of x and of y,
  (b) write down the negative of **a** as a column vector,
  (c) show that $|\mathbf{a}| = |\mathbf{b}| = \sqrt{\frac{145}{4}}$ units.

(ii) If $|\mathbf{a}| = |\mathbf{b}|$,
  (a) express y in terms of x,
  (b) explain why **a** might not be equal to **b**.

Worked Example 3

Drawing vectors to form a parallelogram

The figure below shows the positions of the points P, A and B, where $\overrightarrow{AB} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$.



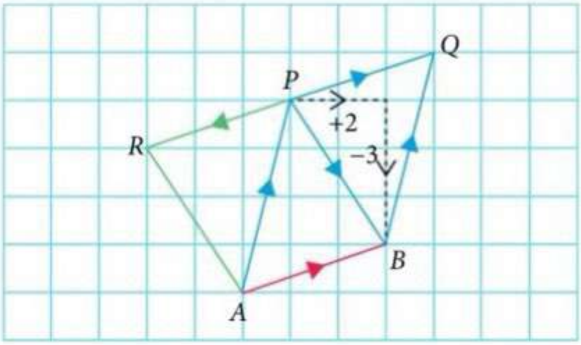
(i) Express $\overrightarrow{PB}$ as a column vector.

(ii) Plot the point Q such that $ABQP$ is a parallelogram. Express $\overrightarrow{BQ}$ as a column vector.

(iii) Plot the point R such that $ABPR$ is a parallelogram. Express $\overrightarrow{PR}$ as a column vector.

(iv) Do the two vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ have the same magnitude? Is $\overrightarrow{PQ} = \overrightarrow{PR}$? Explain.

Solution



(i) $\overrightarrow{PB} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$

(ii) $\overrightarrow{BQ} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$

(iii) $\overrightarrow{PR} = \begin{pmatrix} -3 \\ -1 \end{pmatrix}$

(iv) $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ have the same magnitude but $\overrightarrow{PQ} \neq \overrightarrow{PR}$ because they do not have the same direction.

Problem-solving Tip

(ii) For parallelogram *ABQP*, the vertices must be in this order:

$A \rightarrow B \rightarrow Q \rightarrow P$.

To draw the parallelogram *ABQP*, we note that $AP = BQ$ and $AP \parallel BQ$, i.e. $\overrightarrow{AP} = \overrightarrow{BQ}$.

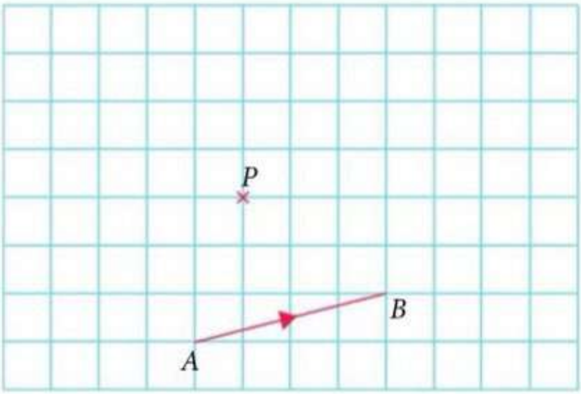
Practise Now 3

Similar and Further Questions

Exercise 4A

Questions 4, 9, 13

The figure below shows the positions of the points *P*, *A* and *B*, where $\overrightarrow{AB} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$.



(i) Express $\overrightarrow{PB}$ as a column vector.

(ii) Plot the point *Q* such that *ABQP* is a parallelogram. Express $\overrightarrow{BQ}$ as a column vector.

(iii) Plot the point *R* such that *ABPR* is a parallelogram. Express $\overrightarrow{PR}$ as a column vector.

(iv) Do the two vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ have the same magnitude? Is $\overrightarrow{PQ} = \overrightarrow{PR}$? Explain.

Reflection

1. How do I tell when two vectors are equal if they are drawn using directed line segments?
2. How do I tell when two vectors are equal if they are in column vector form?
3. What have I learnt in this section that I am still unclear of?

Exercise 4A

1. Find the magnitude of each of the following vectors.

(a) $\begin{pmatrix}3\\4\end{pmatrix}$ **(b)** $\begin{pmatrix}-5\\12\end{pmatrix}$

(c) $\begin{pmatrix}-7\\-2\end{pmatrix}$ **(d)** $\begin{pmatrix}0\\-6\frac{1}{2}\end{pmatrix}$

(e) $\begin{pmatrix}8\\0\end{pmatrix}$

2. Write down the negative of each of the following vectors.

(a) $\begin{pmatrix}12\\-7\end{pmatrix}$ **(b)** $\begin{pmatrix}-2\\0\end{pmatrix}$

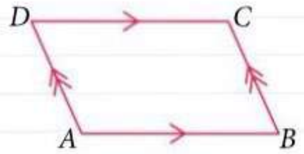
(c) $\begin{pmatrix}4\\8\end{pmatrix}$ **(d)** $\begin{pmatrix}-3\\-1.2\end{pmatrix}$

(e) $\begin{pmatrix}0\\3\frac{1}{4}\end{pmatrix}$

3. If $\mathbf{p} = \begin{pmatrix}a\\3\end{pmatrix}$, $\mathbf{q} = \begin{pmatrix}-2\\a+2b\end{pmatrix}$ and $\mathbf{p} = \mathbf{q}$, find the value of a and of b.

4. $ABCD$ is a parallelogram. It is given that

$\overrightarrow{AB} = \begin{pmatrix}7\\0\end{pmatrix}$ and $\overrightarrow{BC} = \begin{pmatrix}-3\\4\end{pmatrix}$.

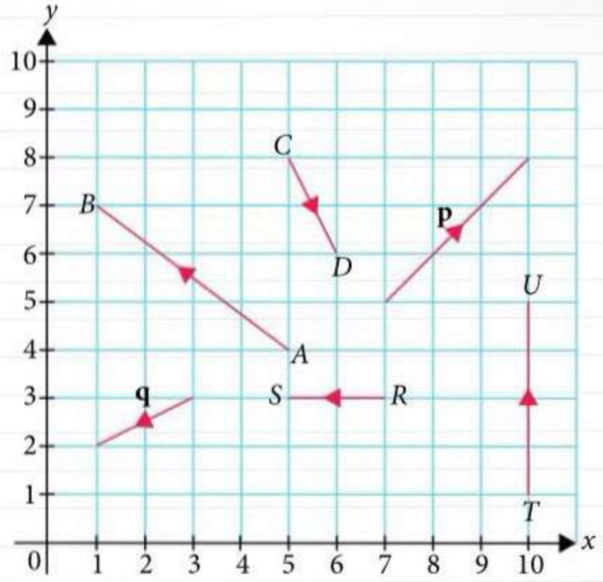


(i) Find the value of $|\overrightarrow{AB}|$.

(ii) Express each of the following as a column vector.

(a) $\overrightarrow{DC}$ **(b)** $\overrightarrow{DA}$

5. Express each of the vectors in the diagram as a column vector and find its magnitude.



6. On a square grid or a sheet of graph paper, draw the following column vectors. You need to draw the x-axis and y-axis, and indicate the scale.

(a) $\begin{pmatrix}3\\2\end{pmatrix}$ **(b)** $\begin{pmatrix}-4.5\\8\end{pmatrix}$

(c) $\begin{pmatrix}-5\\7\end{pmatrix}$ **(d)** $\begin{pmatrix}0\\-2\frac{1}{2}\end{pmatrix}$

(e) Negative of $\begin{pmatrix}6\\-1\end{pmatrix}$

(f) Negative of $\begin{pmatrix}-3\\0\end{pmatrix}$

Exercise 4A

7. Two column vectors **a** and **b** are such that

$$\mathbf{a} = \begin{pmatrix} x-3 \\ 2-y \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} 5-x \\ y-9 \end{pmatrix}.$$

(i) If **a** = **b**,

  (a) find the value of x and of y,

  (b) write down the negative of **a** as a column vector,

  (c) show that $|\mathbf{a}| = |\mathbf{b}| = \sqrt{\frac{53}{4}}$ units.

(ii) If $|\mathbf{a}| = |\mathbf{b}|$,

  (a) express y in terms of x,

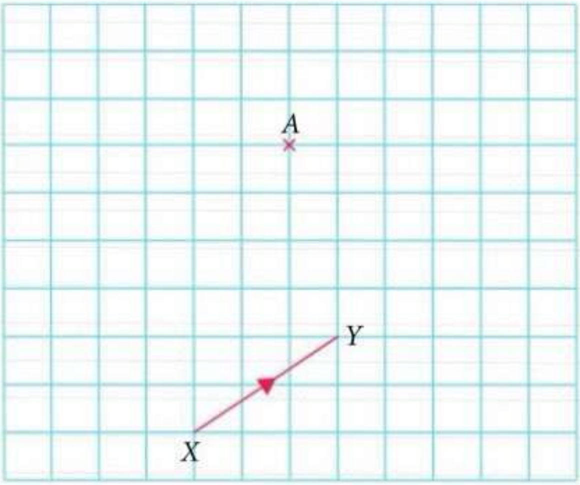
  (b) explain why **a** might not be equal to **b**.

8. If $\overrightarrow{AB} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$ and $\overrightarrow{CD} = \begin{pmatrix} 0 \\ 5 \end{pmatrix}$,

(i) show that $|\overrightarrow{AB}| = |\overrightarrow{CD}|$,

(ii) explain why $\overrightarrow{AB} \neq \overrightarrow{CD}$ even though $|\overrightarrow{AB}| = |\overrightarrow{CD}|$.

9. The figure below shows the positions of the points A, X and Y, where $\overrightarrow{XY} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$.



(i) Express $\overrightarrow{AY}$ as a column vector.

(ii) Plot the point B such that $XYBA$ is a parallelogram. Express $\overrightarrow{YB}$ as a column vector.

(iii) Plot the point C such that $XYAC$ is a parallelogram. Express $\overrightarrow{AC}$ as a column vector.

(iv) Do the two vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ have the same magnitude? Is $\overrightarrow{AB} = \overrightarrow{AC}$? Explain.

10. If $\mathbf{a} = \begin{pmatrix} n \\ -3 \end{pmatrix}$, find the possible values of n such that $|\mathbf{a}| = 7$ units, leaving your answer in *surd form* (i.e. square root form in this case) if necessary.

11. On a square grid or a sheet of graph paper, draw the following column vectors. You need to draw the x-axis and y-axis, and indicate the scale.

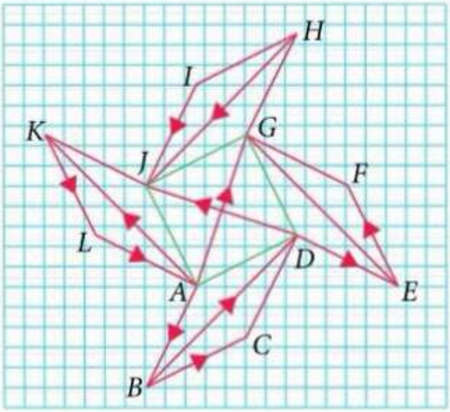
(a) Two times of $\begin{pmatrix} 5 \\ -3 \end{pmatrix}$

(b) Three times of the negative of $\begin{pmatrix} -4 \\ 3 \end{pmatrix}$

12. If $\mathbf{u} = \begin{pmatrix} 13s \\ 4t \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 6t+20 \\ 18-7s \end{pmatrix}$ and $\mathbf{u} = \mathbf{v}$, find the value of s and of t.

Exercise 4A

13. The figure below consists of a square *ADGJ* and four identical rhombuses *AJKL*, *GHIJ*, *DEFG* and *ABCD*.



(i) **(a)** Explain why $\overrightarrow{AB} = \overrightarrow{IJ}$.

(b) Name two other vectors that are equal to $\overrightarrow{AB}$.

(ii) Name all the vectors that are equal to

(a) $\overrightarrow{KL}$, **(b)** $\overrightarrow{DE}$,

(c) $\overrightarrow{BC}$, **(d)** $\overrightarrow{AK}$.

(iii) Give a reason why $\overrightarrow{AG} \neq \overrightarrow{DJ}$.

(iv) The line segments *BD* and *HJ* have the same length and are parallel. Explain why $\overrightarrow{BD} \neq \overrightarrow{HJ}$.

OPEN **(v)** Name a vector that has the same magnitude but is in the opposite direction of

(a) $\overrightarrow{BC}$, **(b)** $\overrightarrow{EF}$,

(c) $\overrightarrow{LA}$.

4.2 Addition of vectors

In this section, we will learn how to add two or more vectors.

A. Addition of vectors

A boat left point *P* for point *Q*, which is 2.1 km away, on a bearing of 298° (see Fig. 4.5). Then it sailed 1.9 km away from *Q* on a bearing of 081°. Another boat left point *P* and travelled 1.3 km north. Did they arrive at the same destination?

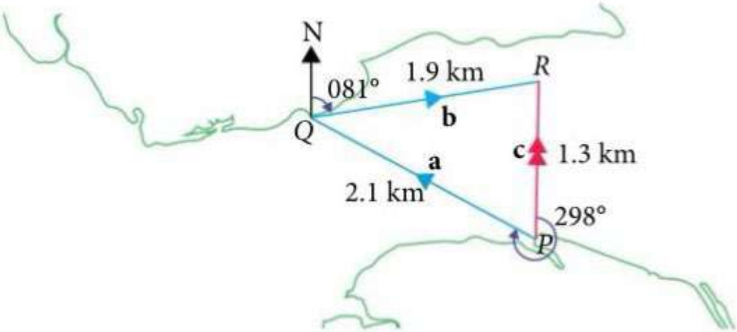
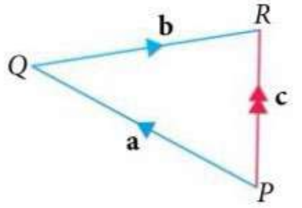


Fig. 4.5

Yes, they arrived at point R.

This is the concept behind the **addition of vectors**. We can think of vector $\overrightarrow{PQ}$ as a *translation* (i.e. movement) from P to Q. Moving from P to Q and subsequently from Q to R is the same as moving from P to R.



We define the addition of two vectors $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ as

$$\overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}$$

start — end — start — end

must be the same point for vector addition

$$\text{or} \qquad \mathbf{a} + \mathbf{b} = \mathbf{c}.$$

$\overrightarrow{PR}$ is called the vector sum or **resultant vector** of $\overrightarrow{PQ}$ and $\overrightarrow{QR}$, and we represent it using a directed line segment with a *double arrow* in the vector diagram.

Worked Example 4

Adding two vectors

The diagram shows two vectors **a** and **b**.

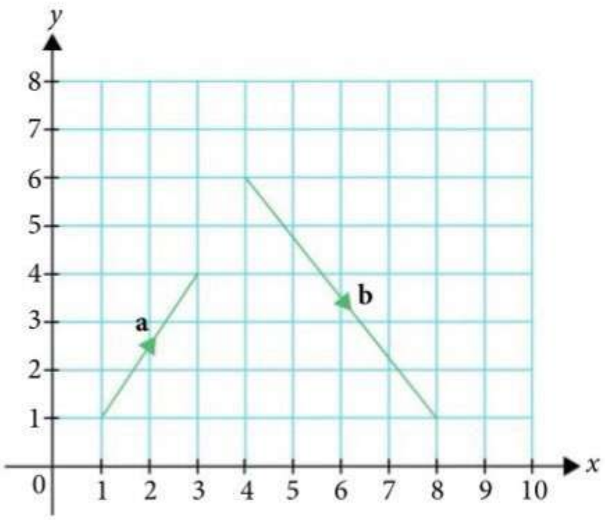
(i) Draw the sum of the two vectors **a** and **b**, i.e. **a** + **b**.

(ii) By looking at the diagram that you have drawn, express each of the vectors **a**, **b** and **a** + **b** as a column vector.

(iii) Find **a** + **b** from **a** and **b** using column vectors directly.

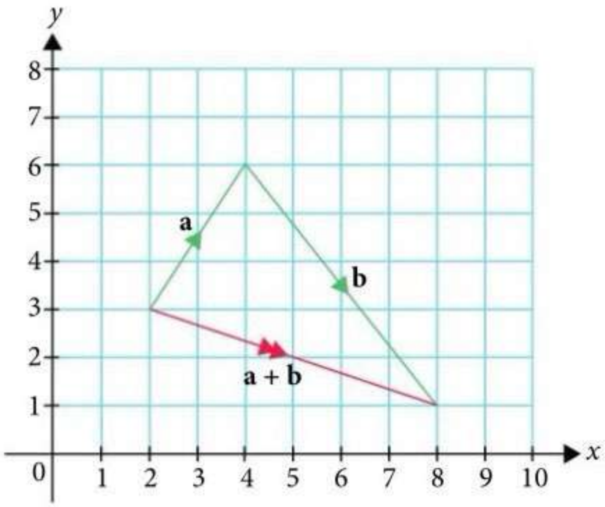
(iv) Find the value of $|\mathbf{a}|$, of $|\mathbf{b}|$ and of $|\mathbf{a} + \mathbf{b}|$, leaving your answers in surd form (i.e. square root form in this case).

(v) Is $|\mathbf{a} + \mathbf{b}| = |\mathbf{a}| + |\mathbf{b}|$? Explain why or why not, using the result in part **(iv)**.



Solution

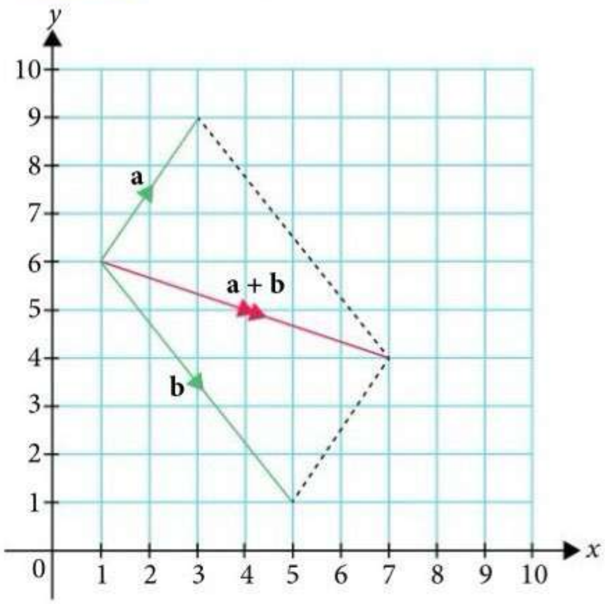
(i) Method 1: Triangle Law of Vector Addition



Problem-solving Tip

Copy the vector **a** on a square grid using its column vector $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ as a guide. From the *ending point of* **a**, start drawing the vector $\mathbf{b} = \begin{pmatrix} 4 \\ -5 \end{pmatrix}$. Then draw a directed line segment from the starting point of **a** to the ending point of **b**. This is the resultant vector of **a** and **b**, i.e. **a** + **b**.
Can you draw **b** first, followed by **a**? Is **b** + **a** = **a** + **b**?

Method 2: Parallelogram Law of Vector Addition



Problem-solving Tip

Draw **a** and **b** from the *same starting point*. Then complete the parallelogram. The resultant vector **a** + **b** is the diagonal of the parallelogram, which has the same starting point as **a** and as **b**.

Reflection

Which method do you prefer? Explain.

(ii) From the diagram, $\mathbf{a} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 4 \\ -5 \end{pmatrix}$ and $\mathbf{a} + \mathbf{b} = \begin{pmatrix} 6 \\ -2 \end{pmatrix}$.

(iii)
$$\begin{aligned}\mathbf{a} + \mathbf{b} &= \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 4 \\ -5 \end{pmatrix} \\ &= \begin{pmatrix} 2 + 4 \\ 3 + (-5) \end{pmatrix} \quad \text{add the respective } x\text{- and } y\text{-components} \\ &= \begin{pmatrix} 6 \\ -2 \end{pmatrix}\end{aligned}$$

(iv) $|\mathbf{a}| = \sqrt{2^2 + 3^2}$

$= \sqrt{13}$ units

$|\mathbf{b}| = \sqrt{4^2 + (-5)^2}$

$= \sqrt{41}$ units

$|\mathbf{a} + \mathbf{b}| = \sqrt{6^2 + (-2)^2}$

$= \sqrt{40}$ units

(v) $|\mathbf{a} + \mathbf{b}| = \sqrt{40}$

$= 6.32$ units (to 3 s.f.)

$|\mathbf{a}| + |\mathbf{b}| = \sqrt{13} + \sqrt{41}$

$= 10.0$ units (to 3 s.f.)

$\therefore |\mathbf{a} + \mathbf{b}| \neq |\mathbf{a}| + |\mathbf{b}|$

Attention

(iv) In this case, $|\mathbf{b}| = \sqrt{41}$ units is larger than $|\mathbf{a} + \mathbf{b}| = \sqrt{40}$ units.

(v) Alternatively, from **Method 1** in **(i)**, the 3 vectors **a**, **b** and **a** + **b** form the sides of a triangle. Since the sum of the lengths of any two sides of a triangle is larger than the length of the third side, $|\mathbf{a} + \mathbf{b}| \neq |\mathbf{a}| + |\mathbf{b}|$.

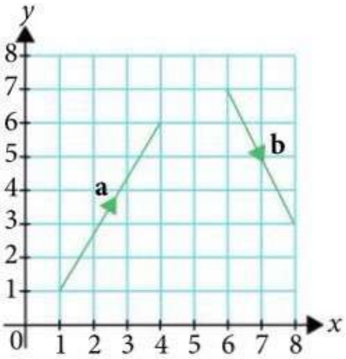
Practise Now 4

Similar and Further Questions

Exercise 4B

Questions 1(a)–(c), 2, 9(a)–(d), 15(a)–(d), 16

1. The diagram shows two vectors, **a** and **b**.



(i) Draw the sum of the two vectors **a** and **b**, using both the Triangle Law of Vector Addition and the Parallelogram Law of Vector Addition.

(ii) By looking at the diagram that you have drawn, express each of the vectors **a**, **b** and **a** + **b** as a column vector.

(iii) Find **a** + **b** from **a** and **b** using column vectors directly.

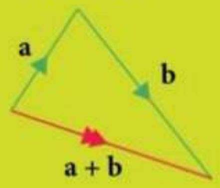
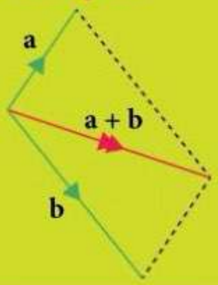
(iv) Find the value of $|\mathbf{a}|$, of $|\mathbf{b}|$ and of $|\mathbf{a} + \mathbf{b}|$, leaving your answers in surd form.

(v) Is $|\mathbf{a} + \mathbf{b}| = |\mathbf{a}| + |\mathbf{b}|$? Explain why or why not, using the diagram that you have drawn.

2. Simplify each of the following.

(a) $\begin{pmatrix} 6 \\ 9 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ **(b)** $\begin{pmatrix} 8 \\ -3 \end{pmatrix} + \begin{pmatrix} -10 \\ -5 \end{pmatrix} + \begin{pmatrix} 4 \\ 8 \end{pmatrix}$

From Worked Example 4 and Practise Now 4, we have learnt two methods to draw the resultant vector for the addition of two vectors:

Triangle Law of Vector Addition	**Parallelogram Law of Vector Addition**
Ending point of first vector **a** = starting point of second vector **b**	Both vectors **a** and **b**, and the resultant vector **a** + **b**, all start from the *same point*.
	

Attention

In practice, it is usually easier to use the Triangle Law of Vector Addition. However, if a parallelogram has already been drawn in a question (e.g. Worked Example 9**(iii)** and Worked Example 16**(i)(a)**), it will be easier to use the Parallelogram Law of Vector Addition.

We have also learnt the following:

- For column vectors, $\begin{pmatrix} p \\ q \end{pmatrix} + \begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} p+r \\ q+s \end{pmatrix}$.
- For two *non-zero* and *non-parallel* vectors, **a** and **b**, $|\mathbf{a}+\mathbf{b}| \neq |\mathbf{a}| + |\mathbf{b}|$.

Reflection

When do you think $|\mathbf{a}+\mathbf{b}|$ will be equal to $|\mathbf{a}| + |\mathbf{b}|$? Explain.

The principle of adding two vectors can be extended to any number of vectors. In Fig. 4.6,

$$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} = \overrightarrow{AD} \qquad \text{or} \qquad \overrightarrow{AD} = \begin{pmatrix} 0 \\ 3 \end{pmatrix} + \begin{pmatrix} 5 \\ 1 \end{pmatrix} + \begin{pmatrix} -2 \\ -5 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix},$$

i.e. $\overrightarrow{AD}$ is the result of the addition of all three vectors.

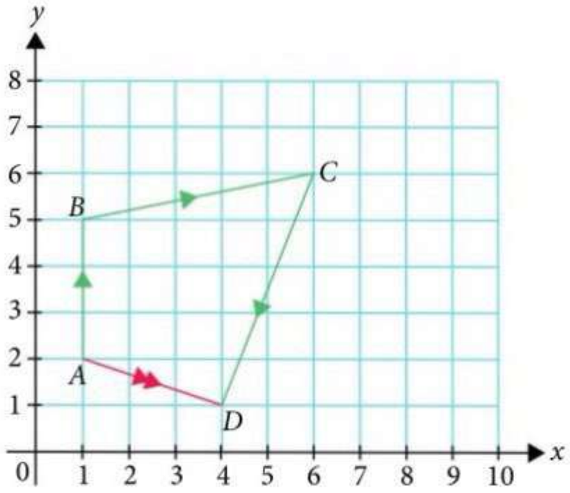


Fig. 4.6

Worked Example 5

Adding two or more vectors

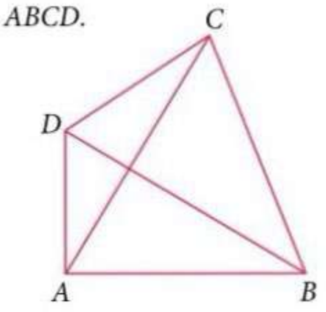
The diagram shows a quadrilateral *ABCD*.

Simplify

(i) $\overrightarrow{AB}+\overrightarrow{BC}$,

(ii) $\overrightarrow{DB}+\overrightarrow{AD}$,

(iii) $\overrightarrow{AC}+\overrightarrow{CB}+\overrightarrow{BD}$.



***Solution**

(i) $\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$ Triangle Law of Vector Addition

(ii) $\overrightarrow{DB}+\overrightarrow{AD}$

$=\overrightarrow{AD}+\overrightarrow{DB}$ vector addition is commutative

$=\overrightarrow{AB}$ Triangle Law of Vector Addition

(iii) $\overrightarrow{AC}+\overrightarrow{CB}+\overrightarrow{BD}$

$=\overrightarrow{AB}+\overrightarrow{BD}$ Triangle Law of Vector Addition

$=\overrightarrow{AD}$ Triangle Law of Vector Addition

Problem-solving Tip

We can simplify the vector additions without looking at the quadrilateral by matching the vertices:

(i) $\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$

start end start end

must be the same

(iii) $\overrightarrow{AC}+\overrightarrow{CB}+\overrightarrow{BD}=\overrightarrow{AD}$

start end start end

must be the same must be the same

Big Idea

Invariance

Adding two or more vectors in any order will give the same resultant vector, e.g.
$\overrightarrow{AC}+\overrightarrow{CB}+\overrightarrow{BD}=\overrightarrow{AC}+\overrightarrow{BD}+\overrightarrow{CB}$.
We say that the resultant vector is **invariant** regardless of the order in which we add the vectors.

Practise Now 5

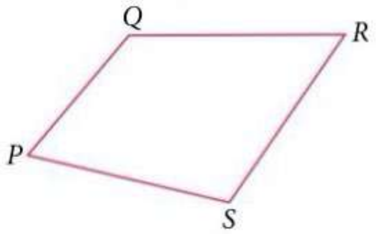
Similar and Further Questions
Exercise 4B
Questions 3, 10

The diagram shows a quadrilateral *PQRS*. Simplify

(i) $\overrightarrow{PQ}+\overrightarrow{QR}$,

(ii) $\overrightarrow{SR}+\overrightarrow{PS}$,

(iii) $\overrightarrow{PR}+\overrightarrow{RS}+\overrightarrow{SQ}$.



B. Zero vectors

Class Discussion **The zero vector**

In Fig. 4.5 at the start of Section 4.2 on page 123, one of the boats travelled from point *P* to point *R*, and its journey is represented by the vector $\overrightarrow{PR}$.

Suppose the boat travelled back from point *R* to point *P*. Then its journey could be represented by the vector $\overrightarrow{RP}$.

1. What do you think is the meaning of $\overrightarrow{PR}+\overrightarrow{RP}$?
2. How should you simplify $\overrightarrow{PR}+\overrightarrow{RP}$?

For the above Class Discussion, because the boat travelled from point *P* to point *R* and then back to point *P*, the result of the whole journey is a zero displacement of the boat from point *P*.

In other words, $\overrightarrow{PR}+\overrightarrow{RP}=\mathbf{0}$.

$\mathbf{0}$ is called the **zero vector**. It has a magnitude of 0, but it has *no direction*.

Attention

The zero vector $\mathbf{0}$ is *not a point*. It is still a vector but it has no direction.
The zero vector $\mathbf{0}$ is necessary to make vector addition 'closed', so that the addition of two or more vectors will always be a vector.

Nevertheless, **0** is still a vector, unlike the scalar 0.
To distinguish between the two, notice that the zero vector **0** is in bold, unlike the scalar 0. Since it is tedious to write in bold by hand, we write the zero vector as $\underset{\sim}{0}$.

The column vector form of the zero vector **0** is $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Worked Example 6

Solving problem involving zero vector

(a) Simplify $\begin{pmatrix} 3 \\ -4 \end{pmatrix} + \begin{pmatrix} -3 \\ 4 \end{pmatrix}$.

(b) Copy and complete the following vector equation.

$$\begin{pmatrix} 5 \\ \square \end{pmatrix} + \begin{pmatrix} \square \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

***Solution**

(a) $\begin{pmatrix} 3 \\ -4 \end{pmatrix} + \begin{pmatrix} -3 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

(b) $\begin{pmatrix} 5 \\ 2 \end{pmatrix} + \begin{pmatrix} -5 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

In Worked Example 6(**a**), $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$ and $\begin{pmatrix} -3 \\ 4 \end{pmatrix}$ are the negatives of each other, and $\begin{pmatrix} 3 \\ -4 \end{pmatrix} + \begin{pmatrix} -3 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

In Worked Example 6(**b**), $\begin{pmatrix} 5 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} -5 \\ -2 \end{pmatrix}$ are the negatives of each other, and $\begin{pmatrix} 5 \\ 2 \end{pmatrix} + \begin{pmatrix} -5 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

In general,

The addition of a vector **a** and its negative **–a** will give the zero vector **0**, i.e. **a** + (**–a**) = **0** = (**–a**) + **a**.

Practise Now 6

Similar and Further Questions
Exercise 4B
Questions 4(a)–(c), 5(a)–(c), 6(a)–(c)

(a) Simplify $\begin{pmatrix} 8 \\ -1 \end{pmatrix} + \begin{pmatrix} -8 \\ 1 \end{pmatrix}$.

(b) Copy and complete the following vector equation.

$$\begin{pmatrix} -6 \\ \square \end{pmatrix} + \begin{pmatrix} \square \\ 7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Reflection

1. How do I draw a diagram in two ways to add two vectors if they are drawn using directed line segments?
2. How do I add two vectors in vertex form, e.g. $\overrightarrow{AB} + \overrightarrow{BC}$?
3. How can I add two vectors in column vector form?
4. How is **0** different from 0 or a point?
5. What have I learnt in this section that I am still unclear of?

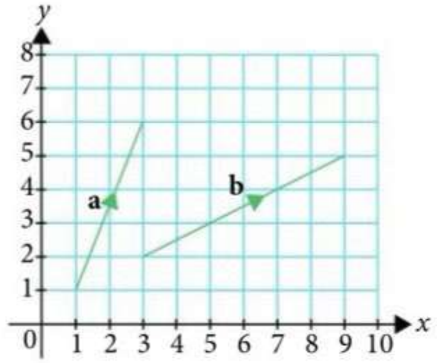
4.3 Vector subtraction

In this section, we will learn how to subtract one vector from another, e.g. **a** – **b** and **b** – **a**.

Worked Example 7

Subtracting one vector from another

The diagram shows two vectors **a** and **b**.



(i) Draw the vector **a** – **b**.

(ii) By looking at the diagram that you have drawn, express each of the vectors **a**, **b** and **a** – **b** as a column vector.

(iii) Find **a** – **b** from **a** and **b** using column vectors directly.

(iv) Find the value of |**a**|, of |**b**| and of |**a** – **b**|, leaving your answers in surd form.

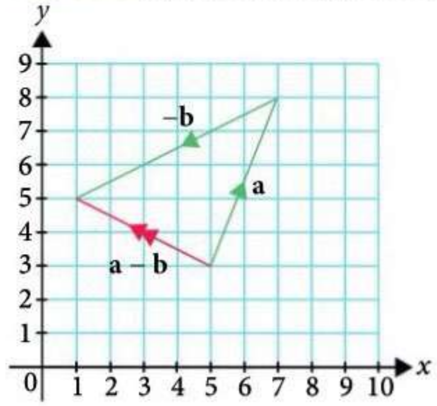
(v) Is |**a** – **b**| = |**a**| – |**b**|?

Attention

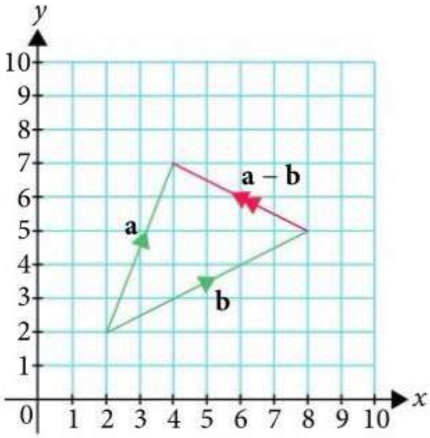
It is not clear if we only say 'the difference of two vectors **a** and **b**'. We need to specify whether we mean **a** – **b** or **b** – **a**.

Solution

(i) Method 1: **Addition of negative vector**



Method 2: Triangle Law of Vector Subtraction



(ii) From the diagram, $\mathbf{a} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 6 \\ 3 \end{pmatrix}$ and

$\mathbf{a} - \mathbf{b} = \begin{pmatrix} -4 \\ 2 \end{pmatrix}$.

(iii)
$$\begin{aligned} \mathbf{a} - \mathbf{b} &= \begin{pmatrix} 2 \\ 5 \end{pmatrix} - \begin{pmatrix} 6 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} 2-6 \\ 5-3 \end{pmatrix} \quad \text{subtract the respective } x\text{- and } y\text{-components} \\ &= \begin{pmatrix} -4 \\ 2 \end{pmatrix} \end{aligned}$$

(iv)
$$\begin{aligned} |\mathbf{a}| &= \sqrt{2^2 + 5^2} \\ &= \sqrt{29} \text{ units} \\ |\mathbf{b}| &= \sqrt{6^2 + 3^2} \\ &= \sqrt{45} \text{ units} \\ |\mathbf{a} - \mathbf{b}| &= \sqrt{(-4)^2 + 2^2} \\ &= \sqrt{20} \text{ units} \end{aligned}$$

(v)
$$\begin{aligned} |\mathbf{a} - \mathbf{b}| &= \sqrt{20} \\ &= 4.47 \text{ units (to 3 s.f.)} \\ |\mathbf{a}| - |\mathbf{b}| &= \sqrt{29} - \sqrt{45} \\ &= -1.32 \text{ units (to 3 s.f.)} \end{aligned}$$
$\therefore |\mathbf{a} - \mathbf{b}| \neq |\mathbf{a}| - |\mathbf{b}|$

Problem-solvin Tip

In Method 1, since we can view $\mathbf{a} - \mathbf{b}$ as $\mathbf{a} + (-\mathbf{b})$, we can use the Triangle Law of Vector Additio to add the two vectors **a** and $-\mathbf{b}$ **(negative vector of b)**. Copy the vector **a** on a square grid using its column vector $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$ as a guide. From the *ending point of* **a**, start drawing the vector $-\mathbf{b} = \begin{pmatrix} -6 \\ -3 \end{pmatrix}$. Then draw a directed line segment from the starting point of **a** to the ending point of $-\mathbf{b}$. This is the resultant vector of **a** and $-\mathbf{b}$, i.e. $\mathbf{a} + (-\mathbf{b}) = \mathbf{a} - \mathbf{b}$.
Can you draw $-\mathbf{b}$ first, followed by **a**? What if you draw **b** first, followed by $-\mathbf{a}$?

Problem-solving Tip

In Method 2, draw **a** and **b** from the *same starting point*. Then draw a directed line segment from the ending point of **b** to the ending point of **a**. This is the vector $\mathbf{a} - \mathbf{b}$ since $\mathbf{b} + (\mathbf{a} - \mathbf{b}) = \mathbf{a}$. What if you draw the red vector in the opposite direction?

Attention

The **resultant vector** is only used for **vector addition**, i.e. the resultant vector for **a** and **b** is always $\mathbf{a} + \mathbf{b}$. In Method 2, we can say that **a** is the resultant vector of **b** and $\mathbf{a} - \mathbf{b}$.

Practise Now 7

Similar and Further Questions
Exercise 4B
Questions 11(a)–(d)

The diagram shows two vectors **r** and **s**.

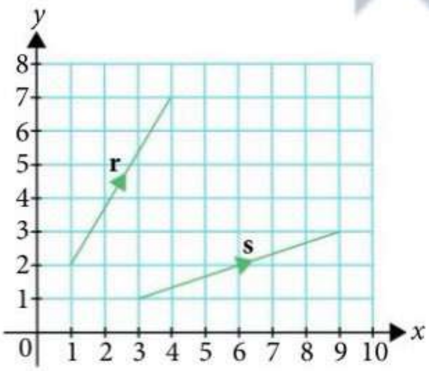
(i) Draw the vector **r** – **s**.

(ii) By looking at the diagram that you have drawn, express each of the vectors **r**, **s** and **r** – **s** as a column vector.

(iii) Find **r** – **s** from **r** and **s** using column vectors directly.

(iv) Find the value of |**r**|, of |**s**| and of |**r** – **s**|, leaving your answers in surd form.

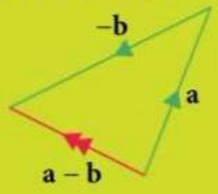
(v) Is |**r** – **s**| = |**r**| – |**s**|?



From Worked Example 7 and Practise Now 7, we have learnt two methods to draw the vector **a** – **b**:

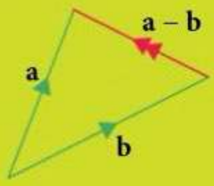
Addition of negative vector

Ending point of first vector **a**
= starting point of second vector **–b**
(negative vector of **b**)



Triangle Law of Vector Subtraction

Both vectors **a** and **b** start from the *same point*.



We have also learnt the following:

- For column vectors, $\begin{pmatrix} p \\ q \end{pmatrix} - \begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} p - r \\ q - s \end{pmatrix}$.
- For two *non-zero* and *non-parallel* vectors, **a** and **b**, |**a** – **b**| ≠ |**a**| – |**b**|.

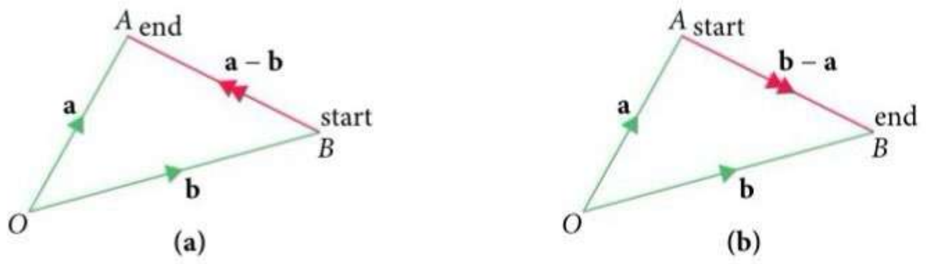
Thinking Time

When do you think |**a** – **b**| will be equal to |**a**| – |**b**|? Explain.

In many vector problems with a diagram (see Worked Example 9, Fig. 4.16 in Section 4.6 and Worked Ex pl 7 in Section 4.7), the diagram will look like the diagram for the Triangle Law of Vector Subtraction on page 132. So there is a need to learn how to apply the Triangle Law of Vector Subtraction to find **a** – **b** directly.

In the diagram for the Triangle Law of Vector Subtraction on page 132, if we had drawn the red vector in the opposite direction, then we would have drawn the vector **b** – **a** instead, because the negative vector of **a** – **b** is $-(\mathbf{a} - \mathbf{b}) = -\mathbf{a} + \mathbf{b} = \mathbf{b} - \mathbf{a}$.

The following shows a shortcut to determine the direction of **a** – **b** or **b** – **a** in the Triangle Law of Vector Subtraction: just remember **'end minus start'**.



(a)

The arrow for **a** – **b** starts from the ending point of **b** and ends at the ending point of **a**:

$$\underset{\text{start}\ \text{end}}{\overrightarrow{BA}} = \underset{\text{end}}{\mathbf{a}} - \underset{\text{start}}{\mathbf{b}}$$

(b)

The arrow for **b** – **a** starts from the ending point of **a** and ends at the ending point of **b**:

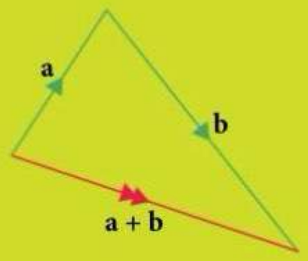
$$\underset{\text{start}\ \text{end}}{\overrightarrow{AB}} = \underset{\text{end}}{\mathbf{b}} - \underset{\text{start}}{\mathbf{a}}$$

Fig. 4.7

To summarise the difference between the Triangle Law of Vector Addition and the Triangle Law of Vector Subtraction:

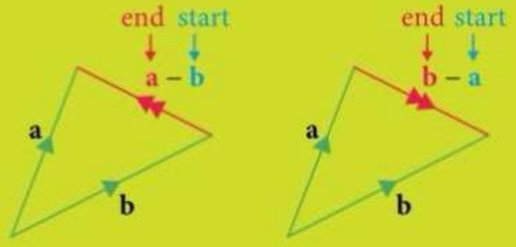
Triangle Law of Vector Addition

Ending point of first vector **a**
= starting point of second vector **b**



Triangle Law of Vector Subtraction

Both vectors **a** and **b** start from the *same point*.



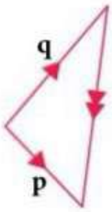
Worked Example 8

Adding and subtracting vectors

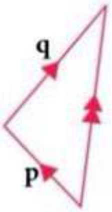
Find the vector represented by the double arrow in each of the diagrams below.

(i)

(ii)



(iii)



(iv)

Solution

(i) $\underset{\substack{\uparrow\\ \text{end}}}{\mathbf{q}} - \underset{\substack{\uparrow\\ \text{start}}}{\mathbf{p}}$ Triangle Law of Vector Subtraction

(ii) $\underset{\substack{\uparrow\\ \text{end}}}{\mathbf{p}} - \underset{\substack{\uparrow\\ \text{start}}}{\mathbf{q}}$ Triangle Law of Vector Subtraction

(iii) $\mathbf{p} + \mathbf{q}$ Triangle Law of Vector Addition

(iv) $-(\mathbf{p} + \mathbf{q}) = -\mathbf{p} - \mathbf{q}$ negative vector

Problem-solving Tip

(i) **p** and **q** start from the *same point*, so we use the **Triangle Law of Vector Subtraction.**

(iii) **p** and **q** do not start from the same point, but **q** starts from where **p** ends. So we use the **Triangle Law of Vector Addition.**

(iv) The vector represented by the double arrow is the *negative* of the resultant vector in **(iii)**, so the vector is $-(\mathbf{p} + \mathbf{q}) = -\mathbf{p} - \mathbf{q}$.
In fact, the arrows show a 'round trip' along the sides of the triangle starting from the initial point of **p**, i.e. $\mathbf{p} + \mathbf{q} + (-\mathbf{p} - \mathbf{q}) = \mathbf{0}$.

Reflection

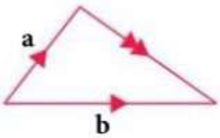
Do you see the usefulness of applying the Triangle Law of Vector Subtraction to find the answers to **(i)** and **(ii)** directly?

Practise Now 8

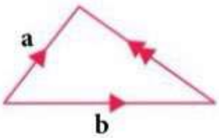
Similar and Further Questions
Exercise 4B
Question 7

Find the vector represented by the double arrow in each of the diagrams below.

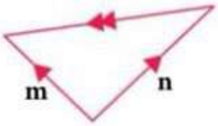
(i)



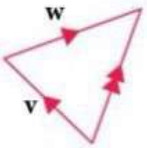
(ii)



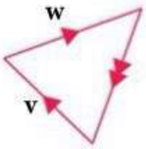
(iii)



(iv)



(v)

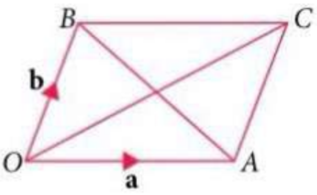


Worked Example 9

Solving problem involving vector addition and subtraction

The diagram shows a parallelogram $OACB$, where $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.
Express the following vectors in terms of **a** and/or **b**.

(i) $\overrightarrow{BC}$
(ii) $\overrightarrow{CA}$
(iii) $\overrightarrow{OC}$
(iv) $\overrightarrow{CO}$
(v) $\overrightarrow{AB}$
(vi) $\overrightarrow{BA}$



Solution

(i) $\overrightarrow{BC} = \overrightarrow{OA}$ equal vectors

$= \mathbf{a}$

(ii) $\overrightarrow{CA} = \overrightarrow{BO}$

$= -\overrightarrow{OB}$ negative vector

$= -\mathbf{b}$

(iii) Method 1:

$\overrightarrow{OC} = \mathbf{a} + \mathbf{b}$ Parallelogram Law of Vector Addition

Method 2:

$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC}$ Triangle Law of Vector Addition

$= \mathbf{a} + \mathbf{b}$

(iv) $\overrightarrow{CO} = -\overrightarrow{OC}$ negative vector

$= -(\mathbf{a} + \mathbf{b})$

$= -\mathbf{a} - \mathbf{b}$

(v) Method 1:

$$\overrightarrow{\underset{\substack{\uparrow\\\text{start}}}{A}\underset{\substack{\uparrow\\\text{end}}}{B}} = \underset{\substack{\uparrow\\\text{end}}}{\mathbf{b}} - \underset{\substack{\uparrow\\\text{start}}}{\mathbf{a}}$$

Triangle Law of Vector Subtraction

Method 2:

$\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$ Triangle Law of Vector Addition

$= -\overrightarrow{OA} + \overrightarrow{OB}$ negative vector

$= -\mathbf{a} + \mathbf{b}$

$= \mathbf{b} - \mathbf{a}$

Attention

Notice that $\overrightarrow{OC}$ and $\overrightarrow{AB}$ are vectors along the diagonals of the parallelogram. The vector along the diagonal *OC* represents vector addition $\overrightarrow{OC} = \mathbf{a} + \mathbf{b}$ or the negative vector $\overrightarrow{CO} = -\mathbf{a} - \mathbf{b}$, depending on which direction the arrow is pointing; while the vector along the other diagonal *AB* represents vector subtraction $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$ or $\overrightarrow{BA} = \mathbf{a} - \mathbf{b}$, depending on which direction the arrow is pointing.

(vi) Method 1:

$$\overrightarrow{\underset{\substack{\uparrow\\\text{start}}}{B}\underset{\substack{\uparrow\\\text{end}}}{A}} = \underset{\substack{\uparrow\\\text{end}}}{\mathbf{a}} - \underset{\substack{\uparrow\\\text{start}}}{\mathbf{b}}$$

Triangle Law of Vector Subtraction

Method 2:

$\overrightarrow{BA} = -\overrightarrow{AB}$ negative vector

$= -(\mathbf{b} - \mathbf{a})$

$= \mathbf{a} - \mathbf{b}$

Reflection

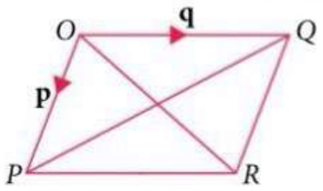
Do you see the usefulness of applying the Triangle Law of Vector Subtraction to find the answers to **(v)** and **(vi)** directly?

Practise Now 9

Similar and Further Questions
Exercise 4B
Questions 12, 17, 18(a)–(f)

The diagram shows a parallelogram *OPRQ*, where $\overrightarrow{OP} = \mathbf{p}$ and $\overrightarrow{OQ} = \mathbf{q}$.

Express the following vectors in terms of **p** and/or **q**.



(i) $\overrightarrow{PR}$ **(ii)** $\overrightarrow{RQ}$

(iii) $\overrightarrow{OR}$ **(iv)** $\overrightarrow{RO}$

(v) $\overrightarrow{PQ}$ **(vi)** $\overrightarrow{QP}$

For the addition of vectors, we have seen at the start of Section 4.2 on page 123 that:

$$\underset{\text{start}\quad\text{end}}{\overrightarrow{PQ}} + \underset{\text{start}\quad\text{end}}{\overrightarrow{QR}} = \underset{\text{start}\quad\text{end}}{\overrightarrow{PR}}$$

must be the same

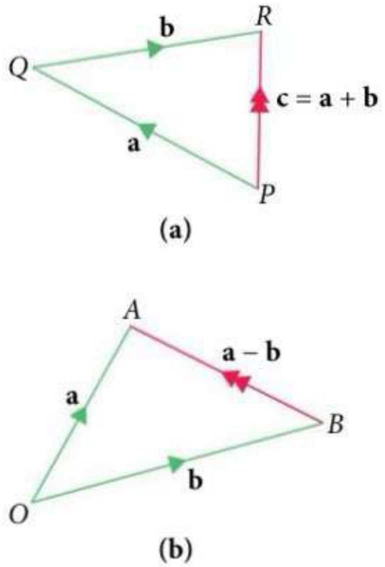


Fig. 4.8

The first vector starts at P and ends at Q, and the second vector starts at Q and ends at R.
So the resultant vector starts at P and ends at R, as shown in Fig. 4.8(**a**).

For the subtraction of vectors, in terms of the vertices of the triangle in Fig. 4.8(**b**):

must be the same

$$\underset{\text{start}\quad\text{end}}{\overrightarrow{BA}} = \underset{\text{end}}{\mathbf{a}} - \underset{\text{start}}{\mathbf{b}} = \underset{\text{end}}{\overrightarrow{OA}} - \underset{\text{start}}{\overrightarrow{OB}}$$

Notice it still has the same idea of **'end minus start'**.

Worked Example 10

Adding and subtracting vectors using starting and ending points without a diagram

Simplify the following if possible.

(a) $\overrightarrow{PR} + \overrightarrow{RQ}$ (b) $\overrightarrow{OQ} - \overrightarrow{OR}$ (c) $\overrightarrow{PQ} - \overrightarrow{QR}$

Solution

(a) $\underset{\text{start}\quad\text{end}}{\overrightarrow{PR}} + \underset{\text{start}\quad\text{end}}{\overrightarrow{RQ}} = \underset{\text{start}\quad\text{end}}{\overrightarrow{PQ}}$ Triangle Law of Vector Addition

check that these are the same

(b) **Method 1:**

$$\begin{aligned} \overrightarrow{OQ} - \overrightarrow{OR} &= \overrightarrow{OQ} + \overrightarrow{RO} && \text{negative vector} \\ &= \overrightarrow{RO} + \overrightarrow{OQ} \\ &= \overrightarrow{RQ} && \text{Triangle Law of Vector Addition} \end{aligned}$$

Method 2:

check that these are the same

$\underset{\text{end}}{\overrightarrow{OQ}} - \underset{\text{start}}{\overrightarrow{OR}} = \underset{\text{start}\quad\text{end}}{\overrightarrow{RQ}}$ Triangle Law of Vector Subtraction

(c) $\overrightarrow{PQ} - \overrightarrow{QR}$

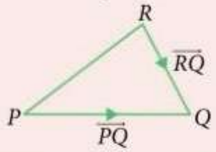
These are the same. But this is not vector addition, so we cannot simplify this further using P, Q and R.

Reflection

(**b**) Which method do you prefer? Explain.

Attention

(**c**) This is how it would look like in a diagram.



Practise Now 10

Similar and Further Questions
Exercise 4B
Questions 13(a)–(f)

Simplify the following if possible.

(a) $\overrightarrow{AB} + \overrightarrow{BC}$ **(b)** $\overrightarrow{AB} - \overrightarrow{AC}$

(c) $\overrightarrow{AB} - \overrightarrow{BC}$ **(d)** $\overrightarrow{PQ} - \overrightarrow{PR}$

(e) $\overrightarrow{PQ} - \overrightarrow{RQ}$ **(f)** $\overrightarrow{PQ} + \overrightarrow{RP} - \overrightarrow{RS}$

Worked Example 11

Adding and subtracting column vectors

(a) Simplify $\begin{pmatrix} 3 \\ 4 \end{pmatrix} - \begin{pmatrix} 5 \\ -2 \end{pmatrix}$.

(b) Find the values of x and y in each of the following equations.

(i) $\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 10 \\ -7 \end{pmatrix} = \begin{pmatrix} -6 \\ 8 \end{pmatrix}$

(ii) $\begin{pmatrix} 5 \\ 4x \end{pmatrix} - \begin{pmatrix} 3 \\ 8 \end{pmatrix} = \begin{pmatrix} x + 2y \\ -2 \end{pmatrix}$

*Solution

(a)
$$\begin{aligned} \begin{pmatrix} 3 \\ 4 \end{pmatrix} - \begin{pmatrix} 5 \\ -2 \end{pmatrix} &= \begin{pmatrix} 3 - 5 \\ 4 - (-2) \end{pmatrix} \\ &= \begin{pmatrix} 3 - 5 \\ 4 + 2 \end{pmatrix} \\ &= \begin{pmatrix} -2 \\ 6 \end{pmatrix} \end{aligned}$$

Problem-solving Tip

Recall that we have learnt earlier on page 132 that $\begin{pmatrix} p \\ q \end{pmatrix} - \begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} p - r \\ q - s \end{pmatrix}$.

(b) (i)
$$\begin{aligned} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 10 \\ -7 \end{pmatrix} &= \begin{pmatrix} -6 \\ 8 \end{pmatrix} \\ \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} -6 \\ 8 \end{pmatrix} - \begin{pmatrix} 10 \\ -7 \end{pmatrix} \\ &= \begin{pmatrix} -16 \\ 15 \end{pmatrix} \end{aligned}$$

$\therefore x = -16$ and $y = 15$

(ii)
$$\begin{aligned} \begin{pmatrix} 5 \\ 4x \end{pmatrix} - \begin{pmatrix} 3 \\ 8 \end{pmatrix} &= \begin{pmatrix} x + 2y \\ -2 \end{pmatrix} \\ \begin{pmatrix} 2 \\ 4x - 8 \end{pmatrix} &= \begin{pmatrix} x + 2y \\ -2 \end{pmatrix} \end{aligned}$$

$\therefore 2 = x + 2y$ — (1)

$4x - 8 = -2$ — (2)

From (2), $4x = 6$

$$x = 1\frac{1}{2}$$

Substitute $x = 1\frac{1}{2}$ into (1):

$$1\frac{1}{2} + 2y = 2$$

$$2y = \frac{1}{2}$$

$$y = \frac{1}{4}$$

$\therefore x = 1\frac{1}{2}$ and $y = \frac{1}{4}$

Practise Now 11

Similar and Further Questions
Exercise 4B
Questions 8(a)–(d), 14(a)–(d)

(a) Simplify each of the following.

(i) $\begin{pmatrix} 1 \\ 3 \end{pmatrix} - \begin{pmatrix} 8 \\ -6 \end{pmatrix}$ **(ii)** $\begin{pmatrix} -2 \\ -3 \end{pmatrix} - \begin{pmatrix} -5 \\ 4 \end{pmatrix} + \begin{pmatrix} -6 \\ 7 \end{pmatrix}$

(b) Find the values of x and y in each of the following equations.

(i) $\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -5 \\ 7 \end{pmatrix} = \begin{pmatrix} 8 \\ -3 \end{pmatrix}$ **(ii)** $\begin{pmatrix} 3 \\ y \end{pmatrix} - \begin{pmatrix} x \\ -9 \end{pmatrix} = \begin{pmatrix} 4 \\ x \end{pmatrix}$

Reflection

1. How do I draw a diagram in two ways to subtract one vector from another vector if they are drawn using directed line segments?
2. How do I subtract two vectors in vertex form, e.g. $\overrightarrow{AC} - \overrightarrow{AB}$?
3. How can I subtract one vector from another vector in column vector form?
4. What have I learnt in this section that I am still unclear of?

Basic Intermediate Advanced

Exercise 4B

1. Simplify each of the following.

(a) $\begin{pmatrix} 2 \\ 4 \end{pmatrix} + \begin{pmatrix} 7 \\ 5 \end{pmatrix}$ **(b)** $\begin{pmatrix} 3 \\ -5 \end{pmatrix} + \begin{pmatrix} -4 \\ 1 \end{pmatrix}$

(c) $\begin{pmatrix} -9 \\ -2 \end{pmatrix} + \begin{pmatrix} 0 \\ -8 \end{pmatrix} + \begin{pmatrix} -3 \\ 7 \end{pmatrix}$

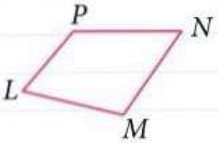
2. If $\mathbf{a} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} -7 \\ 2 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} 1 \\ -5 \end{pmatrix}$, determine whether

(i) $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$,

(ii) $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$.

Exercise 4B

3. The diagram shows a quadrilateral *PLMN*. Simplify



(i) $\overrightarrow{LM} + \overrightarrow{MN}$,

(ii) $\overrightarrow{PN} + \overrightarrow{LP}$,

(iii) $\overrightarrow{LN} + \overrightarrow{NM} + \overrightarrow{MP}$.

4. Simplify each of the following.

(a) $\begin{pmatrix}12\\-6\end{pmatrix}+\begin{pmatrix}-12\\6\end{pmatrix}$ **(b)** $\begin{pmatrix}5\\7\end{pmatrix}+\begin{pmatrix}-5\\-7\end{pmatrix}$

(c) $\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}-x\\-y\end{pmatrix}$

5. Copy and complete the following vector equations.

(a) $\begin{pmatrix}9\\\square\end{pmatrix}+\begin{pmatrix}\square\\-1\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$

(b) $\begin{pmatrix}-3\\\square\end{pmatrix}+\begin{pmatrix}\square\\7\end{pmatrix}=\mathbf{0}$

(c) $\begin{pmatrix}\square\\p\end{pmatrix}+\begin{pmatrix}-q\\\square\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$

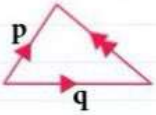
6. Simplify each of the following.

(a) $\overrightarrow{AB} + \overrightarrow{BA}$ **(b)** $\overrightarrow{PQ} + \overrightarrow{QR} + \overrightarrow{RP}$

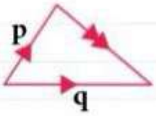
(c) $\overrightarrow{MN} + \overrightarrow{LM} + \overrightarrow{NL}$

7. Find the vector represented by the double arrow in each of the diagrams below.

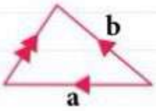
(i)



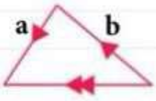
(ii)



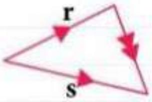
(iii)



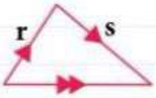
(iv)



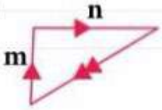
(v)



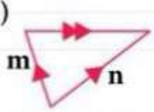
(vi)



(vii)



(viii)



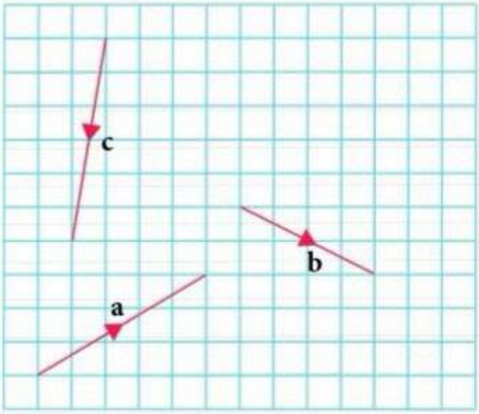
8. Simplify each of the following.

(a) $\begin{pmatrix}5\\4\end{pmatrix}-\begin{pmatrix}3\\2\end{pmatrix}$ **(b)** $\begin{pmatrix}-1\\3\end{pmatrix}-\begin{pmatrix}-3\\-4\end{pmatrix}$

(c) $\begin{pmatrix}2\\3\end{pmatrix}+\begin{pmatrix}5\\-2\end{pmatrix}-\begin{pmatrix}7\\-3\end{pmatrix}$

(d) $\begin{pmatrix}4\\7\end{pmatrix}-\begin{pmatrix}-2\\5\end{pmatrix}-\begin{pmatrix}3\\-6\end{pmatrix}$

9. The diagram shows three vectors **a**, **b** and **c**.

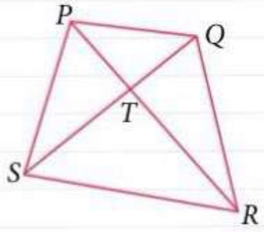


On a square grid or a sheet of graph paper, draw appropriate triangles to illustrate the following vector additions.

(a) **a** + **b** **(b)** **b** + **a**

(c) **a** + **c** **(d)** **b** + **c**

10. The diagram shows a quadrilateral *PQRS* where its diagonals intersect at *T*.



Simplify each of the following.

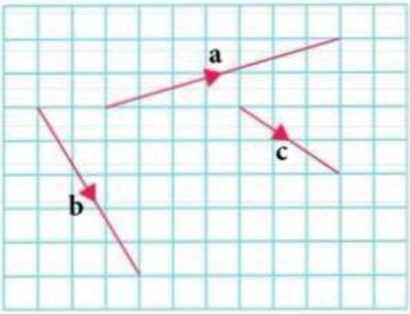
(i) $\overrightarrow{PT} + \overrightarrow{TR}$ **(ii)** $\overrightarrow{SQ} + \overrightarrow{QR}$

(iii) $\overrightarrow{TR} + \overrightarrow{ST}$ **(iv)** $\overrightarrow{SQ} + \overrightarrow{QT}$

(v) $\overrightarrow{SQ} + \overrightarrow{QR} + \overrightarrow{PS}$ **(vi)** $\overrightarrow{RQ} + \overrightarrow{QT} + \overrightarrow{TP} + \overrightarrow{PS}$

Exercise 4B

11. The diagram shows three vectors **a**, **b** and **c**.

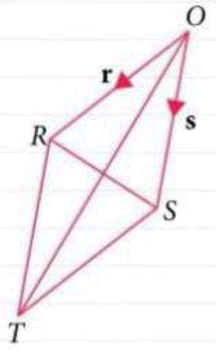


On a square grid or a sheet of graph paper, use the Triangle Law of Vector Subtraction to illustrate the following vector subtractions.

(a) **a** – **b** **(b)** **b** – **a**

(c) **a** – **c** **(d)** **c** – **b**

12. The diagram shows a parallelogram *ORTS* where $\overrightarrow{OR} = \mathbf{r}$ and $\overrightarrow{OS} = \mathbf{s}$.



Express the following vectors in terms of **r** and/or **s**.

(i) $\overrightarrow{RT}$ **(ii)** $\overrightarrow{TS}$

(iii) $\overrightarrow{OT}$ **(iv)** $\overrightarrow{RS}$

(v) $\overrightarrow{SR}$

13. Simplify the following if possible.

(a) $\overrightarrow{RS} + \overrightarrow{ST}$ **(b)** $\overrightarrow{RS} - \overrightarrow{RT}$

(c) $\overrightarrow{RT} - \overrightarrow{RS}$ **(d)** $\overrightarrow{RS} - \overrightarrow{ST}$

(e) $\overrightarrow{RS} - \overrightarrow{TS}$ **(f)** $\overrightarrow{RS} + \overrightarrow{TR} - \overrightarrow{TU}$

14. Find the value of *x* and of *y* in each of the following equations.

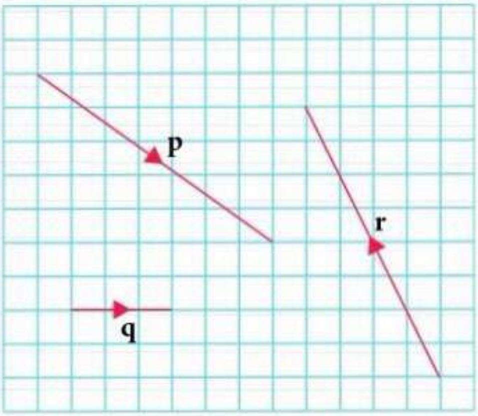
(a) $\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -3 \\ 2 \end{pmatrix} = \begin{pmatrix} 7 \\ -5 \end{pmatrix}$

(b) $\begin{pmatrix} 3 \\ y \end{pmatrix} - \begin{pmatrix} x \\ -8 \end{pmatrix} = \begin{pmatrix} -6 \\ 9 \end{pmatrix}$

(c) $\begin{pmatrix} y \\ 3 \end{pmatrix} + \begin{pmatrix} -4 \\ 2x \end{pmatrix} = \begin{pmatrix} 6 \\ x \end{pmatrix}$

(d) $\begin{pmatrix} 2x \\ 5 \end{pmatrix} - \begin{pmatrix} y-3 \\ -10 \end{pmatrix} = \begin{pmatrix} 4 \\ 3y \end{pmatrix}$

15. The diagram shows three vectors **p**, **q** and **r**.



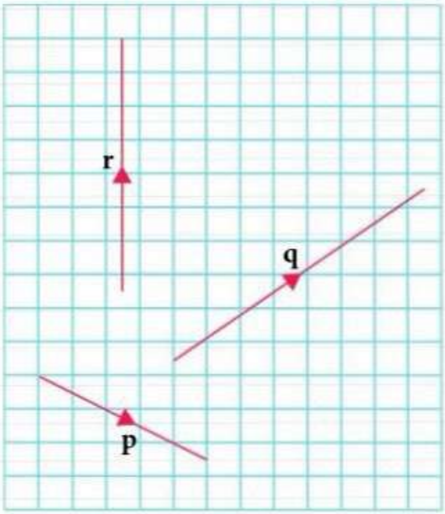
On a square grid or a sheet of graph paper, draw appropriate parallelograms to illustrate the following vector additions.

(a) **p** + **q** **(b)** **q** + **p**

(c) **p** + **r** **(d)** **q** + **r**

Exercise 4B

16. **(i)** Illustrate graphically the following vector sums using the vectors given in the diagram.



(a) $\mathbf{p} + \mathbf{q}$

(b) $\mathbf{q} + \mathbf{p}$

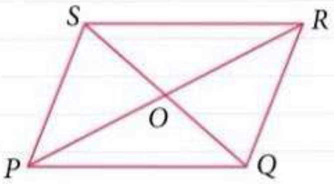
(c) $(\mathbf{p} + \mathbf{q}) + \mathbf{r}$

(d) $\mathbf{p} + (\mathbf{q} + \mathbf{r})$

(ii) Is $\mathbf{p} + \mathbf{q} = \mathbf{q} + \mathbf{p}$? Explain.

(iii) Is $(\mathbf{p} + \mathbf{q}) + \mathbf{r} = \mathbf{p} + (\mathbf{q} + \mathbf{r})$? Explain.

17. *PQRS* is a parallelogram. *O* is the point of intersection of its diagonals.



(i) Simplify

(a) $\overrightarrow{PQ} + \overrightarrow{PS}$,

(b) $\overrightarrow{RO} - \overrightarrow{QO}$,

(c) $\overrightarrow{PR} - \overrightarrow{SR} + \overrightarrow{SQ}$.

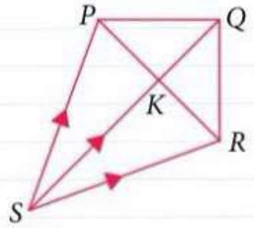
(ii) If $\overrightarrow{PQ} = \mathbf{a}$ and $\overrightarrow{PS} = \mathbf{b}$, find, in terms of $\mathbf{a}$ and/or $\mathbf{b}$,

(a) $\overrightarrow{SR}$,

(b) $\overrightarrow{PR}$,

(c) $\overrightarrow{SQ}$.

18. In the figure below, the diagonals of *PQRS* intersect at *K*. Find, for each of the following equations, a vector which can replace $\mathbf{u}$.



(a) $\overrightarrow{SK} + \mathbf{u} = \mathbf{0}$

(b) $\overrightarrow{SP} + \overrightarrow{PQ} + \mathbf{u} = \mathbf{0}$

(c) $\overrightarrow{PS} + \overrightarrow{SK} + \overrightarrow{KR} = \mathbf{u}$

(d) $\overrightarrow{PK} + \left(-\overrightarrow{SK}\right) = \mathbf{u}$

(e) $\overrightarrow{PS} + \left(-\overrightarrow{RS}\right) = \mathbf{u}$

(f) $\overrightarrow{PQ} + \overrightarrow{QR} + \left(-\overrightarrow{PR}\right) = \mathbf{u}$

4.4 Scalar multiples of a vector

In Section 4.1, we have learnt that two vectors are **parallel** if they have the same or opposite directions, but they can have the same or different magnitudes.

- If the two vectors have the same magnitude and the same direction, they are *equal vectors*.
- If the two vectors have the same magnitude but have opposite directions, they are the *negative vectors* of each other.

Fig. 4.9 shows three parallel vectors where the length of **a** is twice the length of **b**, and thrice the length of **c**.

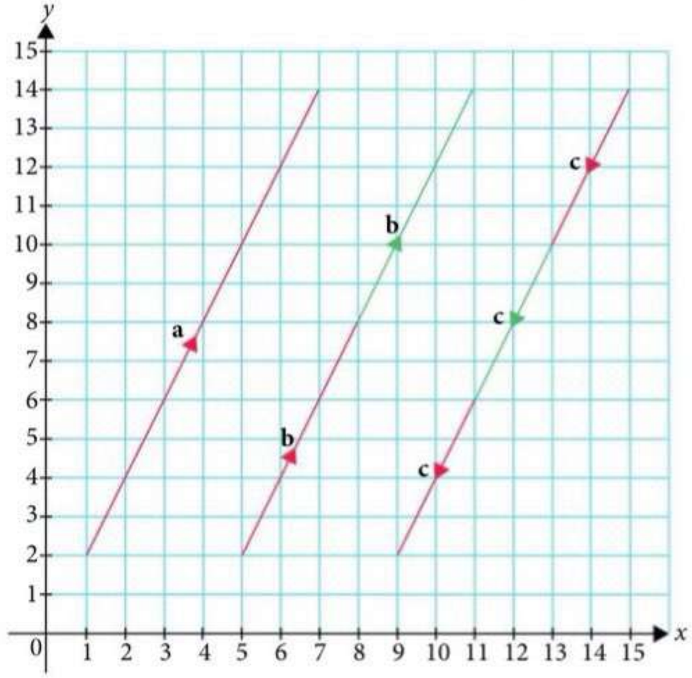


Fig. 4.9

We observe that **a** = **b** + **b** = 2**b** and **a** = (–**c**) + (–**c**) + (–**c**) = –3**c**.

2**b** and 3**c** are called the **scalar multiples** of **b** and **c** respectively. In general,

> If **a** and **b** are two *(non-zero)* **parallel** vectors,
> then **a** = *k***b** for some scalar or real number $k \neq 0$.

In other words, if **a** and **b** are vectors and **a** = *k***b** for some real number *k*, then there are 3 possibilities:

- **a** and **b** are parallel
- **a** = **b** = **0**
- $k = 0$

Thinking Time

If **a** = *k***b**, where **a** ≠ **0**, **b** ≠ **0** and $k \neq 0$, what does it mean if *k* is positive or if *k* is negative?

In Fig. 4.9, in terms of column vectors, $\mathbf{a} = \begin{pmatrix} 6 \\ 12 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$ and $\mathbf{c} = \begin{pmatrix} -2 \\ -4 \end{pmatrix}$.

We observe that $\mathbf{a} = \begin{pmatrix} 6 \\ 12 \end{pmatrix} = 2\begin{pmatrix} 3 \\ 6 \end{pmatrix} = 2\mathbf{b}$ and $\mathbf{a} = \begin{pmatrix} 6 \\ 12 \end{pmatrix} = -3\begin{pmatrix} -2 \\ -4 \end{pmatrix} = -3\mathbf{c}$.

Attention

$2\begin{pmatrix} 3 \\ 6 \end{pmatrix}$ means

$$2 \times \begin{pmatrix} 3 \\ 6 \end{pmatrix} = \begin{pmatrix} 2 \times 3 \\ 2 \times 6 \end{pmatrix} = \begin{pmatrix} 6 \\ 12 \end{pmatrix}.$$

Moreover, $|\mathbf{a}| = |2\mathbf{b}| = 2|\mathbf{b}|$ and $|\mathbf{a}| = |-3\mathbf{c}| = |-3||\mathbf{c}| = 3|\mathbf{c}|$.
In general,

If $\mathbf{a} = \begin{pmatrix} x \\ y \end{pmatrix}$, then $k\mathbf{a} = \begin{pmatrix} kx \\ ky \end{pmatrix}$ and $|k\mathbf{a}| = |k||\mathbf{a}|$
for any real number k.

Big Idea

Notations
In Book 1, we learnt that the absolute value of a number, e.g. −5, is 5. We write $|-5| = 5$. For a positive number, e.g. 5, $|5| = 5$. In other words, $|k|$ is the *absolute value of the scalar* or real number k.
For vectors, the same notation is used to represent a different idea: $|\mathbf{a}|$ is the *magnitude of the vector* **a**.

Worked Example 12

Determining which vectors are parallel

(a) State which of the following pairs of vectors are parallel.

(i) $\begin{pmatrix} 6 \\ 8 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ **(ii)** $\begin{pmatrix} 8 \\ -4 \end{pmatrix}, \begin{pmatrix} -6 \\ 3 \end{pmatrix}$ **(iii)** $\begin{pmatrix} 15 \\ -6 \end{pmatrix}, \begin{pmatrix} -5 \\ 3 \end{pmatrix}$

(b) Write down two vectors that are parallel to $\begin{pmatrix} -5 \\ 2 \end{pmatrix}$, one in the same direction, and the other in the opposite direction.

Solution

(a) (i) Since $\begin{pmatrix} 6 \\ 8 \end{pmatrix} = 2\begin{pmatrix} 3 \\ 4 \end{pmatrix}$, then $\begin{pmatrix} 6 \\ 8 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ are parallel.

(ii) Since $\begin{pmatrix} 8 \\ -4 \end{pmatrix} = -\frac{4}{3}\begin{pmatrix} -6 \\ 3 \end{pmatrix}$, then $\begin{pmatrix} 8 \\ -4 \end{pmatrix}$ and $\begin{pmatrix} -6 \\ 3 \end{pmatrix}$ are parallel.

(iii) If $\begin{pmatrix} 15 \\ -6 \end{pmatrix}$ and $\begin{pmatrix} -5 \\ 3 \end{pmatrix}$ are parallel, then there must be a value of k that satisfies $\begin{pmatrix} 15 \\ -6 \end{pmatrix} = k\begin{pmatrix} -5 \\ 3 \end{pmatrix}$.

$15 = k(-5)$, i.e. $k = 3$

$-6 = k(3)$, i.e. $k = -2$

But $3 \neq -2$.

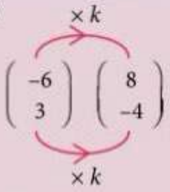
$\therefore \begin{pmatrix} 15 \\ -6 \end{pmatrix}$ and $\begin{pmatrix} -5 \\ 3 \end{pmatrix}$ are not parallel.

(b) A vector in the same direction as $\begin{pmatrix} -5 \\ 2 \end{pmatrix}$ is $2 \times \begin{pmatrix} -5 \\ 2 \end{pmatrix} = \begin{pmatrix} -10 \\ 4 \end{pmatrix}$.

A vector in the opposite direction of $\begin{pmatrix} -5 \\ 2 \end{pmatrix}$ is $-\begin{pmatrix} -5 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$.

Problem-solving Tip

(a) (ii)



Solve for the value of k for each of the equations $-6k = 8$ and $3k = -4$.
If both values of k are equal, the two vectors are **parallel**.
If both values of k are not equal, the two vectors are *not parallel*.

Practise Now 12

Similar and Further Questions

Exercise 4C

Questions 1(a)–(c), 2(a)–(c), 6(a)–(c), 7(a), (b), 13, 14

1. **(a)** State which of the following pairs of vectors are parallel.

(i) $\begin{pmatrix} 6 \\ -9 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \end{pmatrix}$

(ii) $\begin{pmatrix} 14 \\ 18 \end{pmatrix}, \begin{pmatrix} -7 \\ 9 \end{pmatrix}$

(iii) $\begin{pmatrix} -3 \\ 6 \end{pmatrix}, \begin{pmatrix} 4 \\ -8 \end{pmatrix}$

OPEN **(b)** Write down two vectors that are parallel to $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$, one in the same direction, and the other in the opposite direction.

2. Given that $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} 12 \\ p \end{pmatrix}$ are parallel vectors, find the value of p.

Worked Example 13

Solving problem involving addition, subtraction and scalar multiplication of column vectors

(a) If $\mathbf{a} = \begin{pmatrix} 7 \\ -5 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$, express $2\mathbf{a} + 3\mathbf{b}$ as a column vector.

(b) If $\mathbf{u} = \begin{pmatrix} x \\ 4 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} -5 \\ y \end{pmatrix}$ and $\mathbf{u} - 2\mathbf{v} = \begin{pmatrix} 7 \\ 8 \end{pmatrix}$, find the value of x and of y.

*Solution

(a)

$$\begin{aligned} 2\mathbf{a} + 3\mathbf{b} &= 2\begin{pmatrix} 7 \\ -5 \end{pmatrix} + 3\begin{pmatrix} -2 \\ 4 \end{pmatrix} \\ &= \begin{pmatrix} 14 \\ -10 \end{pmatrix} + \begin{pmatrix} -6 \\ 12 \end{pmatrix} \\ &= \begin{pmatrix} 8 \\ 2 \end{pmatrix} \end{aligned}$$

(b)

$$\begin{aligned} \mathbf{u} - 2\mathbf{v} &= \begin{pmatrix} 7 \\ 8 \end{pmatrix} \\ \begin{pmatrix} x \\ 4 \end{pmatrix} - 2\begin{pmatrix} -5 \\ y \end{pmatrix} &= \begin{pmatrix} 7 \\ 8 \end{pmatrix} \\ \begin{pmatrix} x + 10 \\ 4 - 2y \end{pmatrix} &= \begin{pmatrix} 7 \\ 8 \end{pmatrix} \end{aligned}$$

$\therefore x + 10 = 7$ and $4 - 2y = 8$

$x = -3$ $\quad 2y = -4$

$y = -2$

$\therefore x = -3$ and $y = -2$

Practise Now 13

Similar and Further Questions
Exercise 4C
Questions 3(a)–(c), 8(a)–(c), 15

1. If $\mathbf{u} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$, find a single column vector to represent the following.
   (i) $\mathbf{u} + 3\mathbf{v}$ **(ii)** $3\mathbf{u} - 2\mathbf{v} - \mathbf{w}$

2. If $\mathbf{a} = \begin{pmatrix} x \\ y \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$ and $2\mathbf{a} + \mathbf{b} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$, find the value of x and of y.

Class Discussion

Graphical representation of vectors

1. Given that $\mathbf{a} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$, illustrate each of the following on square grid or a sheet of graph paper.
   (i) $2\mathbf{a} + 3\mathbf{b}$ **(ii)** $2\mathbf{a} - 3\mathbf{b}$
2. Do you prefer to use the Triangle Law of Vector Addition or the Parallelogram Law of Vector Addition for Question 1(**i**)?
3. Do you prefer to use the addition of negative vector or the Triangle Law of Vector Subtraction for Question 1(**ii**)?

Similar and Further Questions
Exercise 4C
Question 9

Reflection

1. How do I draw a vector that is a scalar multiple of another vector using a directed line segment?
2. How do I multiply a vector in column vector form by a scalar?
3. If **a** and **b** are vectors and $\mathbf{a} = k\mathbf{b}$ for some real number $k \neq 0$, does that mean that **a** and **b** are parallel? Explain.
4. What have I learnt in this section that I am still unclear of?

4.5 Expression of a vector in terms of two other vectors

We have learnt that the sum or difference of two vectors is also a vector.
Can we do the reverse? That is, can we express a vector as the sum or difference of two other vectors?

Fig. 4.10 shows 2 non-zero and non-parallel vectors **u** and **v**, and the vector $\overrightarrow{AB}$.

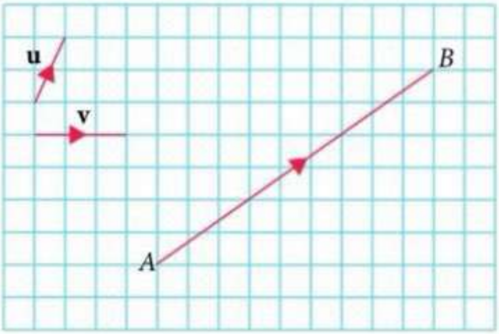


Fig. 4.10

To express $\overrightarrow{AB}$ in terms of **u** and **v**, we start from the point *A* and draw a line parallel to **u** (see Fig. 4.11). Then we draw a line from *B* parallel to **v** (this line must be in the opposite direction of **v** in order to intersect the first line). Name the point of intersection of the two lines *C*.

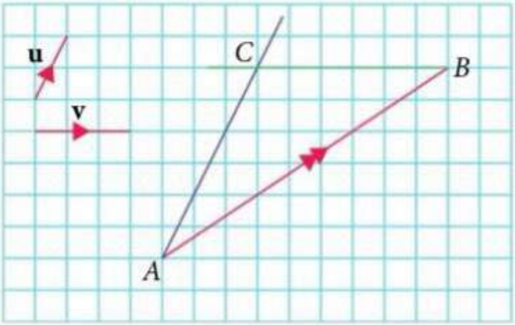


Fig. 4.11

From the diagram, $\overrightarrow{AC} = 3\mathbf{u}$ and $\overrightarrow{CB} = 2\mathbf{v}$ (see Fig. 4.12). Therefore, $\overrightarrow{AB} = 3\mathbf{u} + 2\mathbf{v}$.

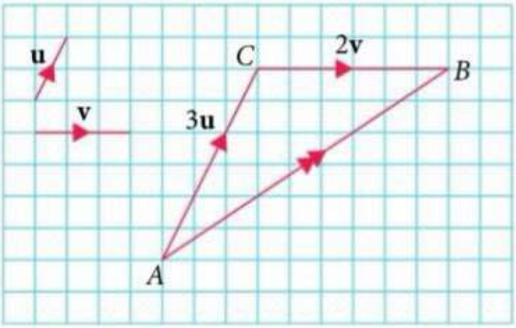


Fig. 4.12

Alternatively, we can start from the point *A* and draw a line parallel to **v** first (see Fig. 4.13). Then draw a line from *B* parallel to **u** that intersects the first line at *D*.

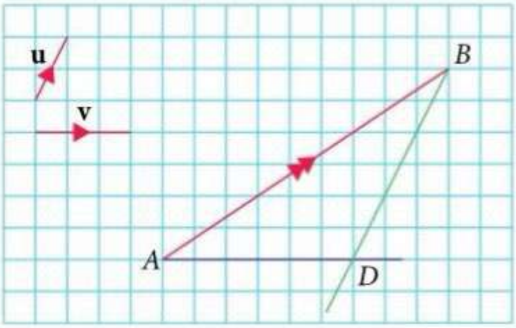


Fig. 4.13

From the diagram, $\overrightarrow{AD} = 2\mathbf{v}$ and $\overrightarrow{DB} = 3\mathbf{u}$. Therefore, $\overrightarrow{AB} = 2\mathbf{v} + 3\mathbf{u} = 3\mathbf{u} + 2\mathbf{v}$. Although there are two ways to draw the vector $\overrightarrow{AB}$ in terms of **u** and **v**, $\overrightarrow{AB} = 3\mathbf{u} + 2\mathbf{v}$ and $\overrightarrow{AB} = 2\mathbf{v} + 3\mathbf{u}$ are exactly the same because $3\mathbf{u} + 2\mathbf{v} = 2\mathbf{v} + 3\mathbf{u}$ (the order of the two vectors does not matter).

Attention

$3\mathbf{u} + 2\mathbf{v} = 2\mathbf{v} + 3\mathbf{u}$ as vector addition is commutative.

Class Discussion Expressing a vector in terms of two other vectors

Fig. 4.14 shows two non-zero and non-parallel vectors **u** and **v**, and the vector $\overrightarrow{PQ}$. Express $\overrightarrow{PQ}$ in terms of **u** and **v** in two ways.

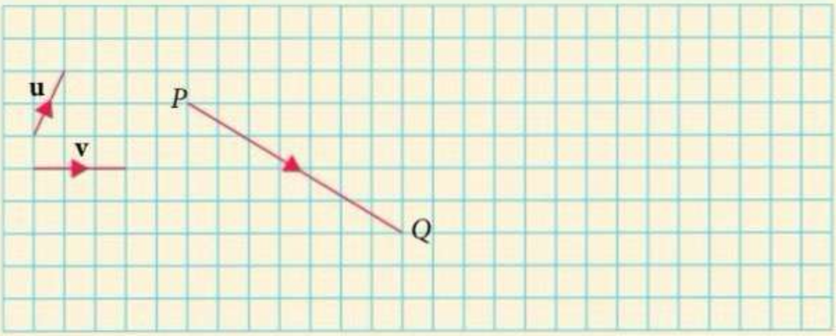


Fig. 4.14

Similar and Further Questions
Exercise 4C
Question 10

Three vectors are **coplanar vectors** if they lie in the same plane. Two vectors will always be coplanar vectors. Why? In general,

Any vector $\overrightarrow{AB}$ can be expressed **uniquely** in terms of two other *non-zero and non-parallel coplanar vectors* **u** and **v**, i.e. $\overrightarrow{AB} = m\mathbf{u} + n\mathbf{v}$, where m and n are real numbers.

Attention

'Coplanar vectors' here means **u** and **v** must lie in the same plane as $\overrightarrow{AB}$.

4.6 Position vectors

In Section 4.1, we have learnt that we can express a vector lying on a Cartesian plane as a **column vector**.

For example, in Fig. 4.15, $\mathbf{a} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$.

In fact, we can draw another vector $\mathbf{b} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$ in the Cartesian plane with a different starting point, as shown in Fig. 4.15.

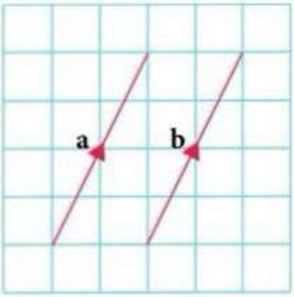


Fig. 4.15

We call vectors **a** and **b** **free vectors** because they do not have a fixed starting point. However, the **position vector** of a point P must have a *fixed starting point*. On a Cartesian plane, this starting point or reference point is usually the origin O.

Fig. 4.16 shows a point $P(2, 5)$. The position vector of P relative to O (or with respect to O) is $\overrightarrow{OP} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$.

What is the position vector of Q relative to O?

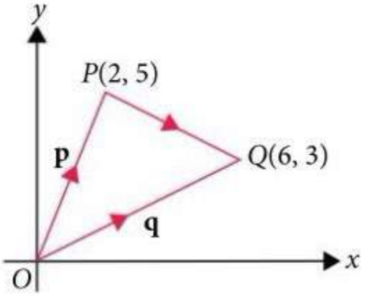


Fig. 4.16

How can we express $\overrightarrow{PQ}$ in terms of the position vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$?

Using the Triangle Law of Vector Subtraction:

$$\underset{\text{start}\ \text{end}}{\overrightarrow{PQ}} = \overbrace{\underset{\text{end}}{\overrightarrow{OQ}} - \underset{\text{start}}{\overrightarrow{OP}}}^{\text{must be the same}} = \underset{\text{end}}{\mathbf{q}} - \underset{\text{start}}{\mathbf{p}}$$

Since the coordinates of P and Q are $(2, 5)$ and $(6, 3)$ respectively, then

$$\begin{aligned}\overrightarrow{PQ} &= \overrightarrow{OQ} - \overrightarrow{OP} \\ &= \begin{pmatrix} 6 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ 5 \end{pmatrix} \\ &= \begin{pmatrix} 4 \\ -2 \end{pmatrix}.\end{aligned}$$

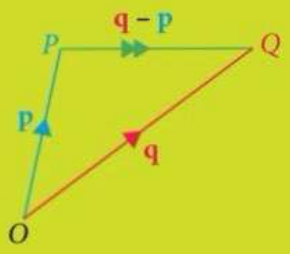
Attention

We can also use the Triangle Law of Vector Addition:

$$\begin{aligned}\overrightarrow{PQ} &= \overrightarrow{PO} + \overrightarrow{OQ} \\ &= -\overrightarrow{OP} + \overrightarrow{OQ} \\ &= \overrightarrow{OQ} - \overrightarrow{OP} \\ &= \mathbf{q} - \mathbf{p}\end{aligned}$$

The **position vector** of a point $P(x, y)$ is $\overrightarrow{OP} = \begin{pmatrix} x \\ y \end{pmatrix}$.

A vector $\overrightarrow{PQ}$ in the Cartesian plane can be expressed in terms of position vectors as follows:



$$\underset{\text{start}\ \text{end}}{\overrightarrow{PQ}} = \overbrace{\underset{\text{end}}{\overrightarrow{OQ}} - \underset{\text{start}}{\overrightarrow{OP}}}^{\text{must be the same}} = \underset{\text{end}}{\mathbf{q}} - \underset{\text{start}}{\mathbf{p}}$$

As mentioned at the start of Section 4.2 on page 123, the vector $\overrightarrow{PQ}$ can be viewed as the movement from P to Q and we call this a **translation** from P to Q. In other words, $\overrightarrow{PQ}$ can be viewed as a **translation vector** which describes the translation from P to Q. Translation vectors can also be expressed as column vectors, as shown in Worked Example 14(**b**).

Worked Example 14

Solving problem involving position vectors

(a) P is the point (2, −3) and Q is the point (4, 1). Write down the position vectors P a d Q Then express $\overrightarrow{PQ}$ as a column vector.

(b) A point $A(-5, 3)$ is translated by the **translation vector** $\overrightarrow{AB} = \begin{pmatrix} 9 \\ -6 \end{pmatrix}$ to the point B. Find the coordinates of B.

Solution

(a) Method 1:

The position vector of P is $\overrightarrow{OP} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$.

The position vector of Q is $\overrightarrow{OQ} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$.

$$\therefore \underset{\text{start}\uparrow\ \uparrow\text{end}}{\overrightarrow{PQ}} = \overbrace{\underset{\uparrow\text{end}}{\overrightarrow{OQ}} - \underset{\uparrow\text{start}}{\overrightarrow{OP}}}^{\text{must be the same}}$$

$$= \begin{pmatrix} 4 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

Method 2:

$$\begin{aligned} \overrightarrow{PQ} &= \overrightarrow{PO} + \overrightarrow{OQ} \\ &= -\overrightarrow{OP} + \overrightarrow{OQ} \\ &= -\begin{pmatrix} 2 \\ -3 \end{pmatrix} + \begin{pmatrix} 4 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ 4 \end{pmatrix} \end{aligned}$$

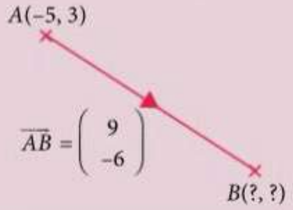
(b) Method 1:

$$\begin{aligned} \overrightarrow{AB} &= \begin{pmatrix} 9 \\ -6 \end{pmatrix} \\ \overrightarrow{OB} - \overrightarrow{OA} &= \begin{pmatrix} 9 \\ -6 \end{pmatrix} \quad \text{express } \overrightarrow{AB} \text{ in terms of position vectors} \\ \overrightarrow{OB} - \begin{pmatrix} -5 \\ 3 \end{pmatrix} &= \begin{pmatrix} 9 \\ -6 \end{pmatrix} \\ \overrightarrow{OB} &= \begin{pmatrix} 9 \\ -6 \end{pmatrix} + \begin{pmatrix} -5 \\ 3 \end{pmatrix} \\ &= \begin{pmatrix} 4 \\ -3 \end{pmatrix} \end{aligned}$$

$\therefore$ coordinates of B are (4, −3).

Problem-solving Tip

(b) Visualisation for Method 2



Method 2:

$$\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$$

point A is translated by vector $\overrightarrow{AB}$ to B

$$\overrightarrow{OB} = \begin{pmatrix} -5 \\ 3 \end{pmatrix} + \begin{pmatrix} 9 \\ -6 \end{pmatrix}$$

$$= \begin{pmatrix} 4 \\ -3 \end{pmatrix}$$

∴ coordinates of B are $(4, -3)$.

Reflection

Method 1 in both **(a)** and **(b)** uses position vectors directly.

Method 2 in both **(a)** and **(b)** uses Triangle Law of Vector Addition.

Which method do you prefer in **(a)** and **(b)**? Explain.

Practise Now 14

Similar and Further Questions

Exercise 4C

Questions 4(a)–(d), 5, 11, 12, 16, 17

(a) P is the point $(8, -2)$ and Q is the point $(-1, 7)$. Write down the position vectors of P and Q. Then express $\overrightarrow{PQ}$ as a column vector.

(b) A point $A(6, -7)$ is translated by the translation vector $\overrightarrow{AB} = \begin{pmatrix} -4 \\ 5 \end{pmatrix}$ to the point B. Find the coordinates of B.

Reflection

1. How do I explain the difference between free vectors and position vectors?
2. How do I express a point A in the Cartesian plane in terms of its position vector?
3. How do I express a vector $\overrightarrow{AB}$ in the Cartesian plane in terms of position vectors?
4. What have I learnt in this section that I am still unclear of?

Basic | Intermediate | Advanced

Exercise 4C

1. State which of the following pairs of vectors are parallel.

(a) $\begin{pmatrix} -2 \\ 1 \end{pmatrix}, \begin{pmatrix} -8 \\ 4 \end{pmatrix}$

(b) $\begin{pmatrix} 9 \\ 7 \end{pmatrix}, \begin{pmatrix} 18 \\ 21 \end{pmatrix}$

(c) $\begin{pmatrix} 6 \\ -8 \end{pmatrix}, \begin{pmatrix} -3 \\ 4 \end{pmatrix}$

2. Write down two vectors that are parallel to each of the following vectors, one in the same direction, and the other in the opposite direction.

(a) $\begin{pmatrix} 8 \\ -7 \end{pmatrix}$

(b) $\begin{pmatrix} 3 \\ 9 \end{pmatrix}$

(c) $\begin{pmatrix} -6 \\ -2 \end{pmatrix}$

Exercise 4C

3. If $\mathbf{p} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$, $\mathbf{q} = \begin{pmatrix} 6 \\ -3 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} -3 \\ -4 \end{pmatrix}$, find a single column vector to represent the following.

(a) $\mathbf{p} + 2\mathbf{q}$ **(b)** $3\mathbf{p} - \frac{1}{2}\mathbf{q}$

(c) $4\mathbf{p} - 3\mathbf{q} + \mathbf{r}$

4. Write down the position vectors of the following points.

(a) $A(4, 7)$ **(b)** $B(-2, 5)$

(c) $C(6, -1)$ **(d)** $D(-4, -9)$

5. If P, Q and R are the points $(3, -2)$, $(2, -4)$ and $(2, 3)$ respectively, express the following as column vectors.

(i) $\overrightarrow{PQ}$ **(ii)** $\overrightarrow{QR}$

(iii) $\overrightarrow{RP}$ **(iv)** $\overrightarrow{PR}$

6. State which of the following pairs of vectors are parallel.

(a) $\begin{pmatrix} 6 \\ -3 \end{pmatrix}, \begin{pmatrix} -4 \\ 2 \end{pmatrix}$ **(b)** $\begin{pmatrix} -5 \\ 15 \end{pmatrix}, \begin{pmatrix} -3 \\ 9 \end{pmatrix}$

(c) $\begin{pmatrix} 7 \\ -8 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \end{pmatrix}$

7. **(a)** Given that $\begin{pmatrix} 5 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 20 \\ p \end{pmatrix}$ are parallel vectors, find the value of p.

(b) Given that $\begin{pmatrix} h \\ 12 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ -9 \end{pmatrix}$ are parallel vectors, find the value of h.

8. For each of the following, find the value of x and of y.

(a) $\mathbf{a} = \begin{pmatrix} x \\ y \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ and $\mathbf{a} + 2\mathbf{b} = \begin{pmatrix} 8 \\ 9 \end{pmatrix}$.

(b) $\mathbf{u} = \begin{pmatrix} 2 \\ y \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} x \\ 2 \end{pmatrix}$ and $4\mathbf{u} + \mathbf{v} = 2\begin{pmatrix} \frac{1}{2} \\ 9 \end{pmatrix}$.

(c) $\mathbf{p} = \begin{pmatrix} x \\ 5 \end{pmatrix}$, $\mathbf{q} = \begin{pmatrix} 6 \\ y \end{pmatrix}$ and $5\mathbf{p} - 2\mathbf{q} = \begin{pmatrix} 3 \\ 23 \end{pmatrix}$.

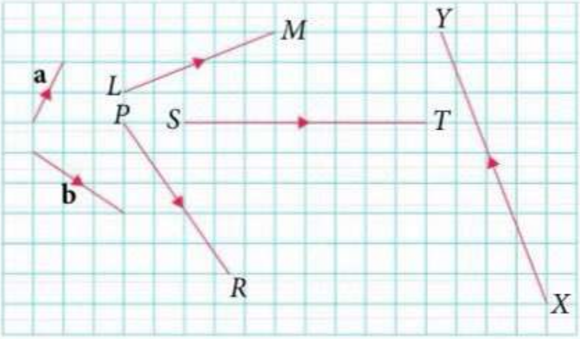
9. Given that $\mathbf{a} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, illustrate each of the following on a square grid or sheet of graph paper.

(i) $2\mathbf{a} + \mathbf{b}$ **(ii)** $3\mathbf{a} + 2\mathbf{b}$

(iii) $\mathbf{a} - 2\mathbf{b}$ **(iv)** $2\mathbf{a} - 3\mathbf{b}$

(v) $4\mathbf{a} + 3\mathbf{b}$ **(vi)** $-3\mathbf{a} + 4\mathbf{b}$

10. The diagram below shows two non-zero and non-parallel vectors **a** and **b**. Express $\overrightarrow{LM}$, $\overrightarrow{PR}$, $\overrightarrow{ST}$ and $\overrightarrow{XY}$ in terms of **a** and **b**.



11. A point $A(-3, 8)$ is translated by the translation vector $\overrightarrow{AB} = \begin{pmatrix} -2 \\ -4 \end{pmatrix}$ to the point B. Find the coordinates of B.

12. $\overrightarrow{AB} = \begin{pmatrix} 9 \\ -15 \end{pmatrix}$ and $\overrightarrow{CD} = \frac{2}{3}\overrightarrow{AB}$.

(i) Express $\overrightarrow{CD}$ as a column vector.

(ii) Given that A is the point $(-2, 7)$, find the coordinates of the point B.

(iii) Given that D is the point $(8, -5)$, find the coordinates of the point C.

Exercise 4C

13. If $\begin{pmatrix} a \\ b \end{pmatrix}$ and $\begin{pmatrix} c \\ d \end{pmatrix}$ are two parallel vectors, explain why $\frac{a}{c} = \frac{b}{d}$.

14. It is given that $\mathbf{u} = \begin{pmatrix} -15 \\ 8 \end{pmatrix}$. If $\mathbf{u} = k\mathbf{v}$, where k is a positive constant, and $|\mathbf{v}| = 51$ units, find the value of k. Hence find $\mathbf{v}$.

15. Given that $\overrightarrow{AB} = \begin{pmatrix} -3 \\ 5 \end{pmatrix}$, $\overrightarrow{CD} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$, $\overrightarrow{EF} = \begin{pmatrix} k \\ 7.5 \end{pmatrix}$ and $\overrightarrow{PQ} = \begin{pmatrix} \frac{1}{4} \\ 1 \end{pmatrix}$,

(i) express $2\overrightarrow{AB} + 5\overrightarrow{CD}$ as a column vector,
(ii) find the value of k if $\overrightarrow{EF}$ is parallel to $\overrightarrow{AB}$,
(iii) explain why $\overrightarrow{PQ}$ is parallel to $\overrightarrow{CD}$.

16. L is the point $(-3, 2)$ and M is the point $(t, 6)$.

(i) Express $\overrightarrow{LM}$ as a column vector.
(ii) If $\overrightarrow{LM}$ is parallel to $\mathbf{p} = \begin{pmatrix} 8 \\ 1 \end{pmatrix}$, find the value of t.
(iii) If instead, $|\overrightarrow{LM}| = |\mathbf{p}|$, find the two possible values of t.

17. P is the point $(2, -3)$ and $\overrightarrow{PQ} = \begin{pmatrix} 8 \\ -2 \end{pmatrix}$.

(i) Find the coordinates of Q.
(ii) Find the gradient of PQ.
(iii) If $\overrightarrow{PQ} = \begin{pmatrix} x \\ y \end{pmatrix}$, express the gradient of PQ in terms of x and y.
(iv) If the gradient of PQ is $\frac{y}{x}$, express $\overrightarrow{PQ}$ in terms of x and y.

4.7 Applications of vectors

A. Vectors in real-world contexts

In the Chapter Opener, we observe that both magnitude and direction are necessary to describe the position of a place from another place.

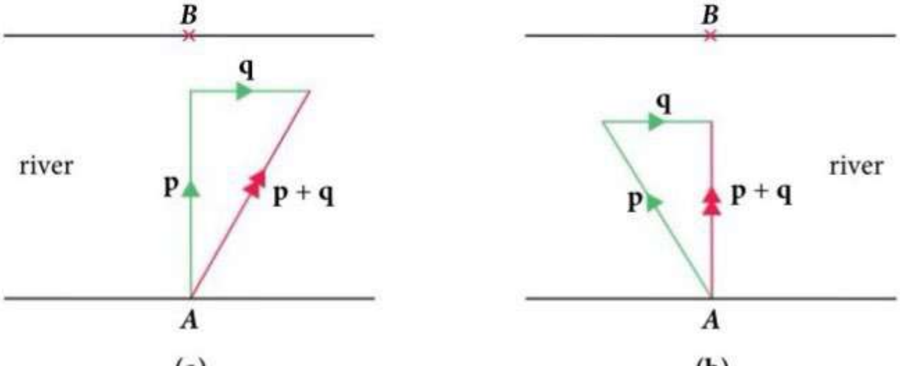
Honeybees have a way of communicating the direction and distance of a new food source with one another. This is done through a waggle dance in the direction of the food source with reference to the direction of the Sun. Amazingly, as the position of the Sun in the sky changes throughout the day, the honeybees are able to adjust the angle between the direction of the Sun and the direction of the food source accordingly. The distance of the food source from the hive is communicated by the duration of the dance. In general, every second of the dance indicates one kilometre from the food source. Therefore, we see the importance of vectors in nature.

In the Introductory Problem, we have discussed some real-life examples of vectors. But what about real-life examples of the resultant vector of two vectors?

Internet Resources

You can watch the video 'The Waggle Dance of the Honeybee' on the Internet for more information.
Another application of vectors is in the Global Positioning System (GPS) which makes use of complex vectors and geometric trilateration to determine the position of objects. Search the Internet for more information.

For example, Fig. 4.17 shows the path of a boat crossing a river from *A* to *B*.
In Fig. 4.17**(a)**, the boat tries to travel towards *B* from *A*, as indicated by **p**. But the water current, as indicated by **q**, causes the boat to travel in the direction indicated by the resultant vector **p** + **q**. So the boat will not reach *B*.



(a) **(b)**

Fig. 4.17

To reach *B* from *A*, the boat must travel in the direction indicated by **p** in Fig. 4.17**(b)**. Then the boat will end up travelling in the direction indicated by the resultant vector **p** + **q**.

Class Discussion **Real-life applications of the resultant vector of two vectors or of the difference between two vectors**

Think of other real-life examples to illustrate the application of the resultant vector of two vectors or of the difference between two vectors.

B. Solving geometric problems involving vectors

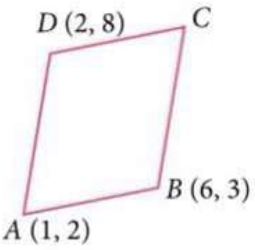
Vectors can also be used to solve some geometric problems.

Worked Example 15

Solving geometric problem with the help of vectors

The coordinates of *A*, *B* and *D* are (1, 2), (6, 3) and (2, 8) respectively.
Find the coordinates of *C* if *ABCD* is a parallelogram.

Solution



Since *ABCD* is a parallelogram, then

$$\overrightarrow{DC} = \overrightarrow{AB}$$
$$\overrightarrow{OC} - \overrightarrow{OD} = \overrightarrow{OB} - \overrightarrow{OA}$$

Problem-solving Tip

We can sketch a parallelogram to help us see which vectors are equal, e.g. $\overrightarrow{DC} = \overrightarrow{AB}$, and which are not, e.g. $\overrightarrow{CD} \neq \overrightarrow{AB}$.
We can also use other pairs of equal vectors, such as $\overrightarrow{CD} = \overrightarrow{BA}$ or $\overrightarrow{AD} = \overrightarrow{BC}$. But the manipulation may be a bit more tedious. Can you try to do so?

$$\overrightarrow{OC} - \begin{pmatrix} 2 \\ 8 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\overrightarrow{OC} = \begin{pmatrix} 6 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ 8 \end{pmatrix}$$

$$= \begin{pmatrix} 7 \\ 9 \end{pmatrix}$$

$\therefore$ the coordinates of C are (7, 9).

Practise Now 15

Similar and Further Questions
Exercise 4D
Questions 1, 8(a), (b), 14

The coordinates of A, B and D are (3, 7), (−1, 2) and (5, −4) respectively. Find the coordinates of C if $ABCD$ is a parallelogram.

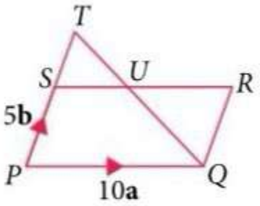
Worked Example 16

Solving geometric problem involving vectors

In the diagram, $SPQR$ is a parallelogram where $\overrightarrow{PQ} = 10\mathbf{a}$ and $\overrightarrow{PS} = 5\mathbf{b}$.

The point U on SR is such that $SU = \frac{2}{5}SR$.

The lines PS and QU, when produced, meet at T.



(i) Express the following in terms of **a** and/or **b**.

(a) $\overrightarrow{PR}$ **(b)** $\overrightarrow{SU}$

(c) $\overrightarrow{RU}$ **(d)** $\overrightarrow{TU}$

(ii) Calculate the value of

(a) $\dfrac{\text{area of } \triangle TSU}{\text{area of } \triangle QRU}$, **(b)** $\dfrac{\text{area of } \triangle TSU}{\text{area of } \triangle PSU}$.

Solution

(i) (a) **Method 1:**

$$\overrightarrow{PR} = \overrightarrow{PQ} + \overrightarrow{PS} \quad \text{Parallelogram Law of Vector Addition}$$
$$= 10\mathbf{a} + 5\mathbf{b}$$

Method 2:

$$\overrightarrow{PR} = \overrightarrow{PQ} + \overrightarrow{QR} \quad \text{Triangle Law of Vector Addition}$$
$$= \overrightarrow{PQ} + \overrightarrow{PS} \quad \text{equal vectors}$$
$$= 10\mathbf{a} + 5\mathbf{b}$$

(b) $SU = \frac{2}{5}SR$

$$\overrightarrow{SU} = \frac{2}{5}\overrightarrow{SR} \quad \overrightarrow{SU} \text{ and } \overrightarrow{SR} \text{ have same direction}$$
$$= \frac{2}{5}\overrightarrow{PQ} \quad \text{equal vectors}$$
$$= \frac{2}{5}(10\mathbf{a})$$
$$= 4\mathbf{a}$$

Reflection

(i) (a) Which method do you prefer? Explain.

Problem-solving Tip

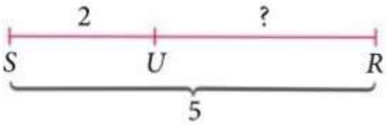
(i) (b) $SU = \frac{2}{5}SR$ does not necessarily imply that $\overrightarrow{SU} = \frac{2}{5}\overrightarrow{SR}$, e.g. it can also imply that $\overrightarrow{SU} = \frac{2}{5}\overrightarrow{RS}$. We need to check the direction of $\overrightarrow{SU}$ and $\overrightarrow{SR}$ or $\overrightarrow{RS}$ in the diagram.

(c) **Method 1:**

$$SU = \frac{2}{5}SR$$

$$\frac{SU}{SR} = \frac{2}{5}$$

From the diagram,



$UR = 3$ parts.

$$\frac{UR}{SR} = \frac{3}{5}$$

$$UR = \frac{3}{5}\ SR$$

$$\overrightarrow{RU} = -\frac{3}{5}\overrightarrow{SR} \qquad \overrightarrow{RU} \text{ and } \overrightarrow{SR} \text{ have opposite directions}$$

$$= -\frac{3}{5}\overrightarrow{PQ} \qquad \text{equal vectors}$$

$$= -\frac{3}{5}(10\mathbf{a})$$

$$= -6\mathbf{a}$$

Method 2:

$$\overrightarrow{RU} = \overrightarrow{RS} + \overrightarrow{SU} \qquad \text{Triangle Law of Vector Addition applied to parallel vectors}$$

$$= -\overrightarrow{SR} + 4\mathbf{a} \qquad \text{from part (i)(b)}$$

$$= -\overrightarrow{PQ} + 4\mathbf{a} \qquad \text{equal vectors}$$

$$= -10\mathbf{a} + 4\mathbf{a}$$

$$= -6\mathbf{a}$$

Problem-solvi ip

(i) (c) Similarly, $UR = \frac{3}{5}\,SR$ does not necessarily imply that $\overrightarrow{RU} = -\frac{3}{5}\,\overrightarrow{SR}$. We need to check the direction of $\overrightarrow{RU}$ and $\overrightarrow{SR}$ or $\overrightarrow{RS}$ in the diagram.

Reflection

(i) (c) Which method do you prefer? Explain.

(d) $\triangle TSU$ and $\triangle QRU$ are similar (AA Similarity Test).

$$\therefore \frac{TU}{QU} = \frac{SU}{RU}$$

$$= \frac{2}{3} \qquad \text{see diagram in part (i)(c)}$$

$$\therefore TU = \frac{2}{3}QU$$

$$\overrightarrow{TU} = \frac{2}{3}\overrightarrow{UQ} \qquad \overrightarrow{TU} \text{ and } \overrightarrow{UQ} \text{ have same direction}$$

$$= \frac{2}{3}\left(\overrightarrow{UR} + \overrightarrow{RQ}\right)$$

$$= \frac{2}{3}\left(\overrightarrow{UR} - \overrightarrow{PS}\right)$$

$$= \frac{2}{3}(6\mathbf{a} - 5\mathbf{b}) \qquad \overrightarrow{RU} = -6\mathbf{a} \text{ from part (i)(c)}$$

(ii) **(a)** $\triangle TSU$ and $\triangle QRU$ are similar.

$$\text{Then } \frac{\text{area of } \triangle TSU}{\text{area of } \triangle QRU} = \left(\frac{TU}{QU}\right)^2$$

$$= \left(\frac{2}{3}\right)^2 \qquad \frac{TU}{QU} = \frac{2}{3} \text{ from part (i)(d)}$$

$$= \frac{4}{9}$$

(b) **Method 1:**

$$\frac{\text{Area of } \triangle TSU}{\text{Area of } \triangle PSU} = \frac{\frac{1}{2} \times ST \times h}{\frac{1}{2} \times PS \times h}\text{, where } h \text{ is the common height of } \triangle TSU \text{ and } \triangle PSU$$

$$= \frac{ST}{PS}$$

$$= \frac{ST}{RQ} \qquad PS = RQ \text{ (opp. sides of parallelogram } PQRS\text{)}$$

$$= \frac{2}{3} \qquad \triangle TSU \text{ is similar to } \triangle QRU$$

Method 2:

$$\begin{aligned} \overrightarrow{ST} &= \overrightarrow{SU} + \overrightarrow{UT} \\ &= \overrightarrow{SU} - \overrightarrow{TU} \qquad \text{negative vector} \\ &= 4\mathbf{a} - \frac{2}{3}(6\mathbf{a} - 5\mathbf{b}) \\ &= 4\mathbf{a} - 4\mathbf{a} + \frac{10}{3}\mathbf{b} \\ &= \frac{10}{3}\mathbf{b} \end{aligned}$$

$$\frac{\text{Area of } \triangle TSU}{\text{Area of } \triangle PSU} = \frac{\frac{1}{2} \times ST \times h}{\frac{1}{2} \times PS \times h}\text{, where } h \text{ is the common height of } \triangle TSU \text{ and } \triangle PSU$$

$$\begin{aligned} &= \frac{ST}{PS} \\ &= \frac{\left|\frac{10}{3}\mathbf{b}\right|}{|5\mathbf{b}|} \\ &= \frac{\frac{10}{3}|\mathbf{b}|}{5|\mathbf{b}|} \\ &= \frac{10}{3} \div 5 \\ &= \frac{10}{3} \times \frac{1}{5} \\ &= \frac{2}{3} \end{aligned}$$

Attention

(ii) (b) For **Method 2**, we cannot divide a vector by another vector, e.g. we *cannot* write $\frac{\overrightarrow{ST}}{\overrightarrow{PS}}$ or $\frac{\frac{10}{3}\mathbf{b}}{5\mathbf{b}}$.

We can only divide the *magnitude* of a vector by the magnitude of another vector, e.g. $\frac{\left|\overrightarrow{ST}\right|}{\left|\overrightarrow{PS}\right|}$ or $\frac{\left|\frac{10}{3}\mathbf{b}\right|}{|5\mathbf{b}|}$.

Reflection

(ii) (b) Which method do you prefer? Explain.

Practise Now 16

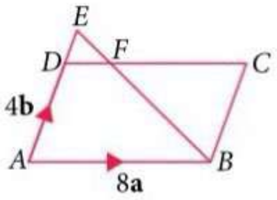
Similar and Further Questions

Exercise 4D

Questions 2–4, 9, 15–17

In the diagram, $DABC$ is a parallelogram where $\overrightarrow{AB} = 8\mathbf{a}$ and $\overrightarrow{AD} = 4\mathbf{b}$.

The point F on DC is such that $DF = \frac{1}{4}DC$. The lines AD and BF, when produced, meet at E.



(i) Express the following in terms of **a** and/or **b**.

(a) $\overrightarrow{AC}$ **(b)** $\overrightarrow{DF}$

(c) $\overrightarrow{CF}$ **(d)** $\overrightarrow{EF}$

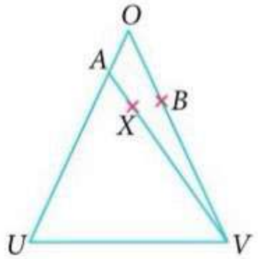
(ii) Calculate the value of

(a) $\dfrac{\text{area of } \triangle EDF}{\text{area of } \triangle BCF}$, **(b)** $\dfrac{\text{area of } \triangle EDF}{\text{area of } \triangle ADF}$.

Worked Example 17

Solving geometric problem involving position vectors

In the diagram, $\overrightarrow{OU} = 15\mathbf{u}$, $\overrightarrow{OV} = 9\mathbf{v}$, $\overrightarrow{OA} = \frac{1}{5}\overrightarrow{OU}$ and $\overrightarrow{OB} = \frac{1}{3}\overrightarrow{OV}$.



(i) Find the vectors $\overrightarrow{UV}$ and $\overrightarrow{AB}$ in terms of **u** and **v**.

(ii) Given that $\overrightarrow{AX} = \frac{1}{5}\overrightarrow{AV}$, express the vector $\overrightarrow{XB}$ in terms of **u** and **v**.

Solution

(i)
$$\begin{aligned}\overrightarrow{UV} &= \overrightarrow{OV} - \overrightarrow{OU} \quad \text{Triangle Law of Vector Subtraction}\\ &= 9\mathbf{v} - 15\mathbf{u}\\ \overrightarrow{OA} &= \frac{1}{5}\overrightarrow{OU}\\ &= \frac{1}{5}(15\mathbf{u})\\ &= 3\mathbf{u}\\ \overrightarrow{OB} &= \frac{1}{3}\overrightarrow{OV}\\ &= \frac{1}{3}(9\mathbf{v})\\ &= 3\mathbf{v}\\ \therefore \overrightarrow{AB} &= \overrightarrow{OB} - \overrightarrow{OA}\\ &= 3\mathbf{v} - 3\mathbf{u}\end{aligned}$$

Attention

(i) You can leave your answer for $\overrightarrow{UV}$ as $9\mathbf{v} - 15\mathbf{u}$ or $3(3\mathbf{v} - 5\mathbf{u})$.

(ii) **Method 1:**
$$\begin{aligned}\overrightarrow{AX} &= \frac{1}{5}\overrightarrow{AV}\\ \overrightarrow{OX} - \overrightarrow{OA} &= \frac{1}{5}\left(\overrightarrow{OV} - \overrightarrow{OA}\right) \quad \text{express in terms of position vectors}\\ \overrightarrow{OX} - 3\mathbf{u} &= \frac{1}{5}(9\mathbf{v} - 3\mathbf{u})\\ \overrightarrow{OX} &= \frac{9}{5}\mathbf{v} - \frac{3}{5}\mathbf{u} + \frac{15}{5}\mathbf{u}\\ &= \frac{9}{5}\mathbf{v} + \frac{12}{5}\mathbf{u}\\ &= \frac{3}{5}(4\mathbf{u} + 3\mathbf{v})\end{aligned}$$

Problem-solving Tip

We can use the *Triangle Law of Vector Subtraction* to express a vector in terms of position vectors directly, e.g. $\overrightarrow{UV} = \overrightarrow{OV} - \overrightarrow{OU}$ in **(i)**.
In **(ii)** Method 1, we express the vectors in the equation $\overrightarrow{AX} = \frac{1}{5}\overrightarrow{AV}$ *in terms of their position vectors*: $\overrightarrow{OX} - \overrightarrow{OA} = \frac{1}{5}\left(\overrightarrow{OV} - \overrightarrow{OA}\right)$.
Position vectors help us to solve the problem easily, *without* having to refer to or draw diagrams, unlike in Method 2.

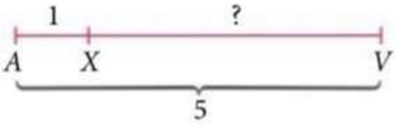
$$\begin{aligned}
\therefore \overrightarrow{XB} &= \overrightarrow{OB} - \overrightarrow{OX} \\
&= 3\mathbf{v} - \frac{3}{5}(4\mathbf{u} + 3\mathbf{v}) \\
&= \frac{15}{5}\mathbf{v} - \frac{12}{5}\mathbf{u} - \frac{9}{5}\mathbf{v} \\
&= \frac{6}{5}\mathbf{v} - \frac{12}{5}\mathbf{u} \\
&= \frac{6}{5}(\mathbf{v} - 2\mathbf{u})
\end{aligned}$$

Method 2:

$$\overrightarrow{AX} = \frac{1}{5}\overrightarrow{AV}$$

$$AX = \frac{1}{5}AV$$

$$\frac{AX}{AV} = \frac{1}{5}$$



From the diagram, $XV = 4$ parts.

$$\frac{XV}{AV} = \frac{4}{5}$$

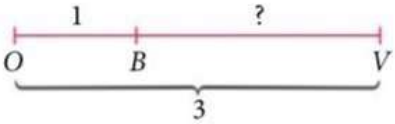
$$XV = \frac{4}{5}AV$$

$$\overrightarrow{XV} = \frac{4}{5}\overrightarrow{AV}$$

$$\overrightarrow{OB} = \frac{1}{3}\overrightarrow{OV}$$

$$OB = \frac{1}{3}OV$$

$$\frac{OB}{OV} = \frac{1}{3}$$



From the diagram, $BV = 2$ parts.

$$\frac{VB}{VO} = \frac{2}{3}$$

$$VB = \frac{2}{3}VO$$

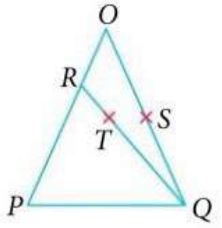
$$\overrightarrow{VB} = \frac{2}{3}\overrightarrow{VO}$$

$$\begin{aligned}
\therefore \overrightarrow{XB} &= \overrightarrow{XV} + \overrightarrow{VB} && \text{Triangle Law of Vector Addition} \\
&= \frac{4}{5}\overrightarrow{AV} + \frac{2}{3}\overrightarrow{VO} \\
&= \frac{4}{5}\left(\overrightarrow{AO} + \overrightarrow{OV}\right) - \frac{2}{3}\overrightarrow{OV} && \text{negative vector} \\
&= \frac{4}{5}\overrightarrow{AO} + \frac{2}{15}\overrightarrow{OV} \\
&= \frac{4}{5}(-3\mathbf{u}) + \frac{2}{15}(9\mathbf{v}) \\
&= -\frac{12}{5}\mathbf{u} + \frac{6}{5}\mathbf{v} \\
&= \frac{6}{5}(\mathbf{v} - 2\mathbf{u})
\end{aligned}$$

Practise Now 17

Similar and Further Questions
Exercise 4D
Questions 5, 6, 10–12

In the diagram, $\overrightarrow{OP} = 9\mathbf{p}$, $\overrightarrow{OQ} = 3\mathbf{q}$, $\overrightarrow{OR} = \frac{1}{3}\overrightarrow{OP}$ and $\overrightarrow{OS} = \frac{1}{2}\overrightarrow{OQ}$.



(i) Find the vectors $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ in terms of **p** and **q**.

(ii) Given that $\overrightarrow{RT} = \frac{1}{4}\overrightarrow{RQ}$, express the vector $\overrightarrow{TS}$ in terms of **p** and **q**.

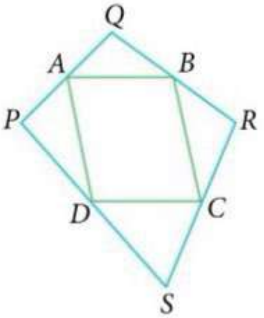
Worked Example 18

Proving geometric properties using vectors

In the diagram, *PQRS* is a quadrilateral, and *A*, *B*, *C* and *D* are the midpoints of *PQ*, *QR*, *RS* and *SP* respectively.
Show that

(i) *PR* is parallel to *AB* and *PR* = 2*AB*,

(ii) *ABCD* is a parallelogram.



***Solution**

(i) Let $\overrightarrow{QA} = \mathbf{a}$ and $\overrightarrow{QB} = \mathbf{b}$.

$$\begin{aligned} \text{Then } \overrightarrow{AB} &= \overrightarrow{QB} - \overrightarrow{QA} && \text{Triangle Law of Vector Subtraction} \\ &= \mathbf{b} - \mathbf{a} \\ \overrightarrow{QP} &= 2\overrightarrow{QA} && A \text{ is the midpoint of } PQ \\ &= 2\mathbf{a} \\ \overrightarrow{QR} &= 2\overrightarrow{QB} && B \text{ is the midpoint of } QR \\ &= 2\mathbf{b} \\ \overrightarrow{PR} &= \overrightarrow{QR} - \overrightarrow{QP} \\ &= 2\mathbf{b} - 2\mathbf{a} \\ &= 2(\mathbf{b} - \mathbf{a}) \\ &= 2\overrightarrow{AB} \end{aligned}$$

Since $\overrightarrow{PR} = 2\overrightarrow{AB}$, then *PR* is parallel to *AB* and *PR* = 2*AB*. (shown)

(ii) Using the same reasoning in part **(i)** for △*SPR*, we can show that $\overrightarrow{PR} = 2\overrightarrow{DC}$.

Since $\overrightarrow{PR} = 2\overrightarrow{AB} = 2\overrightarrow{DC}$, then $\overrightarrow{AB} = \overrightarrow{DC}$.

∴ *AB* is parallel to *DC* and *AB* = *DC*, i.e. *ABCD* is a parallelogram. (shown)

Big Idea

Equivalence

(i) Showing that *PR* // *AB* and *PR* = 2*AB* is equivalent to showing that $\overrightarrow{PR} = 2\overrightarrow{AB}$, where $\overrightarrow{PR}$ and $\overrightarrow{AB}$ are in the same direction.

(ii) Showing that *ABCD* is a parallelogram is equivalent to showing that *AB* is parallel to *DC* and *AB* = *DC*, which is equivalent to showing that $\overrightarrow{AB} = \overrightarrow{DC}$.

Thus we observe the usefulness of **equivalent statements** or *equivalent properties* in solving geometric problems.
How can we show that three points *M*, *N* and *O* are collinear, i.e. they lie on a straight line?

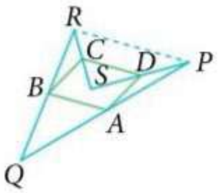
Practise Now 18

Similar and Further Questions
Exercise 4D
Questions 7, 13, 18

In the diagram, *PQRS* is a quadrilateral, and *A*, *B*, *C* and *D* are the midpoints of *PQ*, *QR*, *RS* and *SP* respectively.
Show that

(i) *PR* is parallel to *AB* and *PR* = 2*AB*,

(ii) *ABCD* is a parallelogram.



Reflection

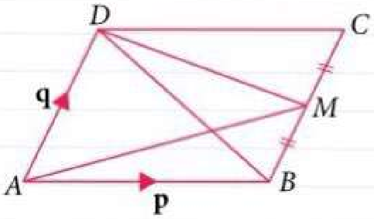
1. How do I go about solving geometric questions involving areas of similar triangles and areas of triangles with the same height?
2. How do I go about solving geometric questions involving position vectors?
3. What have I learnt in this section or chapter that I am still unclear of?

Basic Intermediate Advanced

Exercise 4D

1. The coordinates of A, B and D are (2, 3), (7, 5) and (4, 9) respectively. Find the coordinates of C if $ABCD$ is a parallelogram.

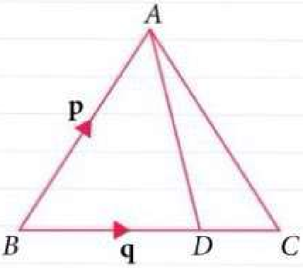
2. $ABCD$ is a parallelogram with M as the midpoint of BC.



If $\overrightarrow{AB} = \mathbf{p}$ and $\overrightarrow{AD} = \mathbf{q}$, express in terms of **p** and/or **q**,

(i) $\overrightarrow{CM}$, **(ii)** $\overrightarrow{DB}$,
(iii) $\overrightarrow{AM}$, **(iv)** $\overrightarrow{MD}$.

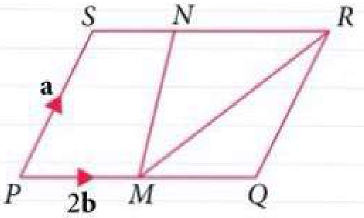
3. In the diagram, D is a point on BC such that $BD = 3DC$.



Given that $\overrightarrow{BA} = \mathbf{p}$ and $\overrightarrow{BD} = \mathbf{q}$, express in terms of **p** and/or **q**,

(i) $\overrightarrow{BC}$, **(ii)** $\overrightarrow{AD}$,
(iii) $\overrightarrow{CA}$.

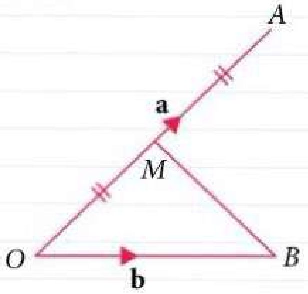
4. In the diagram, $PQRS$ is a parallelogram. M is the midpoint of PQ and N is a point on SR such that $SR = 3SN$.



Given that $\overrightarrow{PS} = \mathbf{a}$ and $\overrightarrow{PM} = 2\mathbf{b}$, express in terms of **a** and/or **b**,

(i) $\overrightarrow{MR}$, **(ii)** $\overrightarrow{RN}$,
(iii) $\overrightarrow{NM}$.

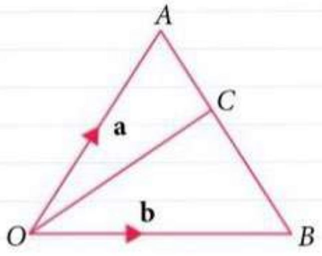
5. In the diagram, $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and M is the midpoint of OA.



Find $\overrightarrow{BM}$ in terms of **a** and **b**.

Exercise 4D

6. In the diagram, $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{AC} = \frac{2}{3}\overrightarrow{CB}$.

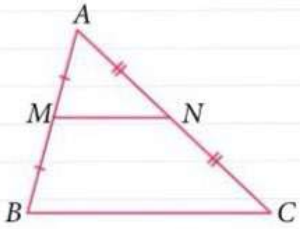


Find in terms of **a** and **b**,

(i) $\overrightarrow{AB}$, (ii) $\overrightarrow{AC}$,

(iii) $\overrightarrow{OC}$.

7.



In the diagram, $\overrightarrow{AB} = \mathbf{u}$, $\overrightarrow{AC} = \mathbf{v}$, and M and N are the midpoints of AB and AC respectively.

Express in terms of **u** and/or **v**,

(i) $\overrightarrow{BC}$, (ii) $\overrightarrow{AM}$,

(iii) $\overrightarrow{AN}$, (iv) $\overrightarrow{MN}$.

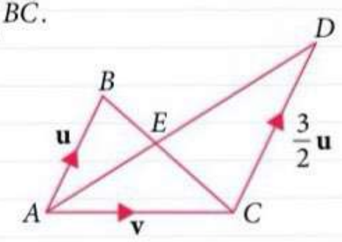
What can you say about $\overrightarrow{BC}$ and $\overrightarrow{MN}$?

8. The coordinates of P, Q and R are (1, 0), (4, 2) and (5, 4) respectively. Use a vector method to determine the coordinates of S if

(a) $PQRS$ is a parallelogram,

(b) $PRQS$ is a parallelogram.

9. In the diagram, $\overrightarrow{AB} = \mathbf{u}$, $\overrightarrow{AC} = \mathbf{v}$, $\overrightarrow{CD} = \frac{3}{2}\mathbf{u}$ and $\overrightarrow{BE} = \frac{2}{5}\overrightarrow{BC}$.



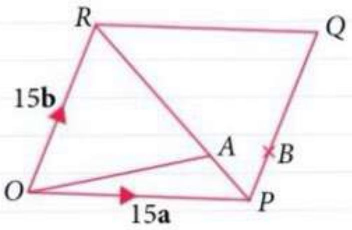
Express the following in terms of **u** and **v**.

(i) $\overrightarrow{BC}$ (ii) $\overrightarrow{BE}$

(iii) $\overrightarrow{AD}$ (iv) $\overrightarrow{AE}$

(v) $\overrightarrow{BD}$

10.



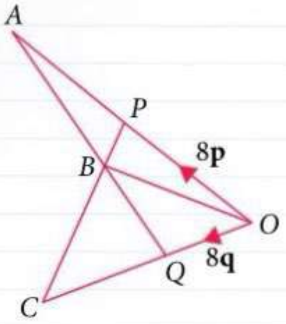
$OPQR$ is a parallelogram. The point A on PR is such that $\overrightarrow{AR} = \frac{3}{4}\overrightarrow{PR}$. The point B on PQ is such that $\overrightarrow{PB} = \frac{1}{3}\overrightarrow{PQ}$. Given that $\overrightarrow{OP} = 15\mathbf{a}$ and $\overrightarrow{OR} = 15\mathbf{b}$, express the following vectors in terms of **a** and **b**.

(i) $\overrightarrow{PR}$ (ii) $\overrightarrow{PA}$

(iii) $\overrightarrow{OA}$ (iv) $\overrightarrow{OB}$

Exercise 4D

11.

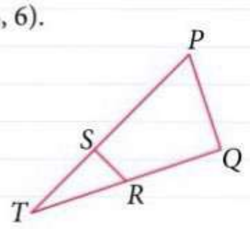


In the diagram, *OPA* and *OQC* are straight lines, and *PC* intersects *QA* at *B*. Given that $\overrightarrow{OQ} = \frac{2}{3}\overrightarrow{QC}$, $\frac{PB}{BC} = \frac{1}{3}$, $\overrightarrow{OP} = 8\mathbf{p}$ and $\overrightarrow{OQ} = 8\mathbf{q}$, express the following vectors, as simply as possible, in terms of **p** and **q**.

(i) $\overrightarrow{PC}$ **(ii)** $\overrightarrow{PB}$

(iii) $\overrightarrow{OB}$ **(iv)** $\overrightarrow{QB}$

12. Relative to the origin *O*, which is not shown in the diagram, *P* is the point (1, 11), *Q* is the point (2, 8), *R* is the point (–1, 7), *S* is the point (–2, 8) and *T* is the point (–4, 6).



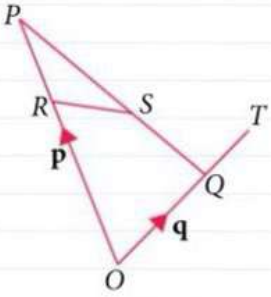
(i) Express the following as column vectors.

(a) $\overrightarrow{PQ}$ **(b)** $\overrightarrow{SR}$

(c) $\overrightarrow{RQ}$ **(d)** $\overrightarrow{TQ}$

(ii) Find the value of the ratio $\frac{RQ}{TQ}$.

13. In the diagram, *ORP*, *OQT* and *PSQ* are straight lines. $\overrightarrow{OP} = \mathbf{p}$, $\overrightarrow{OQ} = \mathbf{q}$, $PS : SQ = 3 : 2$, $OQ : QT = 2 : 1$ and $OR : RP = 2 : 1$.



(i) Express, as simply as possible, in terms of **p** and/or **q**,

(a) $\overrightarrow{QP}$, **(b)** $\overrightarrow{QS}$,

(c) $\overrightarrow{OS}$, **(d)** $\overrightarrow{ST}$.

(ii) **(a)** Show that $\overrightarrow{RS} = k\overrightarrow{ST}$, where *k* is a constant.

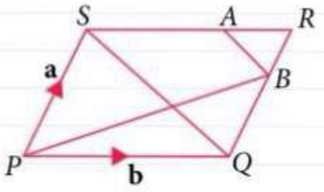
(b) Write down two facts about the points *R*, *S* and *T*.

14. Given that *A* is the point (1, 2), $\overrightarrow{AB} = \begin{pmatrix} 4 \\ -5 \end{pmatrix}$, $\overrightarrow{AC} = \begin{pmatrix} 6 \\ 3 \end{pmatrix}$ and that *M* is the midpoint of *BC*, find

(i) $\overrightarrow{BC}$, **(ii)** $\overrightarrow{AM}$,

(iii) the coordinates of the point *D* such that *ABCD* is a parallelogram.

15. *PQRS* is a parallelogram. $\overrightarrow{BQ} = 2\overrightarrow{RB}$, $\overrightarrow{AR} = \frac{1}{3}\overrightarrow{SR}$, $\overrightarrow{PS} = \mathbf{a}$ and $\overrightarrow{PQ} = \mathbf{b}$.



(i) Express in terms of **a** and/or **b**,

(a) $\overrightarrow{SA}$, **(b)** $\overrightarrow{QB}$,

(c) $\overrightarrow{PB}$, **(d)** $\overrightarrow{QS}$,

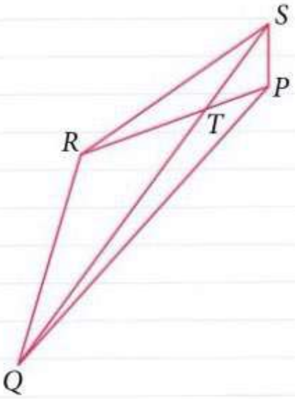
(e) $\overrightarrow{BA}$.

(ii) Calculate the value of

(a) $\dfrac{BA}{QS}$, **(b)** $\dfrac{\text{area of } \triangle ABR}{\text{area of } \triangle SQR}$,

(c) $\dfrac{\text{area of } \triangle ABR}{\text{area of } PQRS}$.

16. In the diagram, T is the point of intersection of the diagonals of the quadrilateral $PQRS$. $\overrightarrow{PR} = 3\overrightarrow{PT}$, $\overrightarrow{PS} = 5\mathbf{b}$, $\overrightarrow{PQ} = 4\mathbf{a} + \mathbf{b}$ and $\overrightarrow{PR} = 3\mathbf{a} + 12\mathbf{b}$.



(i) Express, as simply as possible, in terms of **a** and **b**,

(a) $\overrightarrow{RS}$, **(b)** $\overrightarrow{RT}$,

(c) $\overrightarrow{RQ}$.

(ii) Show that $\overrightarrow{QT} = 3(\mathbf{b} - \mathbf{a})$.

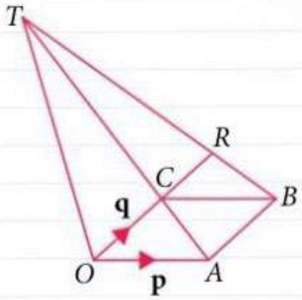
(iii) Express $\overrightarrow{QS}$ as simply as possible, in terms of **a** and **b**.

(iv) Calculate the value of

(a) $\dfrac{QT}{QS}$, **(b)** $\dfrac{\text{area of } \triangle PQT}{\text{area of } \triangle PQS}$,

(c) $\dfrac{\text{area of } \triangle PQT}{\text{area of } \triangle RQT}$.

17. $OABC$ is a parallelogram and ACT is a straight line. OC is produced to meet BT at R. $BT = 4BR$, $\overrightarrow{OA} = \mathbf{p}$, $\overrightarrow{OC} = \mathbf{q}$ and $\overrightarrow{TC} = 3(\mathbf{p} - \mathbf{q})$.



(i) Express, as simply as possible, in terms of **p** and **q**,

(a) $\overrightarrow{OT}$, **(b)** $\overrightarrow{AT}$,

(c) $\overrightarrow{OB}$, **(d)** $\overrightarrow{BT}$,

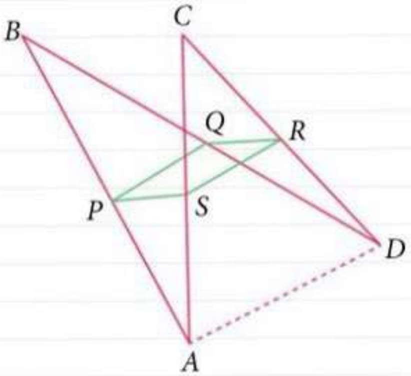
(e) $\overrightarrow{TR}$.

(ii) Show that $\overrightarrow{CR} = \dfrac{3}{4}\mathbf{q}$.

(iii) Find the value of

(a) $\dfrac{CR}{OC}$, **(b)** $\dfrac{\text{area of } \triangle TCR}{\text{area of } \triangle TAB}$.

18.



In the diagram, P, Q, R and S are the midpoints of AB, BD, CD and AC respectively. Show that

(i) PQ is parallel to AD and $PQ = \dfrac{1}{2}AD$,

(ii) $PQRS$ is a parallelogram.

Looking Back

In this chapter, we begin by examining how we can represent quantities with **measures** of both magnitude and direction using vector notations and vector diagrams.

Vector **notations** allow us to express both the magnitude and direction of vectors in a concise and precise manner. This makes it easy for us to manipulate vectors. We can add and subtract vectors, and work with a scalar multiple of a vector. Vector **diagrams** help us to visualise the relationships between two or more vectors and may also make it easier to solve problems involving vectors.

Vectors are useful because they can be used to **model** real-world phenomena, such as the relative location of a city from another, or even how a honeybee communicates the direction and distance of a food source to other honeybees. The connection between vectors and geometry enables us to solve geometric problems using vectors, and vice versa. The study of vectors is an excellent example of how mathematicians extend ideas to include additional features so that we can develop powerful tools to solve more complex problems.

Summary

1. A **scalar** quantity only has a magnitude while a **vector** quantity has both a magnitude and a direction. A non-zero vector can be represented by a directed line segment.

2. The magnitude of a column vector $\mathbf{a} = \begin{pmatrix} x \\ y \end{pmatrix}$ is given by $|\mathbf{a}| = \sqrt{x^2 + y^2}$.

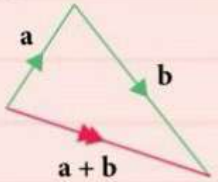
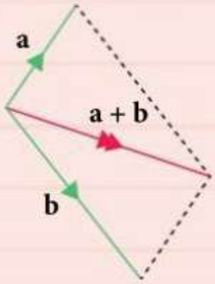
3. Two vectors **a** and **b** are **equal** (i.e. **a** = **b**) if and only if they have the same magnitude and the same direction. In column vector forms, $\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} r \\ s \end{pmatrix}$ if and only if $p = r$ and $q = s$.

4. If two vectors **a** and **c** have the same magnitude but opposite directions, vector **c** is called the **negative vector** of vector **a** (and vice versa) and we write $\mathbf{a} = -\mathbf{c}$ (or $\mathbf{c} = -\mathbf{a}$).

5. Two vectors are **parallel** if they have the same or opposite directions.

Summary

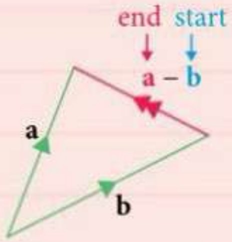
6. The Triangle Law or the Parallelogram Law of Vector Addition can be used to find the sum of two vectors:

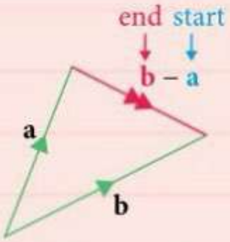
Triangle Law of Vector Addition	**Parallelogram Law of Vector Addition**
Ending point of first vector **a** = starting point of second vector **b**	Both vectors **a** and **b**, and the resultant vector **a** + **b**, all start from the *same point*.
	

- Which of the above two laws is easier to apply? When do you use the Parallelogram Law of Vector Addition?

7. The addition of a vector **a** and its negative –**a** will give the zero vector **0**, i.e. **a** + (–**a**) = **0** = (–**a**) + **a**.

8. The Triangle Law of Vector Subtraction can be used to find the difference of two vectors:
Triangle Law of Vector Subtraction
Both vectors **a** and **b** start from the *same point*.





- How do you determine at a glance whether the vector is **a** – **b** or **b** – **a**?

9. For column vectors,

- $\begin{pmatrix} p \\ q \end{pmatrix} + \begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} p+r \\ q+s \end{pmatrix}$,
- $\begin{pmatrix} p \\ q \end{pmatrix} - \begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} p-r \\ q-s \end{pmatrix}$.

10. For two *non-zero* and *non-parallel* vectors, **a** and **b**,

- |**a** + **b**| ≠ |**a**| + |**b**|,
- |**a** – **b**| ≠ |**a**| – |**b**|.

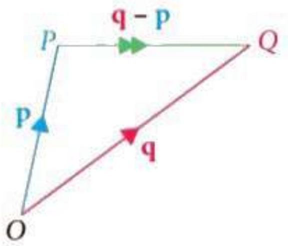
11. If **a** and **b** are two (*non-zero*) **parallel** vectors, then **a** = *k***b** for some scalar or real number $k \neq 0$.

12. If $\mathbf{a} = \begin{pmatrix} x \\ y \end{pmatrix}$, then $k\mathbf{a} = \begin{pmatrix} kx \\ ky \end{pmatrix}$ and $|k\mathbf{a}| = |k||\mathbf{a}|$ for any real number k.

13. Any vector $\overrightarrow{AB}$ can be expressed **uniquely** in terms of two other *non-zero and non-parallel coplanar vectors* $\mathbf{u}$ and $\mathbf{v}$, i.e. $\overrightarrow{AB} = m\mathbf{u} + n\mathbf{v}$, where m and n are real numbers.

14. The **position vector** of a point $P(x, y)$ is $\overrightarrow{OP} = \begin{pmatrix} x \\ y \end{pmatrix}$.

15. A vector $\overrightarrow{PQ}$ in the Cartesian plane can be expressed in terms of position vectors as follows:



$$\underset{\text{start}\ \text{end}}{\overrightarrow{PQ}} = \overbrace{\underset{\text{end}}{\overrightarrow{OQ}} - \underset{\text{start}}{\overrightarrow{OP}}}^{\text{must be the same}} = \underset{\text{end}}{\mathbf{q}} - \underset{\text{start}}{\mathbf{p}}$$

Alternatively, we can also consider

$$\underset{\text{start}\ \text{end}}{\overrightarrow{PQ}} = \underbrace{\underset{\text{start}}{\overrightarrow{PO}} + \underset{\text{end}}{\overrightarrow{OQ}}}_{\text{must be the same}} = -\overrightarrow{OP} + \overrightarrow{OQ} = -\mathbf{p} + \mathbf{q} = \mathbf{q} - \mathbf{p}$$

CHAPTER 5

Relations and Functions

Everytime we put an orange into a blender, orange juice is produced. If we put starfruit into it, we get starfruit juice. Under no circumstances will we get orange juice by putting in some other types of fruits.

In Mathematics, we can define an operation on a set of numbers so that every time we apply the operations on a given number *x*, say we will always get a result *y*. Such an operation is known as a **function**.

Learning Outcomes

What will we learn in this chapter?

- What a relation and a function are
- How to illustrate a relation using a mapping diagram
- How to verify if a given relation is a function
- How to determine the inverse of a given function
- What a composite function is

Introductory Problem

In Book 2, we learnt that a function is a relationship between two variables x and y such that every specified input x produces *exactly one output* y, where

- the input x and the output y of a function can be written as an ordered pair (x, y),
- the function can be represented using words, an equation, a table of values and a graph.

Consider a function which can be represented by the equation $y = 2x + 1$.

(i) Write down the rule of the function.

(ii) Write down the output y for each of the following inputs x.

  (a) Input $x = 2 \rightarrow$ Output $y =$

  (b) Input $x = -1 \rightarrow$ Output $y =$

(iii) Write down the input x for each of the following outputs y.

  (a) Input $x =$ $\rightarrow$ Output $y = 11$

  (b) Input $x =$ $\rightarrow$ Output $y = -5$

(iv) Now, consider $y = x^2$. Write down the value of y for each value x.

  (a) $x = 5$

  (b) $x = -5$

(v) Given $y = \pm\sqrt{x}$, what is/are the value(s) of y for $x = 25$?

(vi) Explain if $y = x^2$ is a function. What about $y = \pm\sqrt{x}$? What is the difference between the two equations?

In this chapter, we will learn about the notations used in a function.

5.1 Relations

We have come across many relations between two sets in everyday life. Examples of such relations include "is the father of", "is the daughter of" and others.

Similarly, many relations exist in Mathematics such as "is less than", "is perpendicular to", "is higher than" and "is equal to". In fact, some mathematicians have described Mathematics as the study of relations.

Let us consider the following common examples of relations between two sets.

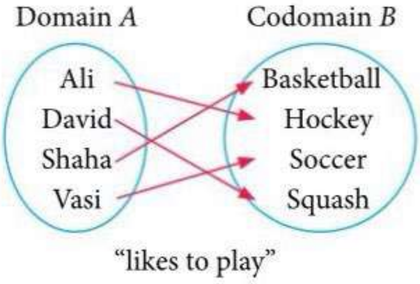


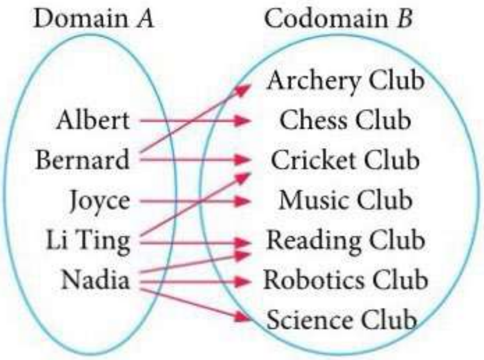
Fig. 5.1

Attention

The arrow always points from the domain to the codomain.

In Fig. 5.1, the members in Set B represent the sports which the members in Set A like to play. We observe the relations that Ali likes to play Hockey, David likes to play Squash, Shaha likes to play basketball and Vasi likes to play soccer. We can also see that a relation has a direction in which it goes. This is conveniently indicated by the arrowheads. Diagrams like Fig. 5.1 are known as **mapping diagrams**.

In any **relation**, we will have a **domain** and a **codomain**. In Fig. 5.1, Set A is the domain while Set B is the codomain of the relation. The member in the codomain that is matched to an element in the domain is referred to as the **image** of the element. Therefore, Squash is the image of David and Soccer is the image of Vasi.



"is a member of the"

Fig. 5.2

Fig. 5.2 shows the relation between each person in Set A and the clubs in which he/she is a member of in Set B. From the arrow diagram, we observe that Bernard is a member of both the Archery Club and the Cricket Club, and Li Ting is a member of the Cricket Club and the Reading Club. The above relation shows that a member in a domain may be related to more than one member in the codomain. In other words, a member in a domain may have more than one image. For example, Archery Club and Cricket Club are the images of Bernard, while Cricket Club and Reading Club are the images of Li Ting.

5.2 Functions

A. Functions

Fig. 5.3 shows the arrow diagram of a relation.

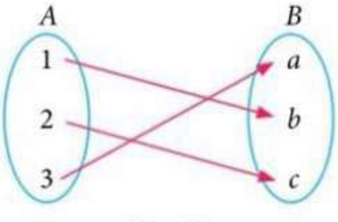


Fig. 5.3

We notice that *only one arrow* leaves each element in the domain. Thus *every* element in the domain of the relation has a *unique* (exactly one) image in the codomain.

The relations, whose arrow diagrams are shown in Fig. 5.4 and Fig. 5.5, also satisfy the property that *every* element in the domain has a *unique* image in the codomain.

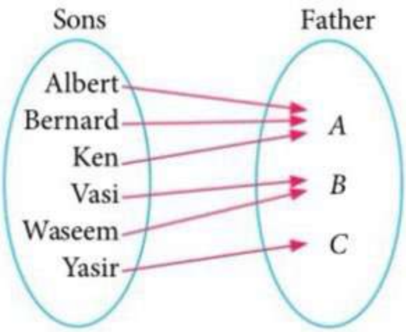


Fig. 5.4

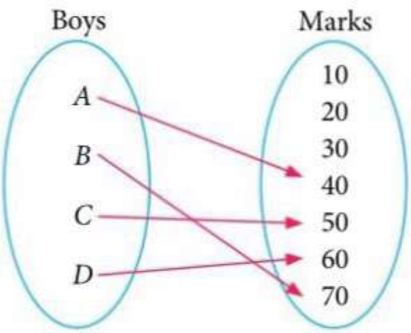


Fig. 5.5

Relations in Fig. 5.3, Fig. 5.4 and Fig. 5.5 are examples of a special kind of relation that we call **functions**. In particular, Fig. 5.3 is a one-to-one correspondence which we call a **one-to-one** function.

A **function** is a **relation** in which every element in the domain has a unique image in the codomain.

Thinking Time

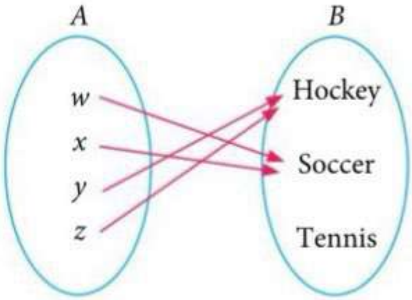
What are the differences among the relations shown in Fig. 5.3, Fig. 5.4 and Fig. 5.5?

Worked Example 1

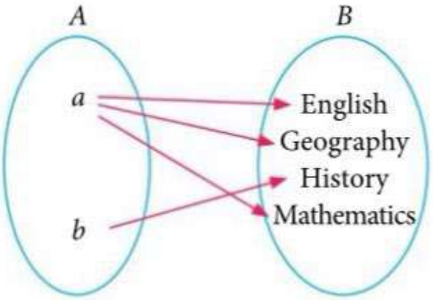
Verifying if a relation is a function

State, with reason, whether each of the following arrow diagrams defines a function.

(a)



(b)



Solution

(a) The relation is a function since every element in the domain *A* has a unique image in the codomain *B*.

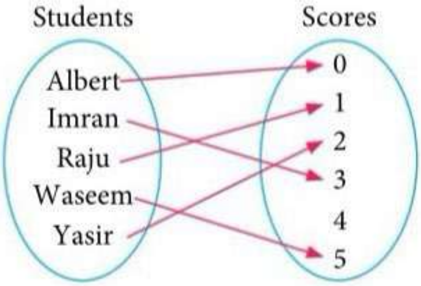
(b) The relation is not a function since the element *a* in the domain *A* has three images, English, Geography and Mathematics, in the codomain *B*.

Practise Now 1

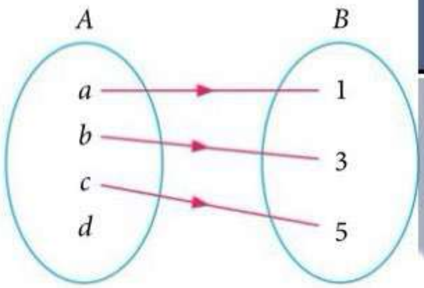
Similar and Further Questions
Exercise 5A
Questions 1(a)-(f), 2(a)-(c)

State, with reason, whether each of the following arrow diagrams defines a function.

(a)



(b)



B. Notations of a function

Consider the function f whose arrow diagram is displayed in Fig. 5.6. The domain of the function is the set $X = \{1, 2, 3, 4\}$ and its codomain is the set $Y = \{1, 2, 3, 4, 5, 6, 7, 8\}$.

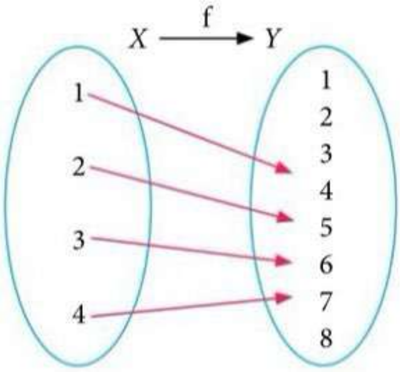


Fig. 5.6

We often use a lower case letter such as f to name a function. The notation $f: X \to Y$ is used to indicate that the function f has domain X and codomain Y. We read this as "a function f from X to Y". We can also write $X \xrightarrow{f} Y$ as illustrated in Fig. 5.7.

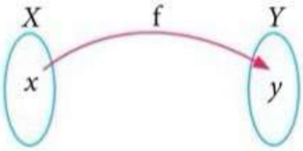


Fig. 5.7

For a function $f: X \to Y$, each element x in the domain X has a unique image y in the codomain Y. We often say y is a function of x and write it as $y = f(x)$.
The function may be written as $f: x \mapsto f(x)$, linking an element x in the domain to its image $f(x)$ in the codomain. Note that the vertical stroke on the arrow distinguishes it from $f: X \to Y$. $f(x)$ is also called the value of the function f at x.

Attention

$f(x)$ is read as "f of x".

Consider the example in Fig. 5.3. Suppose f represents the function, then

$$f(1) = b, f(2) = c \text{ and } f(3) = a.$$

In Fig. 5.4, if g represents the function, then

$$g(\text{Albert}) = g(\text{Bernard}) = g(\text{Ken}) = A, g(\text{Vasi}) = g(\text{Waseem}) = B \text{ and } g(\text{Yasir}) = C.$$

In Fig. 5.5, if h represents the function, then

$$h(A) = 40, h(B) = 70, h(C) = 50 \text{ and } h(D) = 60.$$

Now consider the function f in Fig. 5.6. We have

$$f(1) = 4 = 1 + 3, f(2) = 5 = 2 + 3, f(3) = 6 = 3 + 3 \text{ and } f(4) = 7 = 4 + 3.$$

Thus, in general,

$$f(x) = x + 3$$

i.e. the function f assigns the image $f(x)$ to each x in the domain by adding 3 to x.
The function f can be completely described as follows:

$$f : x \mapsto x + 3, x = 1, 2, 3 \text{ or } 4 \text{ or } x \in \{1, 2, 3, 4\}$$

C. Range of a function

Let $f : X \to Y$ be a function. The set of values of $f(x)$ is called the range of f. The range of the function f in Fig. 5.6 is $\{4, 5, 6, 7\}$. The range may or may not consist of all of the elements of the codomain. The range $\{4, 5, 6, 7\}$ consists of only some of the elements of $Y = \{1, 2, 3, 4, 5, 6, 7, 8\}$.

Very often, we are interested in the range and not the codomain of a function. Hence, it is often inadequate to define a function f by stating its domain and the rule which determines the unique image $f(x)$ of each x in the domain.

In Book 2, we have learnt how to determine the output of a function given an input value. Let us now revisit this using the notations we have learnt in this chapter.

Worked Example 2

Determining an element and image of a function

Given the functions $f : x \mapsto 3x + 2$ and $g : x \mapsto 5x - 4$, find the value of each of the following.

(i) $f(2)$ **(ii)** $f(-5)$ **(iii)** $f\left(\frac{1}{3}\right)$

(iv) $g(3)$ **(v)** $2g(7)$ **(vi)** $g\left(\frac{3}{5}\right)$

(vii) $f(2) + g(2)$ **(viii)** x for which $f(x) = 17$ **(ix)** x for which $f(x) = g(x)$

Solution

$f : x \mapsto 3x + 2$, i.e. $f(x) = 3x + 2$
$g : x \mapsto 5x - 4$, i.e. $g(x) = 5x - 4$

(i) $f(2) = 3(2) + 2$
$= 8$

(ii) $f(-5) = 3(-5) + 2$
$= -13$

(iii) $f\left(\frac{1}{3}\right) = 3\left(\frac{1}{3}\right) + 2$
$= 3$

(iv) $g(3) = 5(3) - 4$
$= 11$

(v) $2g(7) = 2[5(7) - 4]$
$= 2(31)$
$= 62$

(vi) $g\left(\frac{3}{5}\right) = 5\left(\frac{3}{5}\right) - 4$
$= -1$

(vii) $f(1) + g(2) = [3(1) + 2] + [5(2) - 4]$
$= 5 + 6$
$= 11$

(viii) When $f(x) = 17$, we have $3x + 2 = 17$
$3x = 15$
$x = 5$

(ix) When $f(x) = g(x)$, we have $3x + 2 = 5x - 4$
$2x = 6$
$x = 3$

Practise Now 2

Similar and Further Questions
Exercise 5A
Questions 3, 4(a)-(d), 5(a)-(e), 8(a), (b), 9(a)-(f), 12(a)-(d), 13(a)-(c)

Given the functions $f : x \mapsto 10x + 4$ and $g : x \mapsto 4x - 6$, find the value of each of the following.

(a) $f(4)$ **(b)** $f(-7)$ **(c)** $f\left(-\frac{2}{3}\right)$

(d) $g(2)$ **(e)** $2g(6)$ **(f)** $g\left(\frac{7}{8}\right)$

(g) $f\left(\frac{1}{2}\right) + g(1)$ **(h)** x for which $f(x) = g(x)$ **(i)** x for which $f(x) = 34$

Worked Example 3

Expressing image of a function as algebraic expression

If $f(x) = 7x - 4$ and $F(x) = 6x + 5$, express

(a) $f(a)$, **(b)** $F(a + 2)$, **(c)** $f(3a) + F(2a + 1)$,

in terms of a.

Solution

(a) $f(a) = 7a - 4$

(b) $F(a + 2) = 6(a + 2) + 5$
$= 6a + 12 + 5$
$= 6a + 17$

(c) $f(3a) + F(2a + 1) = 7(3a) - 4 + 6(2a + 1) + 5$
$= 21a - 4 + 12a + 6 + 5$
$= 33a + 7$

Practise Now 3

Similar and Further Questions
Exercise 5A
Questions 13(d)-(f)

If $f(x) = 2x - 5$ and $F(x) = 7x + 12$, express

(a) $f(b)$, **(b)** $F(b - 1)$, **(c)** $f(2b) + F(2b- 5)$,

in terms of b.

Worked Example 4

Problem involving functions with higher order expressions

If $f(x) = x^2 + 3x + 2$, express each of the following in terms of x.

(a) $f(2x)$ **(b)** $f(2x + 1)$ **(c)** $f(x^2 + 1)$

Solution

(a) $f(2x) = (2x)^2 + 3(2x) + 2$
$= 4x^2 + 6x + 2$

(b) $f(2x + 1) = (2x + 1)^2 + 3(2x + 1) + 2$
$= (4x^2 + 4x + 1) + (6x + 3) + 2$ apply $(a + b)^2 = a^2 + 2ab + b^2$
$= 4x^2 + 10x + 6$

(c) $f(x^2 + 1) = (x^2 + 1)^2 + 3(x^2 + 1) + 2$
$= (x^4 + 2x^2 + 1) + (3x^2 + 3) + 2$ apply $(a + b)^2 = a^2 + 2ab + b^2$ and Law 3 of Indices
$= x^4 + 5x^2 + 6$

Practise Now 4

Similar and Further Questions
Exercise 5A
Questions 6(a)-(c), 7(a)-(d), 10(a)-(c)

If $f(x) = 4x^2 - 5x + 2$, express each of the following in terms of x.

(a) $f(3x)$ **(b)** $f(2x + 3)$ **(c)** $f(x^2 - 3)$

Worked Example 5

Problem involving functions and simultaneous equations

Given the function $g(x) = hx + d$ and that $g(2) = 6$ and $g(7) = 16$, find the value of h and of d. Hence, evaluate $g(5)$ and $g(-3)$.

Solution

$g(2) = 2h + d = 6$ — (1)
$g(7) = 7h + d = 16$ — (2)
(2) – (1): $5h = 10$
$h = 2$
Substitute $h = 2$ into (1): $2(2) + d = 6$
$d = 6 - 4$
$= 2$
$\therefore h = 2, d = 2$
$g(5) = 2(5) + 2 = 12$
$g(-3) = 2(-3) + 2 = -4$

Practise Now 5

Similar and Further Questions
Exercise 5A
Questions 11, 14

Given the function $f(x) = ax^2 + bx$ and that $f(3) = 15$ and $f(-2) = 8$, find the value of a and of b. Hence, evaluate $f(1)$ and $f(-5)$.

Reflection

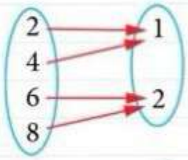
1. What are the differences between a relation and a function?
2. Given a mapping diagram, how can I tell if a relation is a function?

Advanced Intermediate Basic

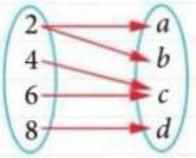
Exercise 5A

1. Each of the following relations has the set of integers {2, 4, 6, 8} as its domain. State whether each of the following arrow diagrams defines a function. If the answer is no, state the reason.

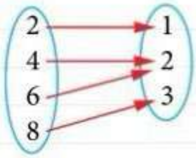
(a)



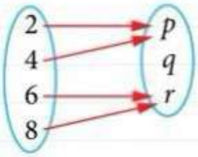
(b)



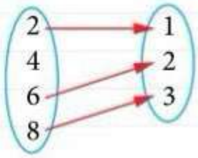
(c)



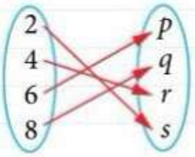
(d)



(e)



(f)



2. Draw mapping diagrams for the following functions, where each has a domain of {−2, −1, 0, 1, 2, 3}.

(a) $\mathrm{f}: x \mapsto x + 2$ **(b)** $\mathrm{g}: x \mapsto 2x^2 - 2$

(c) $\mathrm{h}: x \mapsto 3 - 2x^2$

3. A function f is defined by $\mathrm{f}: x \mapsto 6x - 4$ for all real values of x. What are the images of 2, −4, $\frac{1}{2}$ and $-\frac{1}{2}$ under f?

4. Given the function $\mathrm{f}: x \mapsto 5 - 2x$, evaluate each of the following.

(a) $\mathrm{f}(1)$ **(b)** $\mathrm{f}(-2)$

(c) $\mathrm{f}(0)$ **(d)** $\mathrm{f}(3) + \mathrm{f}(-3)$

5. Given the function $\mathrm{g}(x) = 7x + 4$, find the value of each of the following.

(a) $\mathrm{g}(2)$ **(b)** $\mathrm{g}(-3)$

(c) $\mathrm{g}\left(\frac{4}{7}\right)$ **(d)** $\mathrm{g}(0) + \mathrm{g}(-1)$

(e) $\mathrm{g}\left(\frac{1}{7}\right) - \mathrm{g}\left(-\frac{1}{7}\right)$

6. If $\mathrm{g}(x) = x^2 + 5$, express each of the following in terms of a.

(a) $\mathrm{g}(a)$

(b) $\mathrm{g}(a + 1)$

(c) $\mathrm{g}(a + 1) - \mathrm{g}(a - 1)$

7. Given that $\mathrm{F}(x) = \frac{1}{2}x(x + 1)$, find an expression, in terms of x, for each of the following.

(a) $\mathrm{F}(x - 1)$

(b) $\mathrm{F}(x + 1)$

(c) $\mathrm{F}(x) - \mathrm{F}(x - 1)$

(d) $\mathrm{F}(x^2)$

Exercise 5A

8. Given the functions $f(x) = \frac{x}{2} + 3$ and $g(x) = \frac{3}{4}x - 2$,

(a) find the values of

(i) $f(2) + g(2)$, **(ii)** $f(-1) - g(-1)$,

(iii) $2f(4) - 3g(6)$, **(iv)** $5f(-2) - 7g(-4)$.

(b) What are the values of x for which $f(x) = g(x)$ and $f(x) = 17$?

9. Given the functions $f : x \mapsto 5x - 9$ and $g : x \mapsto 2 - 6x$, find the values of x for which

(a) $f(x) = 16$, **(b)** $g(x) = 14$,

(c) $g(x) = x$, **(d)** $f(x) = 2x$,

(e) $f(x) = g(x)$, **(f)** $2f(x) = 3g(x)$.

10. If $h(x) = x^2 - 5x + 4$,

(a) express $h(2a) - h(a)$ in terms of a,

(b) find the values of a which $h(a) = 0$,

(c) express $h(a^2) + h(a)$ in terms of a.

11. Given the function $g(x) = mx + c$ and that $g(1) = 5$ and $g(5) = -4$, find the value of m and of c. Hence, evaluate $g(3)$ and $g(-4)$.

12. Given the function $f(x) = 4x + 9$, evaluate $f(1)$, $f(2)$ and $f(3)$. Is it true that

(a) $f(1) + f(2) = f(1 + 2)$?

(b) $f(3) - f(2) = f(3 - 2)$?

(c) $f(1) \times f(2) = f(1 \times 2)$?

(d) $f(2) \div f(1) = f(2 \div 1)$?

13. Given the functions $f(x) = \frac{3}{4}x + \frac{1}{2}$ and $g(x) = 1\frac{1}{2} - \frac{2}{3}x$, evaluate $f(2)$, $f\left(\frac{1}{2}\right)$, $g(3)$ and $g(-6)$.

(a) Is it true that $f(2) + f(3) = f(2 + 3)$?

(b) Is it true that $g(4) - g(2) = g(4 - 2)$?

(c) Find the value of x for which $f(x) = g(x)$.

(d) Express $f(a)$, $f(2a)$ and $g(3a)$ in terms of a.

(e) Find the value of a for which $f(a + 1) + g(a) = 5$.

(f) Find the value of a for which $f(2a) = g(6a)$.

14. Given the function $h(x) = px^2 + qx + 2$ and that $h(2) = 34$ and $h(-3) = 29$, find the value of p and of q. Hence, evaluate $h(4)$ and $h(-2)$.

5.3 Inverse functions

Consider the function $f : x \mapsto x + 2$ for the domain $A = \{1, 2, 3, 4\}$.

$f(1) = 1 + 2 = 3$, $f(2) = 2 + 2 = 4$, $f(3) = 3 + 2 = 5$, $f(4) = 4 + 2 = 6$

The mapping diagram below shows the function f.

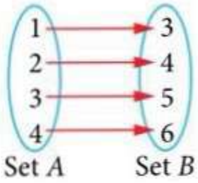


Fig. 5.8

If we reverse the direction of the arrows in Fig. 5.8, that is, map Set B into Set A instead, we get a new function called the **inverse function** of f denoted by f^{-1}.

The mapping diagram below shows the function f^{-1}.

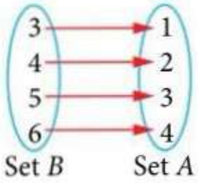


Fig. 5.9

In this case, the domain of f^{-1} is $\{3, 4, 5, 6\}$. Notice that

$$\text{If}\begin{cases} f(1) = 3, \\ f(2) = 4, \\ f(3) = 5, \\ f(4) = 6, \end{cases} \quad \text{then}\begin{cases} f^{-1}(3) = 1 \\ f^{-1}(4) = 2 \\ f^{-1}(5) = 3 \\ f^{-1}(6) = 4 \end{cases}$$

In general, if $y = f(x)$, then $f^{-1}(y) = x$

Attention

Not all functions have inverses. The function $f(x) = x^2$ is an example of a function without an inverse. Recall from the **Introductory Problem** that since every positive number x^2 has two square roots, the inverse of $y = x^2$ is not a function. That is, f^{-1} cannot be defined.

Worked Example 6

Finding the inverse of a function

A function f is defined by $f : x \mapsto x + 2$. Find the inverse function $f^{-1}(x)$.

Solution

$f(x) = x + 2$

Let $y = x + 2$

$f(x) = y$ and $f^{-1}(y) = x$

Expressing x in terms of y, we have $x = y - 2$.

$\therefore f^{-1}(y) = y - 2$ or $f^{-1} : y \mapsto y - 2$.

Note: x and y are dummy variables and it is customary for us to write

$f^{-1}(x) = x - 2$ or $f^{-1} : x \mapsto x - 2$

Attention

$f^{-1}(x) \neq \dfrac{1}{f(x)}$

Practise Now 6

Similar and Further Questions
Exercise 5B
Question 1

A function f is defined by $f : x \mapsto 8x + 3$. Find the inverse function $f^{-1}(x)$.

Worked Example 7

Finding the inverse of a function

A function f is defined by $f : x \mapsto 5x - 4$. Find the inverse function $f^{-1}(x)$.

Hence, evaluate $f^{-1}(5)$, $f^{-1}(-10)$ and $f^{-1}\left(\frac{1}{2}\right)$.

Solution

$f(x) = 5x - 4$
Let $y = 5x - 4$
$f(x) = y$ and $f^{-1}(y) = x$

Expressing x in terms of y, we have $x = \frac{y+4}{5}$.

$\therefore f^{-1}(y) = \frac{y+4}{5}$ or $f^{-1} : y \mapsto \frac{y+4}{5}$

Note: x and y are dummy variables and it is customary for us to write

$$f^{-1}(x) = \frac{x+4}{5} \text{ or } f^{-1} : x \mapsto \frac{x+4}{5}$$

$$f^{-1}(5) = \frac{5+4}{5} = 1\frac{4}{5}$$

$$f^{-1}(-10) = \frac{-10+4}{5} = -1\frac{1}{5}$$

$$f^{-1}\left(\frac{1}{2}\right) = \frac{\frac{1}{2}+4}{12} = \frac{9}{10}$$

Alternatively,
The image of x under f is $5x - 4$. Thus we need to find the value of x for which the image is 5, -10 and $\frac{1}{2}$.
$f(x) = 5x - 4$

If $x = f^{-1}(5)$, then $f(x) = 5$,
i.e. $5x - 4 = 5$

$$5x = 9$$
$$x = 1\frac{4}{5}$$

$\therefore f^{-1}(5) = 1\frac{4}{5}$

If $x = f^{-1}(-10)$, then $f(x) = -10$,

i.e. $5x - 4 = -10$

$$5x = -6$$

$$x = -1\frac{1}{5}$$

$\therefore f^{-1}(-10) = -1\frac{1}{5}$

If $x = f^{-1}\left(\frac{1}{2}\right)$, then $f(x) = \frac{1}{2}$,

i.e. $5x - 4 = \frac{1}{2}$

$$5x = 4\frac{1}{2}$$

$$x = \frac{9}{10}$$

$\therefore f^{-1}\left(\frac{1}{2}\right) = \frac{9}{10}$

Practise Now 7

Similar and Further Questions
Exercise 5B
Questions 2–6, 7(a)–(f)

A function f is defined by $f : x \mapsto 7x - 4$. Find the inverse function $f^{-1}(x)$.

Hence, evaluate $f^{-1}(10)$, $f^{-1}(-4)$ and $f^{-1}\left(\frac{1}{7}\right)$.

Worked Example 8

Defining inverse functions

A function f is defined by $f : x \mapsto \frac{x}{x-2}$ where $x \neq 2$. Find $f^{-1}(x)$ and state the value of x for which f^{-1} is not defined. Evaluate $f^{-1}(5)$, $f^{-1}(-4)$ and $f^{-1}\left(-\frac{1}{2}\right)$.

Solution

$f(x) = \frac{x}{x-2}$

Let $y = \frac{x}{x-2}$

$f(x) = y$ and $f^{-1}(y) = x$

Expressing x in terms of y, we have

$$x = xy - 2y$$

$$2y = xy - x$$

$$2y = x(y - 1)$$

$$x = \frac{2y}{y-1}$$

$\therefore f^{-1}(y) = \frac{2y}{y-1}$

Hence, $f^{-1}(x) = \dfrac{2x}{x-1}$ or $f^{-1} : x \mapsto \dfrac{2x}{x-1}$

f^{-1} is not defined when $x - 1 = 0$, i.e. $x = 1$.

$$f^{-1}(5) = \frac{2(5)}{5-1} = \frac{10}{4} = 2\frac{1}{2}$$

$$f^{-1}(-4) = \frac{2(-4)}{(-4)-1} = \frac{-8}{-5} = 1\frac{3}{5}$$

$$f^{-1}\left(-\frac{1}{2}\right) = \frac{2\left(-\frac{1}{2}\right)}{\left(-\frac{1}{2}\right)-1} = \frac{-1}{-\frac{3}{2}} = \frac{2}{3}$$

Practise Now 8

Similar and Further Questions
Exercise 5B
Questions 9, 10

A function f is defined by $f : x \mapsto \dfrac{2x}{x-5}$ where $x \neq 5$. Find $f^{-1}(x)$ and state the value of x for which f^{-1} is not defined. Evaluate $f^{-1}(6)$, $f^{-1}(-3)$ and $f^{-1}\left(\dfrac{1}{4}\right)$.

Worked Example 9

Problem involving inverse functions and simultaneous equations

A function f is defined by $f : x \mapsto ax + b$.
Given that $f(2) = 10$ and $f^{-1}(3) = 1$, find the value of a and of b.

Solution

$f(x) = ax + b$
$f^{-1}(3) = 1 \Rightarrow f(1) = 3$
Thus, we have $f(1) = 3$ and $f(2) = 10$.
i.e. $3 = a(1) + b$ — (1)
and $10 = a(2) + b$ — (2)
(2) – (1): $a = 7$
Substitute $a = 7$ into (1): $b = 3 - 7$
$= -4$

Practise Now 9

Similar and Further Questions
Exercise 5B
Questions 8, 11–15

A function f is defined by $f : x \mapsto px + q$.
Given that $f(3) = 15$ and $f^{-1}(3) = 6$, find the value of p and of q.

Reflection

1. How do I determine the inverse of a function?
2. Given a function, how do I tell if there is a value of x for which it is undefined, and how do I find this value?

Exercise 5B

1. A function f is defined by $f : x \mapsto \frac{1}{4}x - 3$. Find the inverse function $f^{-1}(x)$.

2. A function f is defined by $f : x \mapsto x - 7$. Find the inverse function $f^{-1}(x)$ and hence evaluate $f^{-1}(3)$, $f^{-1}(7)$, $f^{-1}(-5)$ and $f^{-1}\left(\frac{1}{3}\right)$.

3. A function g is defined by $g : x \mapsto 3x + 4$. Express g^{-1} in a similar form and hence evaluate $g^{-1}(3)$, $g^{-1}(-4)$, $g^{-1}\left(\frac{1}{2}\right)$ and $g^{-1}\left(-\frac{3}{4}\right)$.

4. A function h is defined by $h : x \mapsto 5x + 6$. Express h^{-1} in a similar form and hence evaluate $h^{-1}(6)$, $h^{-1}(10)$, $h^{-1}\left(-\frac{2}{5}\right)$ and $h^{-1}\left(12\frac{1}{2}\right)$.

5. A function f is defined by $f : x \mapsto 8 - 3x$. Find the inverse function of $f^{-1}(x)$. Hence, evaluate $f^{-1}(9)$, $f^{-1}(-12)$, $f^{-1}\left(3\frac{1}{3}\right)$ and $f^{-1}\left(-\frac{3}{16}\right)$.

6. A function g is defined by $g : x \mapsto 6x - 8$ for all real values of x. What are the elements in the domain whose images are 10, 40, −4 and −6?

7. Given the function $f(x) = 7 - \frac{3}{5}x$ and $g(x) = \frac{1}{4}x - 6$, evaluate each of the following.

   (a) $f^{-1}(3)$
   (b) $f^{-1}(-17)$
   (c) $g^{-1}(5)$
   (d) $g^{-1}(-6)$
   (e) $f^{-1}(2) + g^{-1}(1)$
   (f) $f^{-1}(4) - g^{-1}(4)$

8. Given that $f(x) = ax - b$, $f(-2) = 20$ and $f^{-1}(32) = 4$, find the value of a and of b.

9. Given the function $f(x) = \frac{5x}{2 - 4x}$ where $x \neq \frac{1}{2}$, find $f^{-1}(x)$ and state the value of x for which $f^{-1}(x)$ is not defined. Hence, evaluate $f^{-1}(4)$ and $f^{-1}(-6)$.

10. Given the function $f(x) = \frac{3x - 1}{x - 2}$ where $x \neq 2$, find $f^{-1}(x)$ and state the value of x for which $f^{-1}(x)$ is not defined. Hence, evaluate $f^{-1}(5)$ and $f^{-1}(7)$.

11. A function h is defined by $h : x \mapsto px^2 + qx$. Given that $h(1) = 2$ and $h^{-1}(36) = 3$, find the value of p and of q. Hence, evaluate $h(-1)$ and $h(2)$.

12. Given that $f(x) = \frac{ax - b}{4}$, $f(1) = 1$ and $f^{-1}(5) = 2$, find the value of a and of b. Hence, find $f^{-1}(x)$, and evaluate $f^{-1}(7)$ and $f^{-1}\left(-5\frac{1}{2}\right)$.

13. A function f is defined by $f : x \mapsto ax + b$. Given that $f(1) = 3$ and $f^{-1}(7) = 5$, find the value of a and of b. Hence, find $f^{-1}(x)$.

14. A function f is defined by $f : x \mapsto px + q$. Given that $f(1) = -5$ and $f(-2) = -10$, find the value of p and of q. Hence, find $f^{-1}(x)$.

15. A function g is defined by $g : x \mapsto mx + c$. Given that $g^{-1}(-3) = 0$ and $g^{-1}(1) = 2$, find the value of $g(5)$ and $g^{-1}(4)$.

5.4 Composite functions

Consider $A = \{-2, -1, 0, 1, 2\}$, $B = \{0, 1, 4\}$ and $C = \{1, 2, 5\}$. Let f be a function with domain A and codomain B, such that $f : x \mapsto x^2$. Let g be a function with domain B and codomain C, such that $g : x \mapsto x + 1$. Fig. 5.10 and Fig. 5.11 show the mapping diagrams of functions f and g respectively.

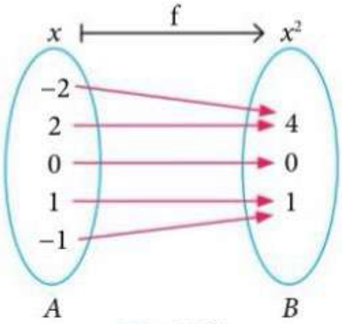


Fig. 5.10

$f(-2) = 4$
$f(2) = 4$
$f(0) = 0$
$f(1) = 1$
$f(-1) = 1$

$\mathbf{f(x) = x^2}$

$g(4) = 5$

$g(0) = 1$
$g(1) = 2$

$\mathbf{g(x^2) = x^2 + 1}$

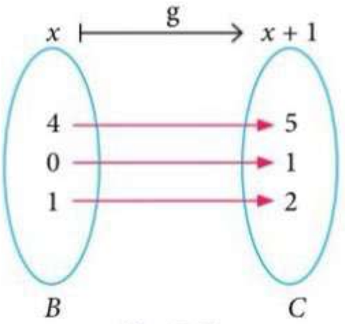


Fig. 5.11

Combining the two mapping diagrams, we obtain the mapping diagram as shown in Fig. 5.12. We see that every element in A is linked to a unique element in C.

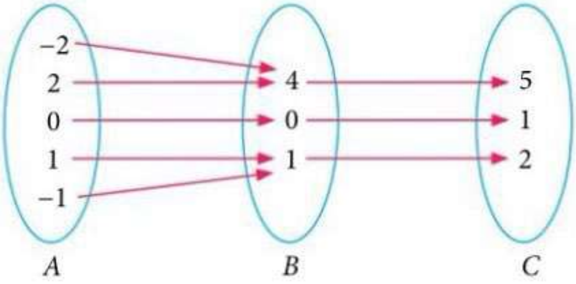


Fig. 5.12

$f(-2) = 4$ and $g(4) = 5 \Rightarrow g(f(-2)) = 5$
$f(2) = 4$ and $g(4) = 5 \Rightarrow g(f(2)) = 5$
$f(0) = 0$ and $g(0) = 1 \Rightarrow g(f(0)) = 1$
$f(1) = 1$ and $g(1) = 2 \Rightarrow g(f(1)) = 2$
$f(-1) = 1$ and $g(1) = 2 \Rightarrow g(f(-1)) = 2$

$\mathbf{f(x) = x^2}$ **and** $\mathbf{g(x^2) = x^2 + 1 \Rightarrow g(f(x)) = x^2 + 1}$

Hence, we can define a new function h as shown in Fig. 5.13.

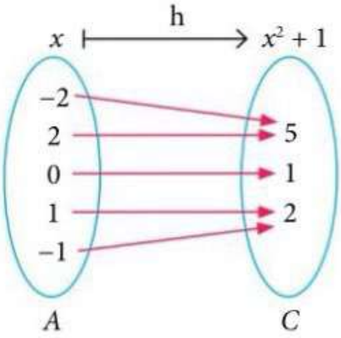


Fig. 5.13

$h(-2) = 5 \Rightarrow g(f(-2)) = gf(-2)$
$h(2) = 5 \Rightarrow g(f(2)) = gf(2)$
$h(0) = 1 \Rightarrow g(f(0)) = gf(0)$
$h(1) = 2 \Rightarrow g(f(1)) = gf(1)$
$h(-1) = 2 \Rightarrow g(f(-1)) = gf(-1)$

$\mathbf{h(x) = x^2 + 1 = g(f(x)) = gf(x)}$

The new function h will have domain A (domain of f) and codomain C (codomain of g), such that $\mathrm{h} : x \mapsto$ [illegible]. Thus, f and g are combined to produce h as shown in Fig. 5.14.

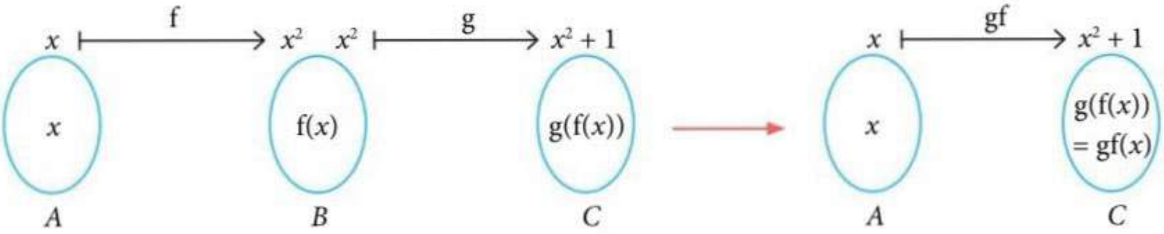


Fig. 5.14

The new function h is known as the **composite function** of f and g, and is denoted by gf. From the process shown in Fig. 5.14, we note that gf(x) = g(f(x)) and h(x) = g(f(x)). Note that the value of f(x) is first found before the function g is applied to it. Algebraically, the formula gf(x) can be obtained as follows:

$f(x) = x^2$ and $g(x) = x + 1$

$$\begin{aligned} gf(x) &= g(f(x)) \\ &= g(x^2) \\ &= x^2 + 1 \end{aligned}$$

If we combine f and g in the reverse order to obtain the composite function fg, will functions fg and gf be the same?

$$\begin{aligned} fg(x) &= f(g(x)) \\ &= f(x + 1) \\ &= (x + 1)^2 \neq gf(x) \end{aligned}$$

Since gf(x) ≠ fg(x), gf and fg are two different functions.

In general, the composite functions fg and gf are different functions. That is, unlike numbers, composition of functions is *not commutative*.

Worked Example 10

Finding composite functions

If $f : x \mapsto x + 3$ and $g : x \mapsto x^2 + 3x + 4$, find the composite functions $gf(x)$ and $fg(x)$. Hence, find the values of $gf(2)$ and $fg(2)$.

Solution

$$
\begin{aligned}
gf(x) &= g(f(x)) \\
&= g(x + 3) \\
&= (x + 3)^2 + 3(x + 3) + 4 \\
&= x^2 + 6x + 9 + 3x + 9 + 4 \\
&= x^2 + 9x + 22 \\
fg(x) &= f(g(x)) \\
&= f(x^2 + 3x + 4) \\
&= (x^2 + 3x + 4) + 3 \\
&= x^2 + 3x + 7 \\
gf(2) &= 2^2 + 9(2) + 22 \\
&= 44 \\
fg(2) &= 2^2 + 3(2) + 7 \\
&= 17
\end{aligned}
$$

Practise Now 10

Similar and Further Questions
Exercise 5C
Questions 1-3

If $f : x \mapsto \dfrac{2}{x - 1}$ and $g : x \mapsto 4(x + 2)^2$, find the composite functions $gf(x)$ and $fg(x)$. Hence, find the values of $gf(-2)$ and $fg(3)$.

Worked Example 11

Finding unknowns in composite functions

If $f : x \mapsto 3x + b$ and $g : x \mapsto 2a - 3x$ such that $fg(x) = gf(x)$, express a in terms of b.

Solution

$$
\begin{aligned}
fg(x) &= gf(x) \\
f(g(x)) &= g(f(x)) \\
f(2a - 3x) &= g(3x + b) \\
3(2a - 3x) + b &= 2a - 3(3x + b) \\
6a - 9x + b &= 2a - 9x - 3b \\
4a &= -4b \\
\therefore\ a &= -b
\end{aligned}
$$

Practise Now 11

Similar and Further Questions
Exercise 5C
Questions 4-8

1. If $f(x) = \dfrac{x}{2} + a$ and $g(x) = 2x + b$ such that $fg(x) = gf(x) + 1$, express a in terms of b.

2. Given the functions $f(x) = x + 1$ and $g(x) = \dfrac{1}{x + 1}$, solve the equation
   (a) $fg(x) = 2$,
   (b) $gf(x) = \dfrac{1}{3}$.

Reflection

1. What do I already know about functions that could guide my learning in this section?
2. Am I able to find the composite function of 1 or 2 functions?
3. What have I learnt in this section or chapter that I am still unclear of?

Advanced Intermediate Basic

Exercise 5C

1. For each of the following pairs of functions, find fg(x) and gf(x).
   - **(a)** $f(x) = 9 - x$, $g(x) = 3x + 4$
   - **(b)** $f(x) = 2x - 3$, $g(x) = x^2 + 5$
   - **(c)** $f(x) = x - 1$, $g(x) = \frac{4}{x}$
   - **(d)** $f(x) = 2x - 1$, $g(x) = \frac{4}{x} - 3$
   - **(e)** $f(x) = x + 2$, $g(x) = \frac{3}{x + 1}$
   - **(f)** $f(x) = 3x$, $g(x) = 4x + 5$
   - **(g)** $f(x) = x - 2$, $g(x) = \frac{2}{x} + 3$
   - **(h)** $f(x) = x^2$, $g(x) = \frac{1 + 2x}{x - 1}$
   - **(i)** $f(x) = 5 - 2x$, $g(x) = \frac{x + 2}{x - 1}$

2. It is given that $f : x \mapsto x + 1$ and $g : x \mapsto 3x + 2$.
   - **(i)** Find the composite functions fg(x) and gf(x).
   - **(ii)** Find the values of fg(3), gf(3), fg(−1) and gf(−1).

3. If $f(x) = 2x^2 + 3$ and $g(x) = 2x + 1$, find
   - **(i)** the composite functions fg(x) and gf(x),
   - **(ii)** the values of fg(−1), gf(−1), fg(3) and gf(3).

4. If $f : x \mapsto ax + b$, where a and b are constants, $g : x \mapsto x + 7$, fg(1) = 7 and fg(2) = 15,
   - **(i)** find gf(5),
   - **(ii)** solve the equation gf(x) = 14.

5. Functions f and g are defined by $f : x \mapsto kx - 3$, where k is a constant, and $g : x \mapsto 2x + 5$. Find
   - **(i)** an expression for fg(x),
   - **(ii)** the value of k for which fg(x) = gf(x).

6. Given that $f(x) = \frac{2}{x}$ and $g(x) = 3x - 4$,
   - **(i)** find fg(x) and gf(x),
   - **(ii)** show that there are no real values of x for which fg(x) = gf(x).

7. It is given that $f(x) = 2x + 3$ and $g(x) = ax + b$.
   - **(i)** Find the composite functions fg(x) and gf(x).
   - **(ii)** Find the values of a and b such that gf(x) = x for all values of x.
   - **(iii)** With these values of a and b, solve the equation fg(x) = 3gf(x).

8. If $f : x \mapsto 2x - 1$ and $g : x \mapsto x^2 + 5$, find
   - **(i)** fg(x) and gf(x),
   - **(ii)** the values of x for which fg(x) = gf(x).

Looking Back

In this chapter, we have learnt about the concepts of a **function** and its **notation**. In fact, we have already begun applying the ideas of functions in Book 1 when finding an output such as speed given distance and time! With sufficient knowledge, functions enable us to **model** many real-world situations. These models can then be used to give useful output in the form of predictions, allowing us to make well-informed decisions. In more advanced levels of mathematics, you might uncover even more useful applications of functions.

Summary

1. A **relation** connects elements in set A (**domain**) to elements in set B (**codomain**) according to the definition of the relation.

2. A **function** is a relation in which every element in the domain has a unique **image** in the codomain.

3. The **range** of a function f is the set of values of $f(x)$ (images under f) for the given domain.

4. If a function f maps x to y, then its **inverse** function f^{-1} maps y to x.

5. Two functions f and g can be combined to produce **composite functions** fg and gf such that

$$fg(x) = f(g(x)) \text{ and } gf(x) = g(f(x)).$$

In general, $fg(x)$ and $gf(x)$ are different functions.

CHAPTER 6

Further Trigonometry

What is the circumference of the Earth? How far is the moon from the Earth?

Such measurements are difficult or impossible to obtain directly.

Around 240 BC, Eratosthenes, a famous mathematician, measured the circumference of the Earth indirectly by modelling it as a triangle (or a sector, as some have suggested) and using trigonometry to determine the distance. The same idea of using triangles as **models** has also been applied to find other real-world measurements and locations.

In this chapter, we will investigate how we can extend our knowledge of trigonometry to find the unknown side or angle of a given triangle.

Learning Outcomes

What will we learn in this chapter?

- What the relationship between trigonometric ratios of acute angles and obtuse angles is
- How to find the area of a triangle
- How to find the unknown sides and angles of a triangle, given
  - two sides and one angle, or
  - two angles and one side, or
  - three sides
- Why trigonometry has useful applications in real life

Introductory Problem

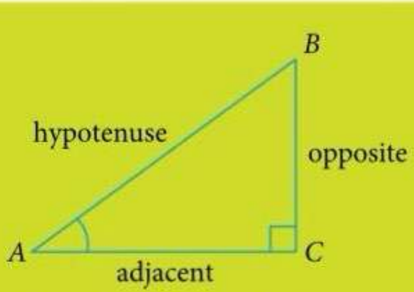
In $\triangle ABC$, $\sin B = \frac{2}{5}$, $AB = 5$ cm and $AC = 10$ cm.

Without finding the angle B, sketch all the possible triangles for $\triangle ABC$.

6.1 Sine and cosine of obtuse angles

A. Trigonometric ratios of a right-angled triangle (Recap)

In Book 2, we learnt about the trigonometric ratios that apply to acute angles in a right-angled triangle.



In a right-angled triangle ABC, if $\angle C = 90°$,

then $\frac{BC}{AB} = \frac{\text{opp}}{\text{hyp}}$ is called the **sine** of $\angle A$, or $\sin A = \frac{\text{opp}}{\text{hyp}}$,

$\frac{AC}{AB} = \frac{\text{adj}}{\text{hyp}}$ is called the **cosine** of $\angle A$, or $\cos A = \frac{\text{adj}}{\text{hyp}}$,

$\frac{BC}{AC} = \frac{\text{opp}}{\text{adj}}$ is called the **tangent** of $\angle A$, or $\tan A = \frac{\text{opp}}{\text{adj}}$.

Big Idea

Notations

Notations such as sin A, cos A and tan A are used to represent mathematical objects or operations in a concise manner. Understanding the mathematical notations used in trigonometry will help in the study of this topic.

What if A is an obtuse angle as shown in Fig. 6.1?

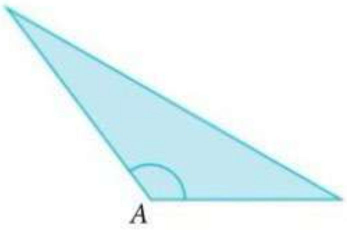


Fig. 6.1

To find the sides and angles of an obtuse-angled triangle, we will need to extend the definitions of trigonometric ratios. In this chapter, we will learn about two trigonometric ratios, namely the sine and cosine ratios, of obtuse angles.

B. Sine and cosine of obtuse angles

Fig. 6.2 shows a circle with centre O and radius r units.
$P(x, y)$ is a point on the circle and $\triangle OPQ$ is a right-angled triangle. A is an acute angle.

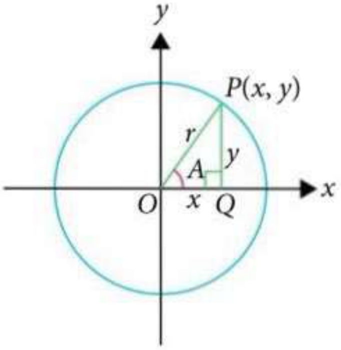


Fig. 6.2

Big Idea

Diagrams
The Cartesian plane allows us to represent and visualise points. The circle in Fig. 6.2 shows how the lengths of OQ and PQ are related to the point $P(x, y)$. Recall that the x- and y-coordinates of a point are either positive or negative with reference to the origin O.

$$\therefore \sin A = \frac{\text{opp}}{\text{hyp}} = \frac{y}{r} \text{ and } \cos A = \frac{\text{adj}}{\text{hyp}} = \frac{x}{r}$$

In other words, we have extended the definition of the sine and cosine ratios of an angle A in terms of the coordinates of a point $P(x, y)$:

$$\sin A = \frac{y}{r} \text{ and } \cos A = \frac{x}{r}$$

The value of r is always positive since it represents the length of the radius. If A is an acute angle, then x and y are positive. In other words, $\sin A$ and $\cos A$ are positive if A is acute.

Fig. 6.3 shows a circle with centre O and radius r units.
$P(x, y)$ is a point on the circle and A is an **obtuse** angle.

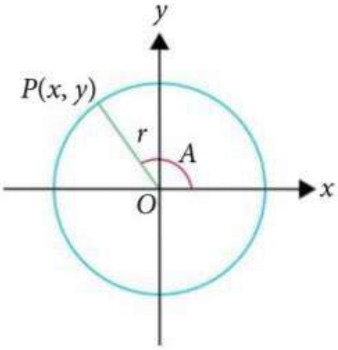


Fig. 6.3

If A is an obtuse angle, x is negative while r and y remain positive.
Using the extended definitions,

$$\sin A = \frac{y}{r} \text{ and } \cos A = \frac{x}{r},$$

$\sin A$ is positive but $\cos A$ is negative.

Investigation

Relationship between trigonometric ratios of acute and obtuse angles

Use your calculator to evaluate the sine and cosine ratios of each of the following pairs of angles, leaving your answers correct to 3 significant figures where necessary.

	A	$180° - A$	$\sin A$	$\sin (180° - A)$	$\cos A$	$\cos (180° - A)$
(a)	30°	150°				
(b)	76°	104°				
(c)	111°	69°				
(d)	167°	13°				

Table 6.1

1. What do you notice about $\sin A$ and $\sin (180° - A)$?
2. What do you notice about $\cos A$ and $\cos (180° - A)$?
3. Which trigonometric ratio is positive, and which one is negative?

From the above Investigation, we observe that the sine and cosine ratios of acute angles are always positive. While the sine ratio of an obtuse angle is still positive, the cosine ratio of an obtuse angle is negative.

In general, for any angle A that is acute or obtuse,

$$\sin A = \sin (180° - A)$$
$$\cos A = -\cos (180° - A)$$

How do we know that this relationship is always true?

From Fig. 6.3, we observe that a right-angled $\triangle OPQ$ can be formed in the 2nd quadrant, as shown in Fig. 6.4. $\angle POQ$ is an acute angle that is equal to $180° - A$ (adj. $\angle$s on a str. line).

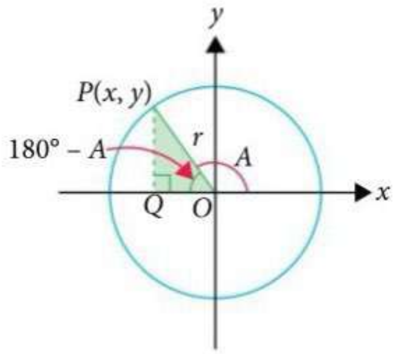


Fig. 6.4

Using the extended definitions,

$$\sin A = \frac{y}{r} \text{ and } \cos A = \frac{x}{r} \qquad \text{— (1)}$$

In the right-angled $\triangle OPQ$,
length $OQ = -x$ units, since $x < 0$ but length > 0
and length $PQ = y$ units.

Since 180° – A is an acute angle, we can use the definitions for acute angles:

$$\sin (180° - A) = \frac{\text{opp}}{\text{hyp}} = \frac{PQ}{OP} = \frac{y}{r}$$

$$\text{and } \cos (180° - A) = \frac{\text{adj}}{\text{hyp}} = \frac{OQ}{OP} = \frac{-x}{r} \qquad \text{— (2)}$$

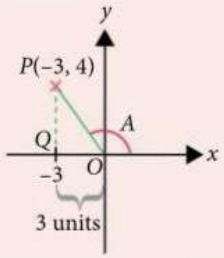
Comparing (1) and (2),

$$\sin A = \frac{y}{r} = \sin (180° - A)$$

$$\text{but } \cos A = \frac{x}{r} = -\cos (180° - A)$$

Attention

For example, if the coordinates of P are (–3, 4), then $x = -3$, but the length of OQ is 3 units (i.e. $-x$ units).



Worked Example 1

Relationship between trigonometric ratios of acute and obtuse angles

Given that sin 55° = 0.819 and cos 136° = –0.719 when corrected to 3 significant figures, find the value of each of the following **without using a calculator**.

(a) sin 125° **(b)** cos 44°

Solution

(a) $\sin 125° = \sin (180° - 125°)$
$= \sin 55°$
$= 0.819$

(b) $\cos 44° = -\cos (180° - 44°)$
$= -\cos 136°$
$= -(-0.719)$
$= 0.719$

Practise Now 1

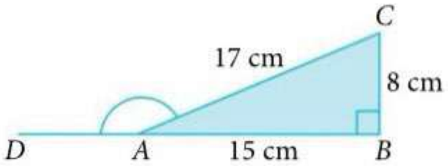
Similar and Further Questions
Exercise 6A
Questions 1(a)–(f), 2(a), (b), 3(a)–(c)

1. Given that sin 84° = 0.995 and cos 129° = –0.629 when corrected to 3 significant figures, find the value of each of the following **without using a calculator**.
   (a) sin 96° **(b)** cos 51°
2. Given that sin 172° = 0.139 and cos 40° = 0.766 when corrected to 3 significant figures, find the value of sin 8° – cos 140° **without using a calculator**.

Worked Example 2

Relationship between trigonometric ratios of acute and obtuse angles

In the figure, DAB is a straight line, $\angle ABC = 90°$, $AB = 15$ cm, $BC = 8$ cm and $AC = 17$ cm.



Find the value of each of the following, giving your answer in exact form.

(a) $\sin \angle DAC$

(b) $\cos \angle DAC$

(c) $\tan \angle ACB$

Solution

(a)
$$\begin{aligned}\sin \angle DAC &= \sin (180° - \angle DAC) \qquad \sin A = \sin (180° - A)\\ &= \sin \angle BAC\\ &= \frac{\text{opp}}{\text{hyp}}\\ &= \frac{BC}{AC}\\ &= \frac{8}{17}\end{aligned}$$

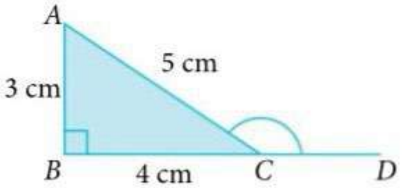
(b)
$$\begin{aligned}\cos \angle DAC &= -\cos (180° - \angle DAC) \qquad \cos A = -\cos (180° - A)\\ &= -\cos \angle BAC\\ &= -\frac{\text{adj}}{\text{hyp}}\\ &= -\frac{AB}{AC}\\ &= -\frac{15}{17}\end{aligned}$$

(c)
$$\begin{aligned}\tan \angle ACB &= \frac{\text{opp}}{\text{adj}}\\ &= \frac{AB}{BC}\\ &= \frac{15}{8}\end{aligned}$$

Practise Now 2

Similar and Further Questions
Exercise 6A
Questions 4, 5, 10–12

1. In the figure, BCD is a straight line, $\angle ABC = 90°$, $AB = 3$ cm, $BC = 4$ cm and $AC = 5$ cm.



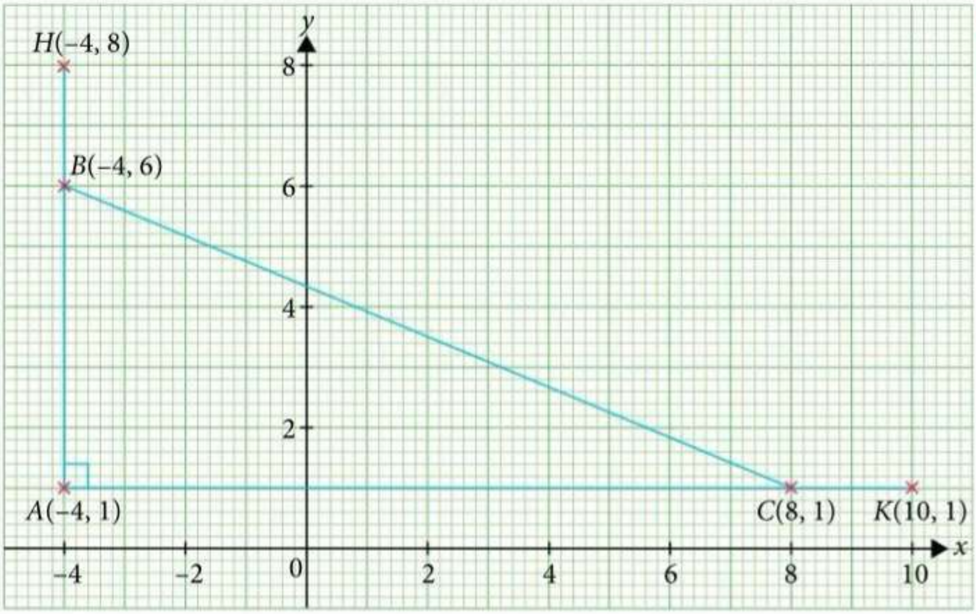
Find the value of each of the following, giving your answer in exact form.

(a) $\sin \angle ACD$

(b) $\cos \angle ACD$

(c) $\tan \angle BAC$

2. The figure shows $\triangle ABC$ with vertices $A(-4, 1)$, $B(-4, 6)$ and $C(8, 1)$. $H(-4, 8)$ a K(0, 1 lie on AB produced and AC produced respectively.



Attention

The scale used for the x-axis and the y-axis must be the same for the trigonometric ratios of the angles to be correct. If the scale used is different, the lengths of the sides of the triangle would not match the distance between two points represented by coordinates.

(i) Find the length of BC.

(ii) State the value of each of the following.

- **(a)** $\sin \angle HBC$
- **(b)** $\cos \angle BCK$
- **(c)** $\tan \angle ABC$

OPEN **(iii)** Given that point D lies on the line $x = 8$, write down a possible set of coordinates of point D such that $\cos \angle ABD$ is negative.

Worked Example 3

Solving simple trigonometric equations

Given that $0° \leqslant x \leqslant 180°$, find the possible value(s) of x for each of the following equations.

(a) $\sin x = 0.45$ **(b)** $\cos x = -0.834$

Solution

(a) Since $\sin x$ is positive, x can either be an acute angle or an obtuse angle.

$\sin x = 0.45$

$x = \sin^{-1} 0.45 = 26.7°$ (to 1 d.p.)

or $180° - 26.7° = 153.3°$ (to 1 d.p.)

$\therefore x = 26.7°$ or $153.3°$

(b) Since $\cos x$ is negative, x is an obtuse angle.

$\cos x = -0.834$

$x = \cos^{-1} (-0.834)$

$= 146.5°$ (to 1 d.p.)

Problem-solving Tip

Computing $\sin^{-1} 0.45$ using a calculator will only give you the acute angle. To find the obtuse angle, subtract the acute angle from 180°.

Attention

Always leave your answer in degrees correct to 1 decimal place, unless otherwise stated.

Practise Now 3

Similar and Further Questions
Exercise 6A
Questions 6(a)–(d), 7(a)–(d), 8(a)–(d), 9(a)–(f), 13(a), (b)

Given that $0° \leqslant x \leqslant 180°$, find the possible value(s) of x for each of the following equations.

(a) $\sin x = 0.415$
(b) $\cos x = -0.234$
(c) $\cos x = 0.104$

Introductory Problem Revisited

How many possible triangles did you sketch in the Introductory Problem?

In $\triangle ABC$, $\sin B = \frac{2}{5}$, $AB = 5$ cm and $AC = 10$ cm.

Since $\sin B$ is a positive ratio, B can be an acute angle or an obtuse angle. Are there more triangles that can be sketched?

Reflection

What have I learnt in this section that builds on my previous understanding of trigonometric ratios of acute angles?

Basic | Intermediate | Advanced

Exercise 6A

1. Express each of the following as a trigonometric ratio of the acute angle.

(a) $\sin 110°$
(b) $\sin 176°$
(c) $\sin 98°$
(d) $\cos 99°$
(e) $\cos 107°$
(f) $\cos 175°$

2. Given that $\sin 32° = 0.530$ and $\cos 145° = -0.819$ when corrected to 3 significant figures, find the value of each of the following without using a calculator.

(a) $\sin 148°$
(b) $\cos 35°$

3. Given that $\sin 45° = \cos 45° = 0.707$ when corrected to 3 significant figures, find the value of each of the following without using a calculator.

(a) $2 \cos 45° + 3 \sin 135°$
(b) $3 \cos 135° + 4 \sin 135°$
(c) $\cos 135° - 2 \sin 45°$

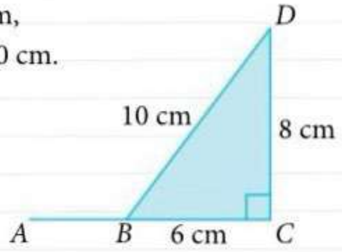
Exercise 6A

4. In the figure, ABC is a straight line, $\angle BCD = 90^\circ$, $BC = 6$ cm, $CD = 8$ cm and $BD = 10$ cm. Find the value of each of the following.

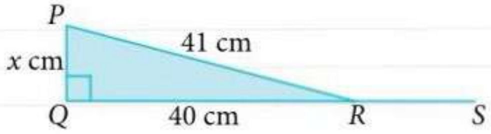
(a) $\sin \angle ABD$

(b) $\cos \angle DBA$

(c) $\tan \angle CBD$



5. In the figure, QRS is a straight line, $\angle PQR = 90^\circ$, $PQ = x$ cm, $QR = 40$ cm and $PR = 41$ cm.



(i) Find the value of x.

(ii) Find the value of each of the following.

(a) $\sin \angle PRS$

(b) $\cos \angle PRS$

(c) $\tan \angle PRQ$

6. Find the acute angle whose sine is

(a) 0.52, **(b)** 0.75,

(c) 0.875, **(d)** 0.3456.

7. Find the obtuse angle whose sine is

(a) 0.52, **(b)** 0.75,

(c) 0.875, **(d)** 0.3456.

8. Find the acute angle whose cosine is

(a) 0.67, **(b)** 0.756,

(c) 0.5, **(d)** 0.985.

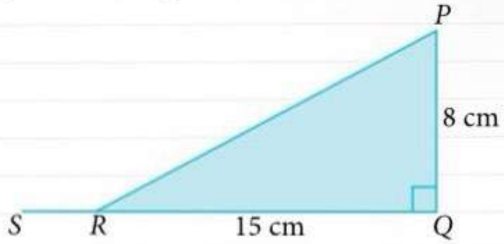
9. Given that $0^\circ \leqslant x \leqslant 180^\circ$, find the possible value(s) of x for each of the following equations.

(a) $\sin x = 0.753$ **(b)** $\sin x = 0.952$

(c) $\sin x = 0.4714$ **(d)** $\cos x = -0.238$

(e) $\cos x = -0.783$ **(f)** $\cos x = 0.524$

10. In the figure, SRQ is a straight line, $\angle PQR = 90^\circ$, $PQ = 8$ cm and $QR = 15$ cm.



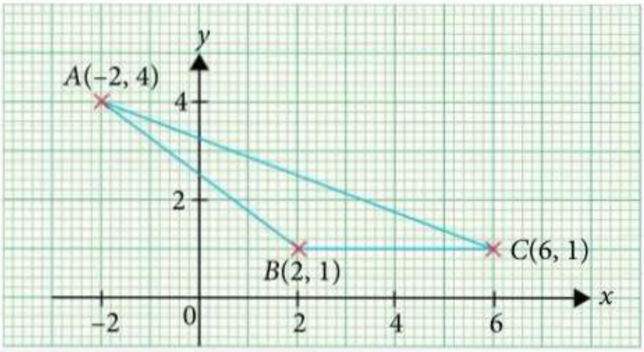
Find the value of each of the following, giving your answer in exact form.

(a) $\sin \angle PRS$

(b) $\cos \angle SRP$

(c) $\tan \angle PRQ$

11. The figure shows $\triangle ABC$ with vertices $A(-2, 4)$, $B(2, 1)$ and $C(6, 1)$.



(i) Find the value of each of the following.

(a) $\sin \angle ABC$

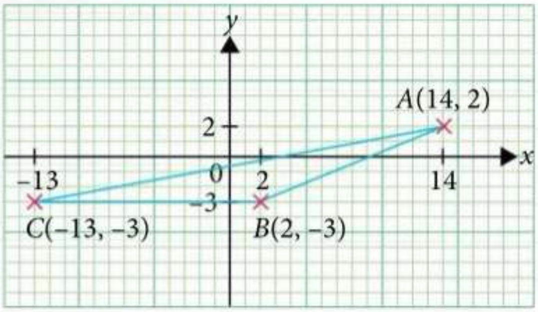
(b) $\cos \angle ABC$

(c) $\tan \angle ACB$

OPEN **(ii)** Given that point D lies on the line $y = 1$, write down a set of possible coordinates of point D such that $\cos \angle ADB$ is negative.

Exercise 6A

12. The figure shows $\triangle ABC$ with vertices $A(14, 2)$, $B(2, -3)$ and $C(-13, -3)$.



Find the value of each of the following.

(a) $\sin \angle ABC$

(b) $\cos \angle ABC$

(c) $\tan \angle ACB$

13. Given that $0° < x < 180°$, find the possible value(s) of x for each of the following equations.

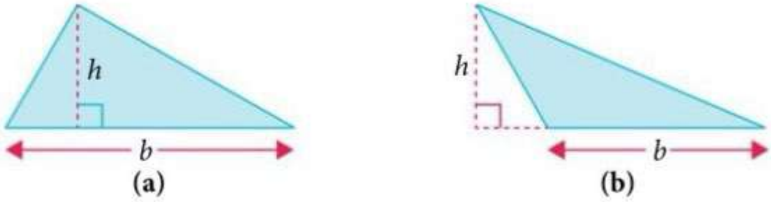
(a) $\sin (x + 10°) = 0.47$

(b) $\cos (x - 10°) = -0.56$

6.2 Area of triangle

The area of a triangle is given by the formula:

$$\text{area of triangle} = \frac{1}{2} \times \text{base} \times \text{height or } \frac{1}{2}bh.$$

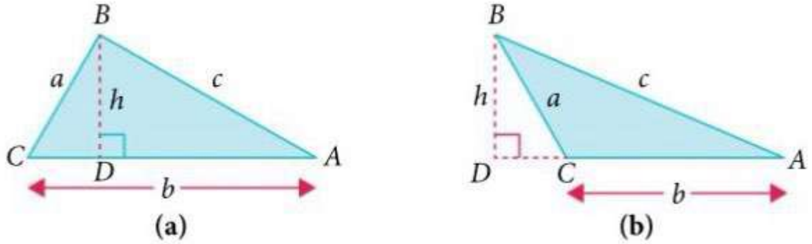


(a) (b)

Fig. 6.5

Can we find the area of a triangle without knowing its height?

Fig. 6.6 shows two triangles.
In Fig. 6.6(**a**), $\angle C$ is acute, while in Fig. 6.6(**b**), $\angle C$ is obtuse.



(a) (b)

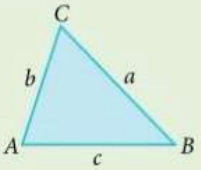
Fig. 6.6

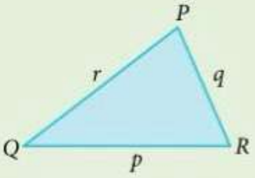
Big Idea

Notations

As a convention, we use capital letters to denote the vertex of a triangle and lowercase letters to denote the lengths of the sides. Each side of a triangle is labelled by the corresponding lowercase letter of the opposite vertex.

For example:





In Fig. 6.6(**a**), consider $\triangle BCD$.

$$\sin C = \frac{\text{opp}}{\text{hyp}}$$
$$= \frac{h}{a}$$
$$h = a \sin C$$
$$\therefore \text{ area of } \triangle ABC = \frac{1}{2}bh$$
$$= \frac{1}{2}b(a \sin C)$$
$$= \frac{1}{2}ab \sin C$$

In Fig. 6.6(**b**), consider $\triangle BCD$.

$$\sin C = \frac{\text{opp}}{\text{hyp}}$$
$$= \frac{h}{a}$$
$$\therefore \sin \angle ACB = \sin (180° - \angle ACB)$$
$$= \sin (\angle BCD)$$
$$= \frac{h}{a}$$
$$h = a \sin \angle ACB$$
$$= a \sin C$$
$$\therefore \text{ area of } \triangle ABC = \frac{1}{2}bh$$
$$= \frac{1}{2}b(a \sin C)$$
$$= \frac{1}{2}ab \sin C$$

Attention

In the formula $\frac{1}{2}ab \sin C$, notice that the angle C is in between the two sides a and b, i.e. C is called the **included** angle.

Big Idea

Equivalence

The formulae $\frac{1}{2}ab \sin C$, $\frac{1}{2}ac \sin B$ and $\frac{1}{2}bc \sin A$ are equivalent as they will give the same area of $\triangle ABC$. We can decide which formula to use based on the information we have been given.

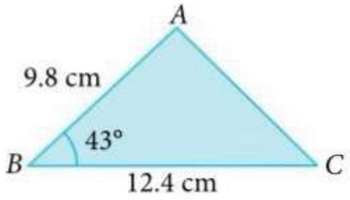
By considering sin A and sin B in a similar way, we can show that:

$$\text{Area of } \triangle ABC = \frac{1}{2}ab \sin C = \frac{1}{2}bc \sin A = \frac{1}{2}ac \sin B$$

Worked Example 4

Finding area of triangle

Find the area of $\triangle ABC$, given that $AB = 9.8$ cm, $BC = 12.4$ cm and $\angle ABC = 43°$.



Solution

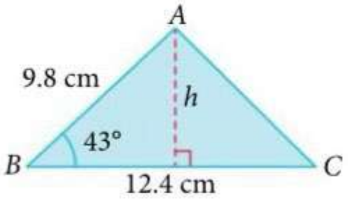
Method 1:

We have $a = 12.4$, $c = 9.8$ and $B = 43°$.

$$\text{Area of } \triangle ABC = \frac{1}{2}ac \sin B$$
$$= \frac{1}{2} \times 12.4 \times 9.8 \times \sin 43°$$
$$= 41.4 \text{ cm}^2 \text{ (to 3 s.f.)}$$

Method 2:

Let the height of the triangle be h cm.



$$\sin 43° = \frac{h}{9.8}$$
$$h = 9.8 \sin 43°$$

$$\text{Area of } \triangle ABC = \frac{1}{2} \times \text{base} \times \text{height}$$
$$= \frac{1}{2} \times 12.4 \times 9.8 \sin 43°$$
$$= 41.4 \text{ cm}^2 \text{ (to 3 s.f.)}$$

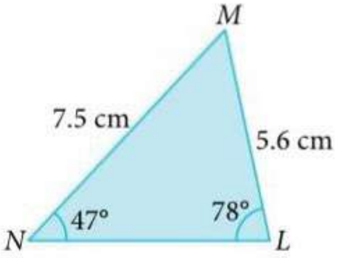
Reflection

Which method do you prefer? Why?

Practise Now 4

Similar and Further Questions
Exercise 6B
Questions 1(a)–(f), 2–7

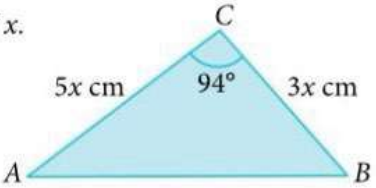
1. Find the area of $\triangle ABC$, given that $BC = 31.8$ m, $AC = 24.8$ m and $\angle ACB = 49°$.
2. Find the area of $\triangle MNL$, given that $MN = 7.5$ cm, $ML = 5.6$ cm, $\angle MNL = 47°$ and $\angle MLN = 78°$.



Worked Example 5

Solving problem involving area of triangle

In $\triangle ABC$, $AC = 5x$ cm, $CB = 3x$ cm and $\angle ACB = 94°$.
Given that the area of $\triangle ABC$ is 145 cm², find the value of x.



Solution

We have $a = 3x$, $b = 5x$ and $C = 94°$.

$$\text{Area of } \triangle ABC = \frac{1}{2}ab \sin C$$
$$145 = \frac{1}{2} \times 3x \times 5x \times \sin 94°$$
$$= 7.5x^2 \sin 94°$$
$$x^2 = \frac{145}{7.5 \sin 94°}$$
$$x = \sqrt{\frac{145}{7.5 \sin 94°}} \text{ (since } x > 0)$$
$$= 4.40 \text{ (to 3 s.f.)}$$

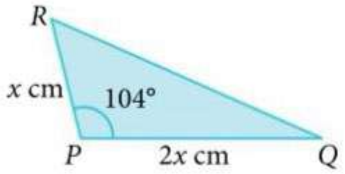
Practise Now 5

Similar and Further Questions

Exercise 6B

Questions 8–13

1. In $\triangle PQR$, $PQ = 2x$ cm, $PR = x$ cm and $\angle QPR = 104°$.
Given that the area of $\triangle PQR$ is 12.5 cm^2, find the value of x.



2. The area of acute-angled triangle XYZ is 12 cm^2. Given that $XY = 5$ cm and $YZ = 6$ cm, find angle XYZ.

Reflection

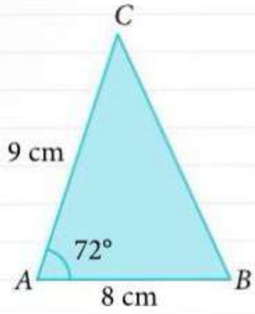
1. What are the different methods of finding the area of a triangle?
2. Compare these methods. Which of these methods do you find more useful? Why?

Advanced Intermediate Basic

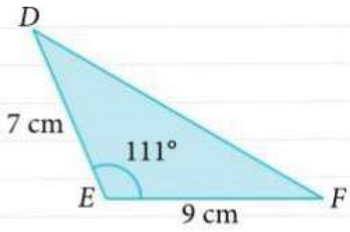
Exercise 6B

1. Find the area of each of the following figures.

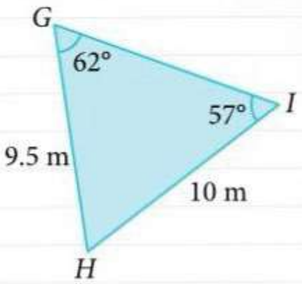
(a)



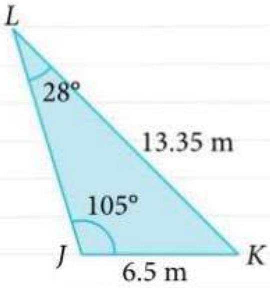
(b)



(c)

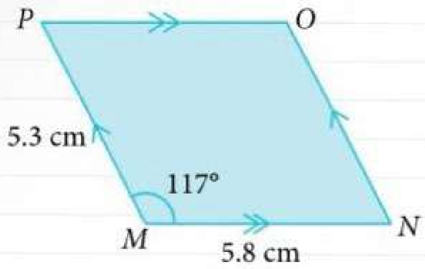


(d)

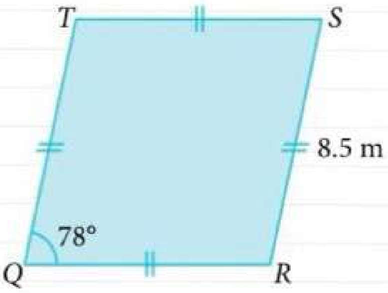


Exercise 6B

(e)



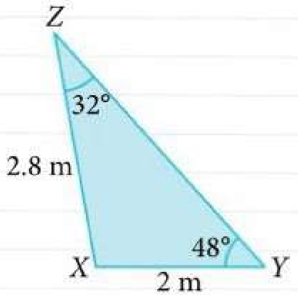
(f)



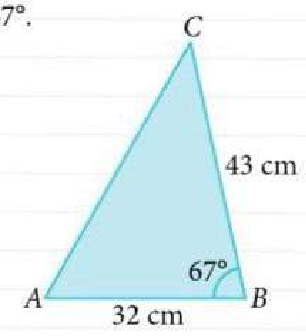
2. Find the area of $\triangle ABC$, given that $AB = 22$ cm, $AC = 15$ cm and $\angle BAC = 45°$.

3. In $\triangle PQR$, $\angle P = 72°$, $q = 152$ cm and $r = 125$ cm. Find the area of $\triangle PQR$.

4. Find the area of $\triangle XYZ$, given that $XY = 2$ m, $XZ = 2.8$ m, $\angle XYZ = 48°$ and $\angle XZY = 32°$.



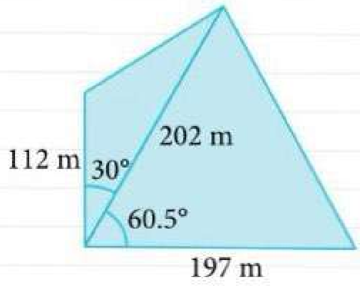
5. In $\triangle ABC$, $AB = 32$ cm, $BC = 43$ cm and $\angle ABC = 67°$.



(i) Find the area of $\triangle ABC$.

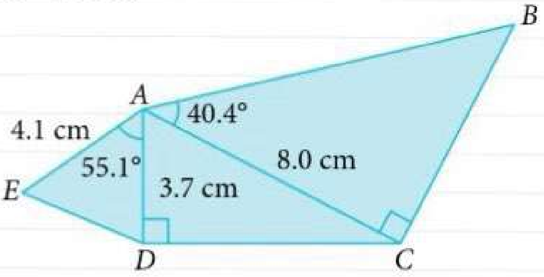
(ii) Hence, find the perpendicular distance from A to BC.

6. The diagram shows the plan of two neighbouring estates in the form of two triangles.



Calculate the total area of the two estates.

7. In the figure, $\angle ADC = \angle ACB = 90°$, $\angle EAD = 55.1°$, $\angle CAB = 40.4°$, $AE = 4.1$ cm, $AD = 3.7$ cm and $AC = 8.0$ cm.



Find

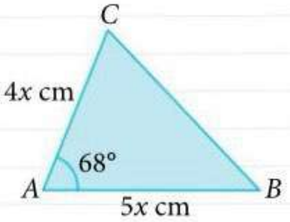
(i) $\angle ACD$,

(ii) the length of AB,

(iii) the area of $\triangle AED$.

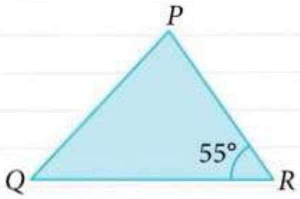
Exercise 6B

8. In $\triangle ABC$, $AB = 5x$ cm, $AC = 4x$ cm and $\angle BAC = 68°$.



Given that the area of $\triangle ABC$ is 97 cm^2, find the value of x.

9. In $\triangle PQR$, $\angle PRQ = 55°$, $3QR = 4PR$ and the area of $\triangle PQR$ is 158 cm^2.



Find the length of QR.

10. In $\triangle XYZ$, $XY = 13$ cm and $YZ = 16.2$ cm. Given that the area of $\triangle XYZ$ is 59.5 cm^2, find the two possible values of $\angle XYZ$.

11. The length of each side of a rhombus is 15 cm. Given that the rhombus has an area of 40 cm^2, find the angles of the rhombus.

12. The diagonals of a parallelogram have lengths x cm and y cm. They intersect at 150°. Given that the area of the parallelogram is 100 cm^2, find a possible value of x and the corresponding value of y.

13. In quadrilateral $ABCD$, $AB = 3.2$ cm, $BC = 5.1$ cm, $\angle CBD = 34.4°$ and the length of the diagonal BD is 7.5 cm. Given further that the area of $\triangle ABD$ is 11.62 cm^2 and $\angle ABD$ is obtuse, find

(i) the area of $\triangle BCD$,

(ii) $\angle ABD$.

6.3 Sine Rule

In Book 2, we learnt to apply Pythagoras' Theorem and trigonometric ratios to find the unknown sides and angles of a right-angled triangle.

However, most of the triangles that we encounter in the real world, such as those shown in Fig. 6.7, are not right-angled triangles.

Can you use triangles to create a **model** of the parts of the structures and objects in Fig. 6.7?

Big Idea

Models

Geometrical shapes such as triangles can help us to model real-world objects. The knowledge of the properties of triangles can then enable us to compute measurements such as perimeter and area.

(a) Electricity tower

(b) Bicycle

(c) Sydney Harbour Bridge

Fig. 6.7

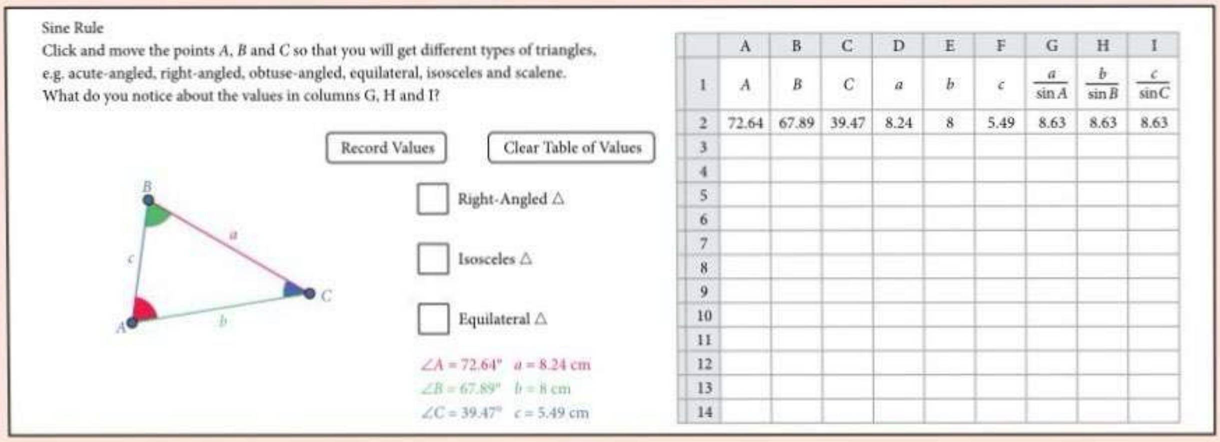
How can we find the length of one side of a non-right angled triangle when given the other dimensions of the triangle?

Investigation

Sine Rule

Go to www.sl-education.com/tmsoupp4/pg202 or scan the QR code on the right and open the geometry software template 'Sine Rule'.

Fig. 6.8 shows a triangle ABC and a table of values.



Sine Rule

Click and move the points A, B and C so that you will get different types of triangles, e.g. acute-angled, right-angled, obtuse-angled, equilateral, isosceles and scalene.

What do you notice about the values in columns G, H and I?

☐ Right-Angled △

☐ Isosceles △

☐ Equilateral △

$\angle A = 72.64°$ $a = 8.24$ cm
$\angle B = 67.89°$ $b = 8$ cm
$\angle C = 39.47°$ $c = 5.49$ cm

	A	B	C	D	E	F	G	H	I
1	A	B	C	a	b	c	$\frac{a}{\sin A}$	$\frac{b}{\sin B}$	$\frac{c}{\sin C}$
2	72.64	67.89	39.47	8.24	8	5.49	8.63	8.63	8.63
3									
4									
5									
6									
7									
8									
9									
10									
11									
12									
13									
14									

Fig. 6.8

1. The labelling of the sides of the triangle with reference to the vertices is important. Copy and complete the following.
   - **(a)** The length of the side of the triangle opposite vertex A is labelled a.
   - **(b)** The length of the side of the triangle opposite vertex B is labelled .
   - **(c)** The length of the side of the triangle opposite vertex C is labelled .
2. Click and drag each of the vertices A, B and C to get different types of triangles. To obtain special triangles such as a right-angled triangle, an isosceles triangle and an equilateral triangle, click on the respective buttons in the template.

   For each triangle, click on 'Record Values' then copy and complete Table 6.2.

Types of triangles	$\angle A$	$\angle B$	$\angle C$	a	b	c	$\frac{a}{\sin A}$	$\frac{b}{\sin B}$	$\frac{c}{\sin C}$
Right-angled triangle									
Isosceles triangle									
Equilateral triangle									
Scalene triangle									
Obtuse-angled triangle									

Table 6.2

3. What do you notice about the last 3 columns in Table 6.2?
4. Hence, write down a formula relating the quantities in the last 3 columns of the table. This is called the **Sine Rule**. Notice that for each fraction, *the side must be opposite the angle*.
5. Do you think the relationship $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$ is also true? Explain your answer.

6. Copy and complete the following:
The lengths of the sides of a triangle are p to the sine of the angles opposite the sides.

Recall

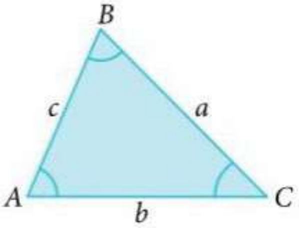
In a triangle,
- the largest angle is opposite the longest side;
- the smallest angle is opposite the shortest side.

From the above Investigation, we can conclude:

The Sine Rule

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \quad \text{or} \quad \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c},$$

where A, B and C are the three interior angles of $\triangle ABC$ opposite the sides whose lengths are a, b and c respectively.



We can prove the Sine Rule using the formula for the area of a triangle obtained from the previous section. For any triangle ABC,

$$\frac{1}{2}ab\sin C = \frac{1}{2}bc\sin A = \frac{1}{2}ac\sin B.$$

Dividing each side by $\frac{1}{2}abc$,

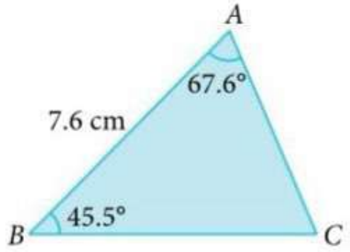
$$\frac{\frac{1}{2}bc\sin A}{\frac{1}{2}abc} = \frac{\frac{1}{2}ac\sin B}{\frac{1}{2}abc} = \frac{\frac{1}{2}ab\sin C}{\frac{1}{2}abc}.$$

$$\therefore \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Worked Example 6

Using Sine Rule when given 2 angles and 1 side

In $\triangle ABC$, $\angle A = 67.6°$, $\angle B = 45.5°$ and $AB = 7.6$ cm.



Find

(i) $\angle C$,

(ii) the length of BC,

(iii) the length of AC.

Solution

(i) $\angle C = 180° - 67.6° - 45.5°$ ($\angle$ sum of $\triangle$)

$= 66.9°$

(ii) Using Sine Rule,

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

$$\frac{a}{\sin 67.6°} = \frac{7.6}{\sin 66.9°}$$

$$a = \frac{7.6 \sin 67.6°}{\sin 66.9°}$$

$$= 7.6390 \text{ (to 5 s.f.)}$$

$$= 7.64 \text{ cm (to 3 s.f.)}$$

$\therefore BC = 7.64$ cm

(iii) **Method 1:**

Using Sine Rule,

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\frac{b}{\sin 45.5°} = \frac{7.6}{\sin 66.9°}$$

$$b = \frac{7.6 \sin 45.5°}{\sin 66.9°}$$

$$= 5.89 \text{ cm (to 3 s.f.)}$$

$\therefore AC = 5.89$ cm

Method 2:

Using Sine Rule,

$$\frac{b}{\sin B} = \frac{a}{\sin A}$$

$$\frac{b}{\sin 45.5°} = \frac{7.6390}{\sin 67.6°}$$

$$b = \frac{7.6390 \sin 45.5°}{\sin 67.6°}$$

$$= 5.89 \text{ cm (to 3 s.f.)}$$

$\therefore AC = 5.89$ cm

Reflection

Which method do you prefer? Why?

Practise Now 6

Similar and Further Questions

Exercise 6C

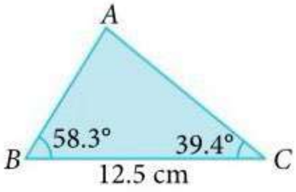
Questions 1(a)–(c), 2, 3, 7–11, 15

In $\triangle ABC$, $\angle B = 58.3°$, $\angle C = 39.4°$ and $BC = 12.5$ cm. Find

(i) $\angle A$,

(ii) the length of AB,

(iii) the length of AC.



Worked Example 7

Using Sine Rule when given 2 sides and 1 non-included angle

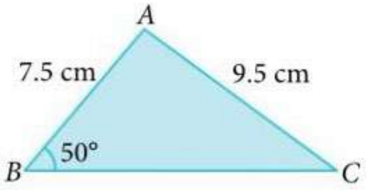
In $\triangle ABC$, $\angle B = 50°$, $AB = 7.5$ cm and $AC = 9.5$ cm.

Find

(i) $\angle C$,

(ii) $\angle A$,

(iii) the length of BC.



Solution

(i) Using Sine Rule,

$$\frac{\sin C}{c} = \frac{\sin B}{b}$$

$$\frac{\sin C}{7.5} = \frac{\sin 50°}{9.5}$$

$$\sin C = \frac{7.5 \sin 50°}{9.5}$$

$$= 0.604\,77 \text{ (to 5 s.f.)}$$

$$\angle C = \sin^{-1} 0.604\,77$$

$$= 37.212° \text{ (to 3 d.p.)}$$

$$\text{or } 180° - 37.212° = 142.788°$$

Since $\angle B$ + obtuse $\angle C = 50° + 142.788° = 192.788° > 180°$, it is not possible for $\angle C$ to be obtuse.

$\therefore \angle C = 37.2°$ (to 1 d.p.)

(ii) $\angle A = 180° - 50° - 37.212°$ ($\angle$ sum of $\triangle$)

$= 92.8°$ (to 1 d.p.)

(iii) Using Sine Rule,

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{a}{\sin 92.788°} = \frac{9.5}{\sin 50°}$$

$$a = \frac{9.5 \sin 92.788°}{\sin 50°}$$

$$= 12.4 \text{ cm (to 3 s.f.)}$$

$\therefore BC = 12.4$ cm

Practise Now 7

Similar and Further Questions

Exercise 6C

Questions 4(a)–(c), 5, 6, 12

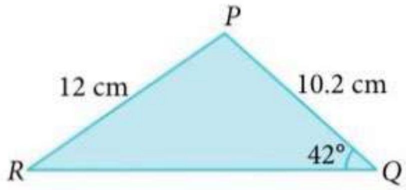
1. In $\triangle PQR$, $\angle Q = 42°$, $PR = 12$ cm and $PQ = 10.2$ cm.

Find

(i) $\angle R$,

(ii) $\angle P$,

(iii) the length of QR.



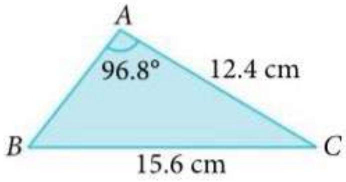
2. In $\triangle ABC$, $\angle BAC = 96.8°$, $AC = 12.4$ cm and $BC = 15.6$ cm.

Find

(i) $\angle ABC$,

(ii) $\angle BCA$,

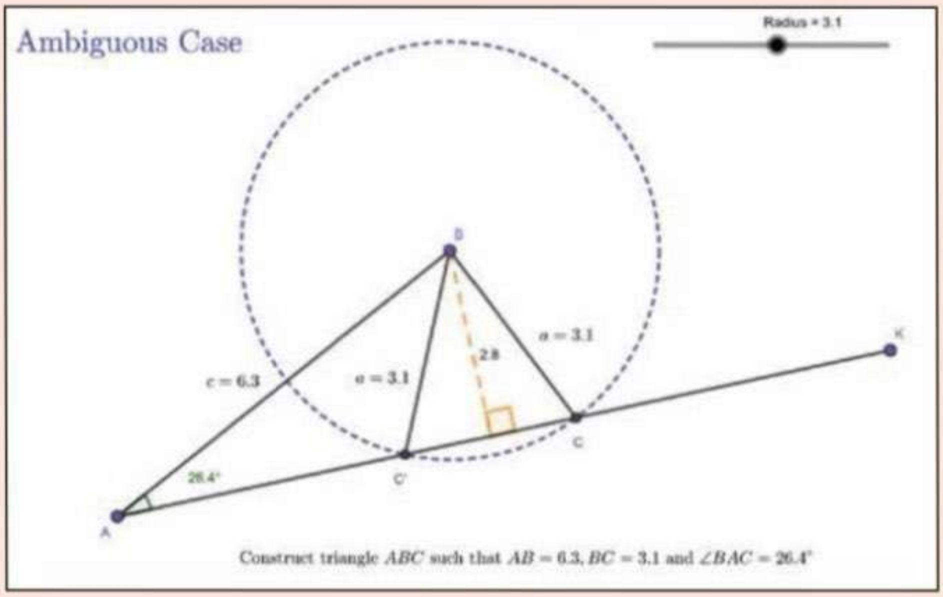
(iii) the length of AB.



Investigation Ambiguous case

When do ambiguous cases arise? Let us investigate.

Go to www.sl-education.com/tmsoupp4/pg206 or scan the QR code on the right and open the geometry software template 'Ambiguous Case'. In triangle ABC, sides a, c and $\angle A$ (i.e. 2 sides and a non-included angle) are given.



Construct triangle ABC such that $AB = 6.3$, $BC = 3.1$ and $\angle BAC = 26.4°$

Fig. 6.9

Explore and record your observations below.

Case 1: $a > c$

Move point A such that it is inside the circle.

$a =$, $c =$

When $a > c$, triangle(s) can be constructed.

Therefore, there is value of $\angle ACB$.

Case 2: $a < c$

Move point A such that it is outside the circle.

$a =$, $c =$

When $a < c$, triangle(s) can be constructed.

Therefore, there are possible values of $\angle ACB$.

From the above Investigation, we see that an ambiguous case will occur if

- the lengths of two sides and a non-included angle in a triangle are given, and
- the length of the side opposite the given angle is shorter than the other given side.

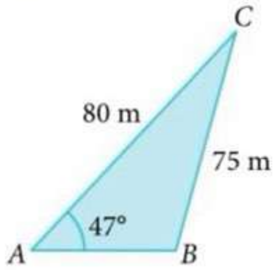
  Note: Implicit in this condition is that the given angle must be acute. Why is this so?

Worked Example 8

Using Sine Rule for obtuse triangle

In $\triangle ABC$, $\angle BAC = 47°$, $AC = 80$ m and $BC = 75$ m. Find the obtuse $\angle ABC$ and the length of AB.

Solution



Big Idea

Diagrams
We can sketch the triangle to help us visualise the relationship between the given angle and sides, as well as the unknown angles and sides.

Using Sine Rule,

$$\frac{\sin \angle ABC}{80} = \frac{\sin 47°}{75}$$

$$\sin \angle ABC = \frac{80 \sin 47°}{75}$$

$$= 0.780\ 11 \text{ (to 5 s.f.)}$$

Since $\angle ABC$ is obtuse,

$$\angle ABC = 180° - \sin^{-1} 0.780\ 11$$

$$= 128.729° \text{ (to 3 d.p.)}$$

$$= 128.7° \text{ (to 1 d.p.)}$$

$$\angle ACB = 180° - 47° - 128.729°$$

$$= 4.271°$$

Using Sine Rule,

$$\frac{AB}{\sin 4.271°} = \frac{75}{\sin 47°}$$

$$AB = \frac{75}{\sin 47°} \times \sin 4.271°$$

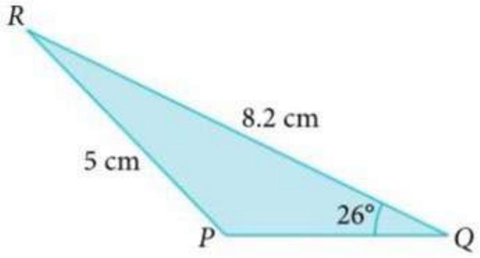
$$= 7.64 \text{ m (to 3 s.f.)}$$

$\therefore \angle ABC = 128.7°$ and $AB = 7.64$ m

Practise Now 8

Similar and Further Questions
Exercise 6C
Questions 13, 14, 16

1. In $\triangle PQR$, QR is the longest side and has a length of 8.2 cm. $PR = 5$ cm and $\angle PQR = 26°$. Find $\angle QPR$ and the length of PQ.



2. In $\triangle ABC$, $\angle ABC = 46°$, $AB = 9.8$ cm and $AC = 7.1$ cm. Find the obtuse $\angle ACB$ and the length of BC.

Reflection

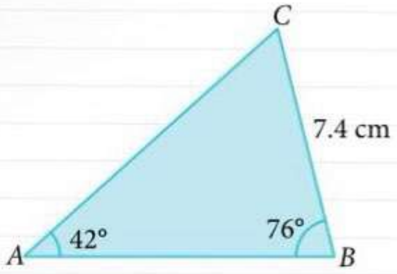
To solve a triangle means to find all the unknown sides and/or angles. What conditions must be given in order to solve a triangle using the Sine Rule?

Basic Intermediate Advanced

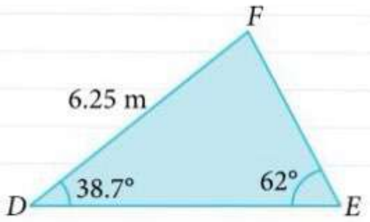
Exercise 6C

1. For each of the following triangles, find the unknown angles and sides.

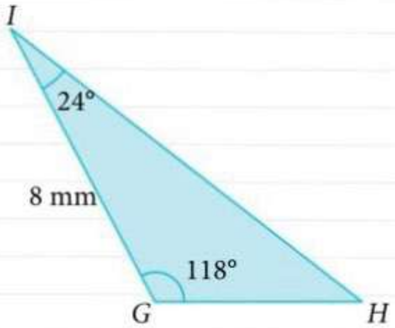
(a)



(b)



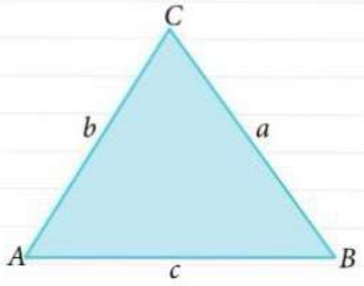
(c)



2. In $\triangle PQR$, $QR = 7$ cm, $\angle PQR = 47°$ and $\angle PRQ = 97°$. Find the length of PQ.

3. In $\triangle PQR$, $\angle P = 75°$, $\angle Q = 60°$ and $q = 14$ cm. Find the length of the longest side.

4. For each of the following triangles ABC, find the unknown angles and sides.

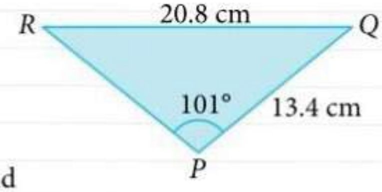


(a) $\angle A = 92.0°$, $b = 6.93$ cm and $a = 15.3$ cm

(b) $\angle B = 98.0°$, $a = 14.5$ m and $b = 17.4$ m

(c) $\angle C = 35.0°$, $b = 8.7$ cm and $c = 9.5$ cm

5. In $\triangle PQR$, $\angle P = 101°$, $PQ = 13.4$ cm and $QR = 20.8$ cm.



Find

(i) $\angle R$,

(ii) $\angle Q$,

(iii) the length of PR.

6. In $\triangle ABC$, $\angle ABC = 91°$, $BC = 7.4$ cm and $AC = 11.6$ cm. Find

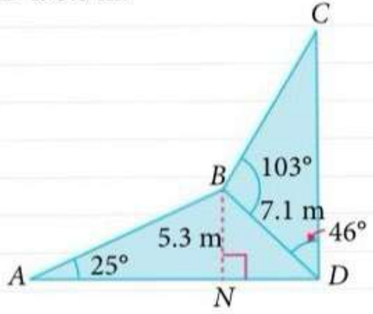
(i) $\angle BAC$,

(ii) $\angle ACB$,

(iii) the length of AB.

Exercise 6C

7. The figure shows a metal framework in which AD is horizontal, $BD = 7.1$ m, $\angle BAD = 25°$, $\angle BDC = 46°$, $\angle DBC = 103°$ and the height of B above AD is 5.3 m.



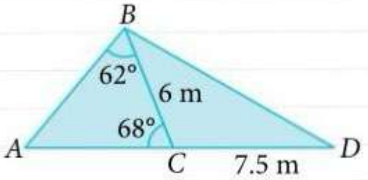
Find

(i) the length of the metal bar AB,

(ii) the angle that BD makes with BN,

(iii) the length of the metal bar CD.

8. In the figure, A, C and D are three points along a straight road where $\angle ABC = 62°$, $\angle ACB = 68°$, $BC = 6$ m and $CD = 7.5$ m.

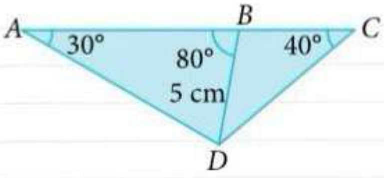


Find

(i) the distance AC,

(ii) the area of the region enclosed by AB, BD and DA.

9. An experiment is carried out to determine the extension of springs. Springs are attached to a horizontal bar at A, B and C and are joined to a mass D.



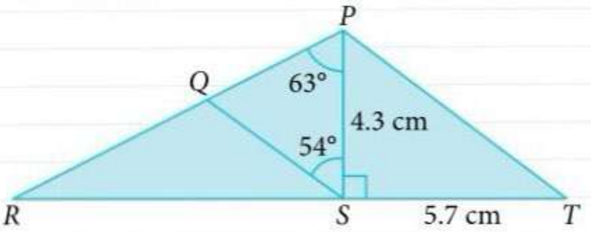
Given that $\angle ACD = 40°$, $\angle CAD = 30°$, $\angle ABD = 80°$ and $BD = 5$ cm, find

(i) the distance between A and B,

(ii) the distance between B and C,

(iii) the vertical distance between the mass and the horizontal bar.

10. In the figure, RST is a straight line, $\angle PST = 90°$, $\angle SPR = 63°$, $\angle PSQ = 54°$, $PS = 4.3$ cm and $ST = 5.7$ cm.

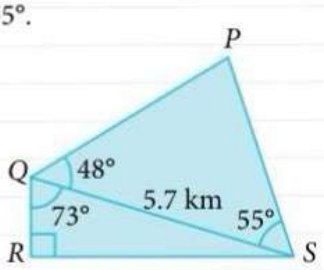


(i) Determine if QS is parallel to PT.

(ii) Find the length of PR.

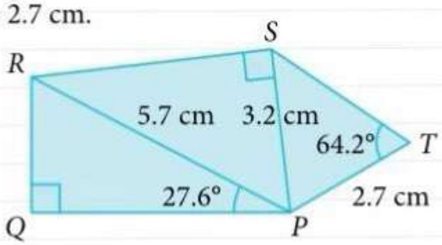
(iii) Find the length of QS.

11. In the figure, $PQRS$ is a nature reserve. A 5.7-km long walkway connects Q to S. It is given that $\angle QRS = 90°$, $\angle SQR = 73°$, $\angle PQS = 48°$ and $\angle PSQ = 55°$.



Find the area of the nature reserve.

12. In the figure, $\angle PQR = \angle PSR = 90°$, $\angle QPR = 27.6°$, $\angle PTS = 64.2°$, $PR = 5.7$ cm, $PS = 3.2$ cm and $PT = 2.7$ cm.



Find

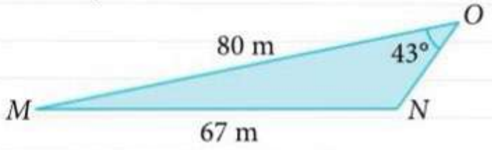
(i) the length of QR, **(ii)** $\angle SPR$,

(iii) $\angle PST$.

Exercise 6C

13. In $\triangle JKL$, $JL = 15$ mm, $KL = 19$ mm and $\angle JKL = 39°$. Find the obtuse $\angle KJL$ and the length of JK.

14. Three buildings on a plot of land form a $\triangle MNO$ in which buildings M and O are the furthest apart, with a distance of 80 m between them. Building N is 67 m away from building M. It is given that $\angle MON = 43°$. Find the distance between buildings O and N.



15. A map has a scale of 8 cm to 1 km. An undeveloped plot of land is shown as a quadrilateral $ABCD$ on the map. The length of the diagonal AC is 7 cm, $\angle BAC = 55°$, $\angle BCA = 77°$, $\angle DAC = 90°$ and $\angle DCA = 40°$. Find

(i) the length, in cm, of the side AB on the map,

(ii) the length, in km, which is represented by AD,

(iii) the area, in km^2, which is represented by $\triangle ADC$.

16. In $\triangle ABC$, $\angle A = 35°$, $BC = 5$ cm and $\sin B = \frac{4}{3} \sin A$.

(i) Given that B is obtuse, find B.

(ii) Find the length of AC.

6.4 Cosine Rule

The Sine Rule can be used to solve a triangle if the following are given:

1. Two angles and the length of one side (see Worked Example 6); or
2. The lengths of two sides and one non-included angle (see Worked Example 7).

What if the lengths of two sides and an included angle are given (see Fig. 6.10)? Can the Sine Rule be used to solve the triangle?

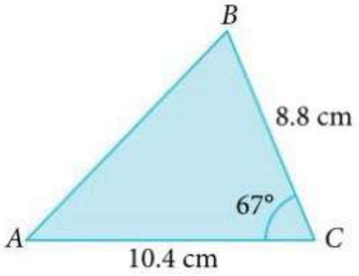
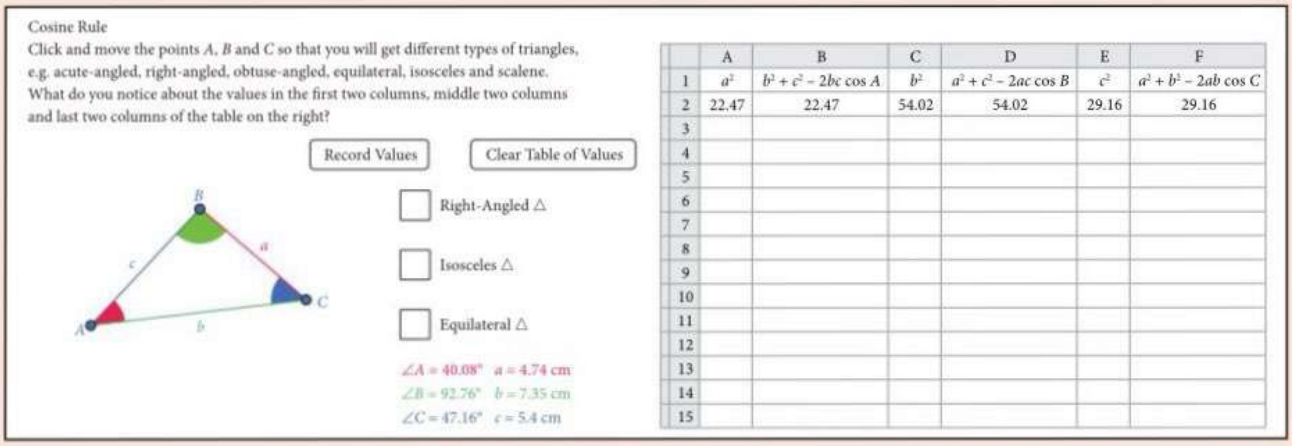


Fig. 6.10

Investigation

Cosine Rule

Go to www.sl-education.com/tmsoupp4/pg211 or scan the QR code on the right and open the geometry software template 'Cosine Rule'.
Fig. 6.11 shows a triangle ABC and a table of values.



Cosine Rule

Click and move the points A, B and C so that you will get different types of triangles, e.g. acute-angled, right-angled, obtuse-angled, equilateral, isosceles and scalene. What do you notice about the values in the first two columns, middle two columns and last two columns of the table on the right?

$\angle A = 40.08°$ $a = 4.74$ cm
$\angle B = 92.76°$ $b = 7.35$ cm
$\angle C = 47.16°$ $c = 5.4$ cm

	A	B	C	D	E	F
1	a^2	$b^2 + c^2 - 2bc \cos A$	b^2	$a^2 + c^2 - 2ac \cos B$	c^2	$a^2 + b^2 - 2ab \cos C$
2	22.47	22.47	54.02	54.02	29.16	29.16
3						
4						
5						
6						
7						
8						
9						
10						
11						
12						
13						
14						
15						

Fig. 6.11

1. The labelling of the sides of the triangle with reference to the vertices is important.
   Copy and complete the following.
   (a) The length of the side of the triangle opposite vertex A is labelled a.
   (b) The length of the side of the triangle opposite vertex B is labelled ______.
   (c) The length of the side of the triangle opposite vertex C is labelled ______.
2. Click and drag each of the vertices A, B and C to get different types of triangles.
   To obtain special triangles such as a right-angled triangle, an isosceles triangle and an equilateral triangle, click on the respective buttons in the template.
   For each triangle, click on 'Record Values', then copy and complete Table 6.3.

Triangle	$\angle A$	$\angle B$	$\angle C$	a^2	$b^2 + c^2 - 2bc \cos A$	b^2	$a^2 + c^2 - 2ac \cos B$	c^2	$a^2 + b^2 - 2ab \cos C$
Right-angled triangle									
Isosceles triangle									
Equilateral triangle									
Scalene triangle									
Obtuse-angled triangle									

Table 6.3

3. What do you notice about the last 6 columns in Table 6.3?
4. Hence, write down a formula relating the quantities in the last 6 columns of the table. This is called the **Cosine Rule**. Notice that for each fraction, the *side by itself must be opposite the angle*.
5. For each of the formulae in Question 4, make the cosine of the angle the subject of the formula.

From the Investigation on page 211, we can conclude:

The Cosine Rule

$$a^2 = b^2 + c^2 - 2bc \cos A, \qquad \cos A = \frac{b^2+c^2-a^2}{2bc},$$

$$b^2 = a^2 + c^2 - 2ac \cos B, \quad \text{or} \quad \cos B = \frac{a^2+c^2-b^2}{2ac},$$

$$c^2 = a^2 + b^2 - 2ab \cos C, \qquad \cos C = \frac{a^2+b^2-c^2}{2ab},$$

where A, B and C are the three interior angles of $\triangle ABC$ opposite the sides whose lengths are a, b and c respectively.

We can prove the Cosine Rule as follows. Without loss of generality, we will just prove $a^2 = b^2 + c^2 - 2bc \cos A$.

There are three cases: $\angle A$ is an acute angle, $\angle A$ is an obtuse angle and $\angle A$ is a right angle.
We will show the case when A is acute.

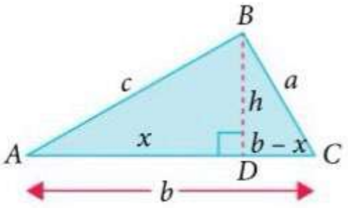


Fig. 6.12

In $\triangle BCD$,

$a^2 = h^2 + (b - x)^2$ Pythagoras' Theorem

$= h^2 + b^2 - 2bx + x^2$

$= b^2 + (h^2 + x^2) - 2bx$ — (1)

In $\triangle BAD$,

$c^2 = h^2 + x^2$ — (2) Pythagoras' Theorem

and $\cos A = \dfrac{x}{c}$, i.e. $x = c \cos A$ — (3)

Substituting (2) and (3) into (1),

$a^2 = b^2 + (h^2 + x^2) - 2bx$

$= b^2 + c^2 - 2bc \cos A$ (proven)

Thinking Time

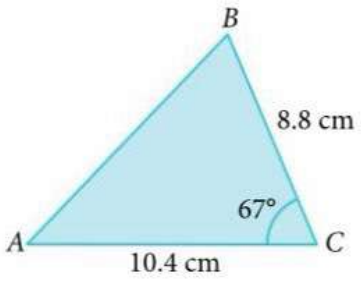
1. Prove the Cosine Rule for Case 2 where $\angle A$ is an obtuse angle.
2. Let us consider Case 3 where $\angle A$ is a right angle.
   - **(a)** What happens to the formula for the Cosine Rule $a^2 = b^2 + c^2 - 2bc \cos A$ if $A = 90°$?
   - **(b)** Is this formula always true if $A = 90°$? Explain your answer.
3. Copy and complete the following:

   ☐ Theorem is a special case of the Cosine Rule.

Worked Example 9

Using Cosine Rule when given 2 sides and 1 included angle

In $\triangle ABC$, $BC = 8.8$ cm, $AC = 10.4$ cm and $\angle ACB = 67°$.



Find

(i) the length of AB,

(ii) $\angle ABC$,

(iii) $\angle BAC$.

Solution

(i) Using Cosine Rule,

$$c^2 = a^2 + b^2 - 2ab \cos C$$
$$AB^2 = 8.8^2 + 10.4^2 - 2 \times 8.8 \times 10.4 \times \cos 67°$$
$$= 114.08 \text{ (to 5 s.f.)}$$
$$\therefore AB = \sqrt{114.08}$$
$$= 10.7 \text{ cm (to 3 s.f.)}$$

(ii) Using Sine Rule,

$$\frac{\sin \angle ABC}{AC} = \frac{\sin \angle ACB}{AB}$$
$$\frac{\sin \angle ABC}{10.4} = \frac{\sin 67°}{\sqrt{114.08}}$$
$$\sin \angle ABC = \frac{10.4 \sin 67°}{\sqrt{114.08}}$$
$$= 0.896\,30 \text{ (to 5 s.f.)}$$
$$\angle ABC = \sin^{-1} 0.896\,30$$
$$= 63.676° \text{ (to 3 d.p.)}$$
$$\text{or } 180° - 63.676° = 116.324°$$

Since obtuse $\angle ABC + \angle ACB = 116.324° + 67° = 183.324° > 180°$, it is not possible for $\angle ABC$ to be obtuse.

$\therefore \angle ABC = 63.7°$ (to 1 d.p.)

Attention

In order for the final answer to be accurate to three significant figures, we use the exact value of AB from **(i)** in the intermediate working. Alternatively, we can substitute $AB = 10.681$ (to 5 s.f.).

(iii) $\angle BAC = 180° - 63.676° - 67°$ ($\angle$ sum of $\triangle$)

$$= 49.3° \text{ (to 1 d.p.)}$$

Practise Now 9

Similar and Further Questions

Exercise 6D

Questions 1-3, 7-10

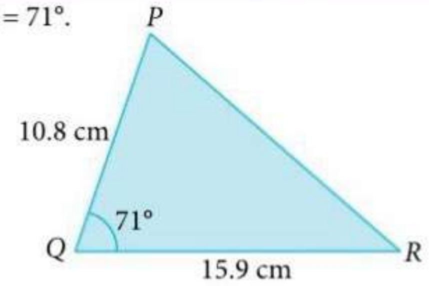
In $\triangle PQR$, $PQ = 10.8$ cm, $QR = 15.9$ cm and $\angle PQR = 71°$.

Find

(i) the length of PR,

(ii) $\angle QPR$,

(iii) $\angle PRQ$.

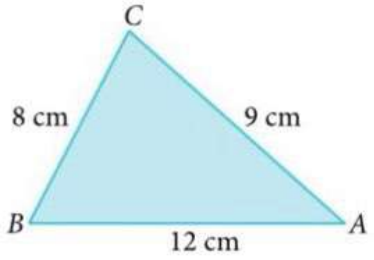


Worked Example 10

Using Cosine Rule when given 3 sides

In $\triangle ABC$, $AB = 12$ cm, $BC = 8$ cm and $AC = 9$ cm.

Find the size of the smallest angle.



Solution

The smallest angle is the angle opposite the shortest side, i.e. $\angle BAC$.

Using Cosine Rule,

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$2bc \cos A = b^2 + c^2 - a^2$$

$$\cos A = \frac{b^2+c^2-a^2}{2bc}$$

$$= \frac{9^2+12^2-8^2}{2\times9\times12}$$

$$= \frac{161}{216}$$

$$A = \cos^{-1}\frac{161}{216}$$

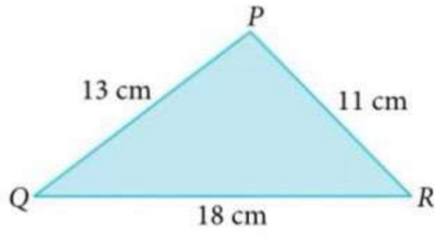
$$= 41.8° \text{ (to 1 d.p.)}$$

$\therefore$ the smallest angle is 41.8°.

Practise Now 10

Similar and Further Questions
Exercise 6D
Questions 4–6, 11–17

In $\triangle PQR$, $PQ = 13$ cm, $QR = 18$ cm and $PR = 11$ cm. Find the size of the largest angle.



Thinking Time

Heron of Alexandria (around AD 10–75) established a formula to find the area of a triangle using only the lengths o its sides. The area of $\triangle ABC$ is given by $\sqrt{s(s-a)(s-b)(s-c)}$, where $s = \frac{a+b+c}{2}$ is half of the perimeter.

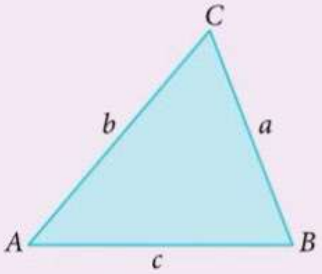


Fig. 6.13

Verify that the above formula is correct for each of the following cases:

(a) $a = 6$ cm, $b = 8$ cm and $c = 10$ cm
(b) $a = 8$ cm, $b = 9$ cm and $c = 10$ cm
(c) $a = 5$ cm, $b = 3$ cm and $c = 7$ cm

Can you find a proof for this formula?

Reflection

1. What is one mistake that I made or one misconception that I held in this chapter? What did I learn from this mistake or misconception?
2. When should I use the Sine Rule and when should I use the Cosine Rule?

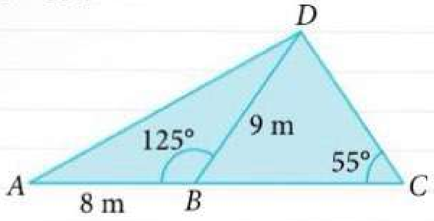
Advanced Intermediate Basic

Exercise 6D

1. In $\triangle ABC$, $a = 5$ cm, $b = 7$ cm and $\angle C = 60°$. Find c.
2. In $\triangle GHI$, $g = 9$ cm, $i = 7$ cm and $\angle H = 30°$. Find h.
3. In $\triangle MNO$, $m = 4.2$ cm, $n = 5.8$ cm and $\angle O = 141.4°$. Find o.
4. In $\triangle XYZ$, $x = 7$ m, $y = 8$ m and $z = 9$ m. Find all the unknown angles.
5. In $\triangle ABC$, $AB = 6.7$ cm, $BC = 3.8$ cm and $AC = 5.3$ cm. Find the size of the smallest angle.
6. In $\triangle PQR$, $PQ = 7.8$ cm, $QR = 9.1$ cm and $PR = 4.9$ cm. Find the size of the largest angle.

Exercise 6D

7. In the figure, the point B lies on AC such that $AB = 8$ m, $BD = 9$ m, $\angle ABD = 125°$ and $\angle BCD = 55°$.

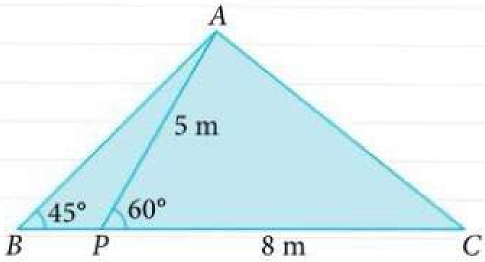


Find

(i) the length of CD,

(ii) the length of AD.

8. The diagram shows the cross section of the roof of an old cottage. It is given that $AP = 5$ m, $PC = 8$ m, $\angle APC = 60°$ and $\angle ABC = 45°$.



Find

(i) the length of AB,

(ii) the length of AC.

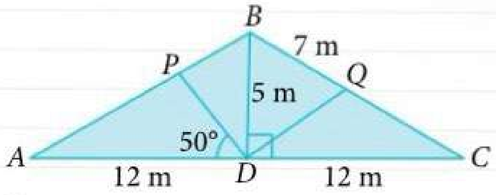
9. In $\triangle ABC$, $BC = 4$ cm. M is the midpoint of BC such that $AM = 4$ cm and $\angle AMB = 120°$. Find

(i) the length of AC,

(ii) the length of AB,

(iii) $\angle ACB$.

10. The diagram shows the support structure of the roof of a building. ADC is a straight line, $BD = 5$ m, $AD = CD = 12$ m, $BQ = 7$ m and $\angle PDA = 50°$.



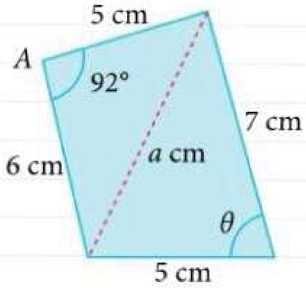
Find

(i) $\angle BAD$,

(ii) the length of the support PD,

(iii) the length of the support DQ.

11. The figure shows a quadrilateral with the dimensions as shown.

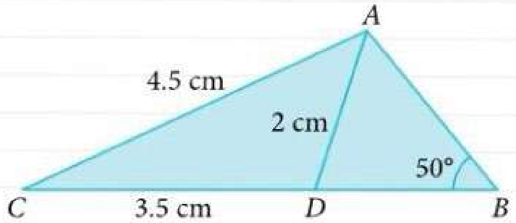


Find

(i) the value of a,

(ii) θ.

12. In the figure, D is a point on CB such that $AD = 2$ cm, $AC = 4.5$ cm, $CD = 3.5$ cm and $\angle ABD = 50°$.



Find

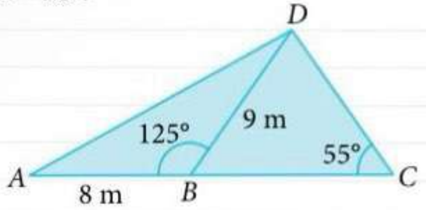
(i) $\angle ADB$,

(ii) the shortest distance from A to CB,

(iii) the length of BD.

Exercise 6D

7. In the figure, the point B lies on AC such that $AB = 8$ m, $BD = 9$ m, $\angle ABD = 125°$ and $\angle BCD = 55°$.

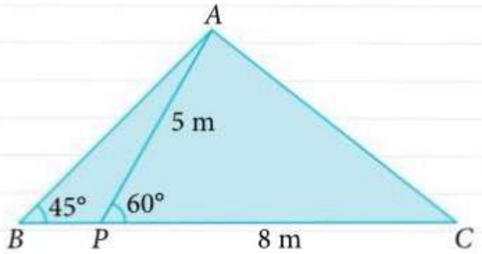


Find

(i) the length of CD,

(ii) the length of AD.

8. The diagram shows the cross section of the roof of an old cottage. It is given that $AP = 5$ m, $PC = 8$ m, $\angle APC = 60°$ and $\angle ABC = 45°$.



Find

(i) the length of AB,

(ii) the length of AC.

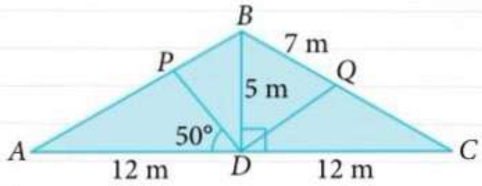
9. In $\triangle ABC$, $BC = 4$ cm. M is the midpoint of BC such that $AM = 4$ cm and $\angle AMB = 120°$. Find

(i) the length of AC,

(ii) the length of AB,

(iii) $\angle ACB$.

10. The diagram shows the support structure of the roof of a building. ADC is a straight line, $BD = 5$ m, $AD = CD = 12$ m, $BQ = 7$ m and $\angle PDA = 50°$.



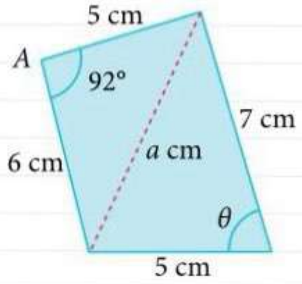
Find

(i) $\angle BAD$,

(ii) the length of the support PD,

(iii) the length of the support DQ.

11. The figure shows a quadrilateral with the dimensions as shown.

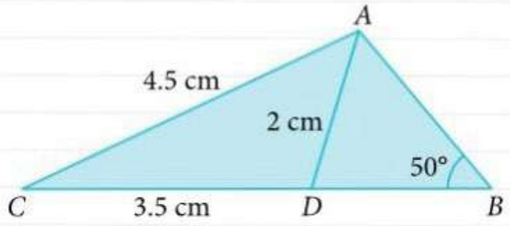


Find

(i) the value of a,

(ii) θ.

12. In the figure, D is a point on CB such that $AD = 2$ cm, $AC = 4.5$ cm, $CD = 3.5$ cm and $\angle ABD = 50°$.



Find

(i) $\angle ADB$,

(ii) the shortest distance from A to CB,

(iii) the length of BD.

Exercise 6D

13. A farm is shown as a triangle XYZ on a map with a scale of 2 cm to 5 km. Given that $XY = 9$ cm, $YZ = 12$ cm and $XZ = 8$ cm, find

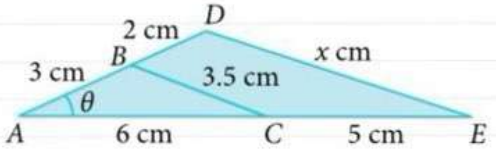
(i) the length, in km, which is represented by XZ,

(ii) $\angle YXZ$,

(iii) the area of the farm, in km².

14. In a trapezium $ABCD$, AB is parallel to DC, $AB = 4.5$ cm, $BC = 5$ cm, $CD = 7.5$ cm and $AD = 6$ cm. The point X lies on CD such that BX is parallel to AD. Find $\angle BCX$ and the length of BD.

15. The figure shows two triangles ABC and ADE.



(i) Determine if $\triangle ADE$ is an enlargement of $\triangle ABC$.

(ii) Find the value of $\cos\theta$.

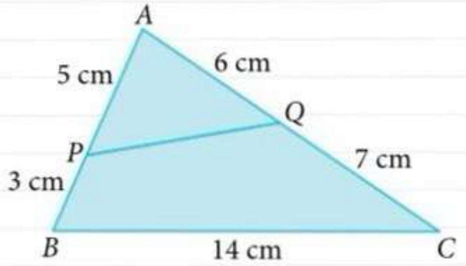
(iii) Hence, find the value of x.

16. In $\triangle ABC$, $AB = 8$ cm, $BC = 5$ cm and $CA = 6$ cm. BC is produced to R so that $CR = 3$ cm.

(i) Express $\cos \angle BCA$ in the form $\frac{p}{q}$, where p and q are integers.

(ii) Hence, find the length of AR.

17. In the figure, the point P lies on AB such that $AP = 5$ cm and $PB = 3$ cm. The point Q lies on AC such that $AQ = 6$ cm and $QC = 7$ cm.



Find the length of PQ.

Looking Back

Triangles are the simplest geometric shapes that we encounter but they are also the most useful. The relationships between the sides and angles of a triangle, from Pythagoras' Theorem for right-angled triangles to the Sine Rule and the Cosine Rule for any triangle, allow us to find out everything about a given triangle.

In this chapter, we see how we can use triangles to **model** many real-life problems and solve the triangles using trigonometry. An important aspect of solving triangles is to be clear about the sides of the triangle in relation to the corresponding angles. The naming convention and **notations** we use in our problems help us to be precise in our communication about the sides and their corresponding angles. The **diagrams** we draw to clarify these relationships help us to make better sense of the problems. Together, these ideas lay down the foundation for us to explore the world around us using different shapes that can be built from triangles.

Summary

1. **Sine and cosine of obtuse angles**

	A is an acute angle	A is an obtuse angle
$\sin A$	Positive	Positive
$\cos A$	Positive	Negative

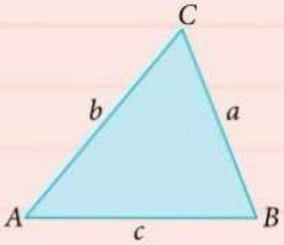
- Give an example of an angle A where $\sin A$ is positive but $\cos A$ is negative.

2. **Relationship between trigonometric ratios of acute and obtuse angles**

$\sin A = \sin (180° - A)$

$\cos A = -\cos (180° - A)$

3. **Area of $\triangle ABC$** $= \frac{1}{2}ab \sin C = \frac{1}{2}bc \sin A = \frac{1}{2}ac \sin B$



4. **Sine Rule**

In any $\triangle ABC$,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \quad \text{or} \quad \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

The Sine Rule can be used to solve a triangle (i.e. find the unknown sides and angles) if the following are given:

- two angles and the length of one side; or
- the lengths of two sides and one *non-included* angle.
- Give an example of a triangle in which the Sine Rule can be applied to solve for an unknown.

5. **Cosine Rule**

In any $\triangle ABC$,

$$a^2 = b^2 + c^2 - 2bc \cos A \qquad \cos A = \frac{b^2+c^2-a^2}{2bc}$$

$$b^2 = a^2 + c^2 - 2ac \cos B \quad \text{or} \quad \cos B = \frac{a^2+c^2-b^2}{2ac}$$

$$c^2 = a^2 + b^2 - 2ab \cos C \qquad \cos C = \frac{a^2+b^2-c^2}{2ab}$$

The Cosine Rule can be used to solve a triangle if the following are given:

- the lengths of all three sides; or
- the lengths of two sides and an *included* angle.
- Give an example of a triangle in which the Cosine Rule can be applied to solve for an unknown.

CHAPTER 7

Applications of Trigonometry

Trigonometry is a branch of mathematics that describes the relationship between the angles and the lengths of the sides of a triangle. Knowledge and understanding of trigonometry helped early explorers to plot the stars and navigate the seas. During the Tang dynasty, special devices for astronomic observation-based navigation known as the "Tang handy ruler" (唐小尺) and the "star-measuring ruler" (天量尺) were invented based on the principles of trigonometry. They were used to estimate the latitude of ships' positions, by measuring the angle of elevation and bearing between the horizon and the Polaris or other stars.

In this chapter, we will learn how trigonometry is applied in navigation.

Learning Outcomes

What will we learn in this chapter?

- What bearings are
- How to find the angle between a line and a plane
- How to solve problems in two and three dimensions including those involving angles of elevation and depression and bearings
- Why trigonometry has useful applications in real life

Introductory Problem

A bus stop is 280 m due north of a taxi stand.
Nadia walks from the taxi stand in the direction 050°.
How far does she walk before she is as close as possible to the bus stop?

In this chapter, we will learn to apply trigonometry to solve such real-world problems involving bearings.

7.1 Angles of elevation and depression

A. Using a clinometer

In Book 2 and Chapter 6 of this book, we have used trigonometry to solve problems involving the heights of tall objects such as trees, buildings and mountains. Let's learn more about how we can find such distances that cannot be measured directly in real life.

Performance Task

How can we use trigonometry to find the height of a tree or a building?

1. Fig. 7.1 shows an instrument known as a **clinometer**.

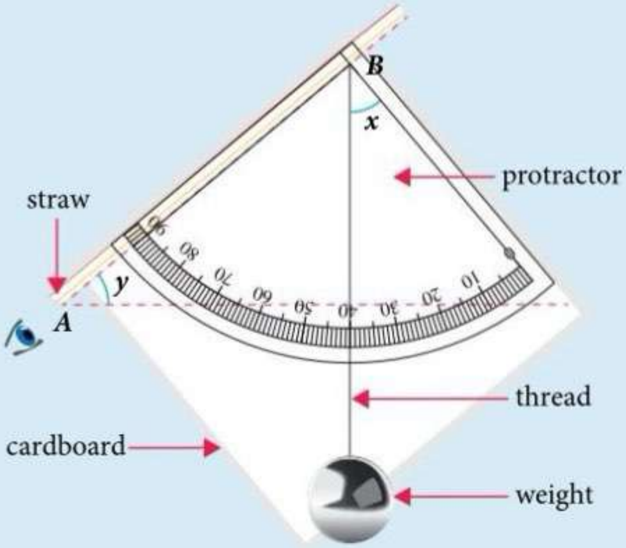


Fig. 7.1

If we look at the top of a tree or a building through the straw, the instrument will give us $\angle y$, which is also equal to $\angle x$. Explain clearly why $\angle y = \angle x$. $\angle y$ is called the angle of elevation.

2. Follow Steps **(i)** – **(iii)** to make a clinometer.
   (i) Photocopy the protractor in Fig. 7.2 and paste it on a piece of cardboard.
   (ii) Make a hole in the cardboard as indicated in Fig. 7.2.
   (iii) Using a straw, a piece of thread, a ball of plasticine (or any suitable weight) and some sticky tape, assemble the clinometer as shown in Fig. 7.1. Ensure that the straw lies at the 90° mark along the line AB, and that the thread touches the straw at the corner B of the protractor as shown in Fig. 7.1.

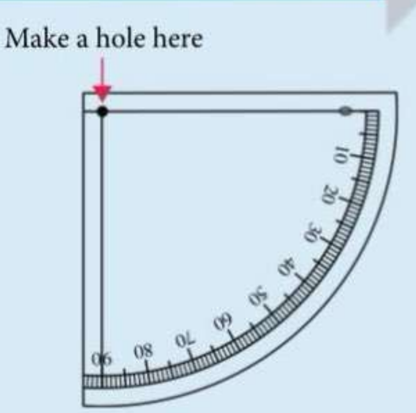


Fig. 7.2

3. Let us use the clinometer to find the height of an object.
   (i) Stand at a spot that is about 10 m to 20 m from an object such as a tree as shown in Fig. 7.3. Measure this distance using a measuring tape.

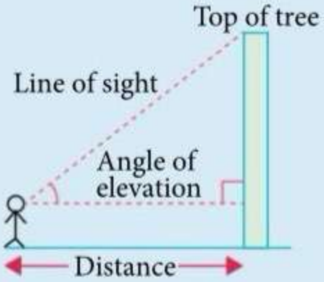


Fig. 7.3

   (ii) Look at the top of the object through the straw of the clinometer. Ask your classmate to read the angle of elevation from your clinometer.
   (iii) Use a trigonometric ratio to find the height of the object, showing your working clearly. Note that the clinometer is at a certain height above the ground.

B. Angles of elevation and depression

Ken stands in front of a vertical wall BC, as shown in Fig. 7.4.
A is the point where his eyes are and AD is an imaginary horizontal line from his eyes to the wall.

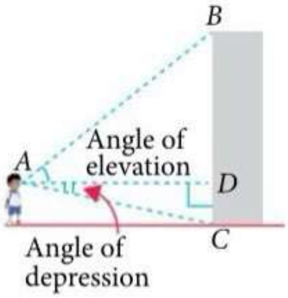


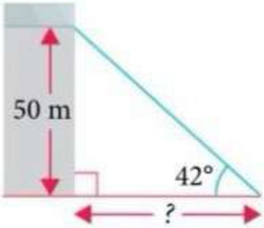
Fig. 7.4

When Ken looks at the top of the wall, B, the angle between the horizontal AD and the line of sight AB, i.e. $\angle BAD$, is called the **angle of elevation**.
When Ken looks at the bottom of the wall, C, the angle between the horizontal AD and the line of sight AC, i.e. $\angle CAD$, is called the **angle of depression**.

Worked Example 1

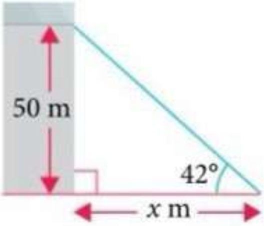
Solving problem involving angle of elevation

A window of a building is 50 m above the ground. Given that the angle of elevation of the window from a point on the ground is 42°, find the distance of the point on the ground from the foot of the building.



***Solution**

Let x m be the distance of the point on the ground from the foot of the building.



$$\tan 42° = \frac{50}{x}$$
$$x \tan 42° = 50$$
$$x = \frac{50}{\tan 42°}$$
$$= 55.5 \text{ (to 3 s.f.)}$$

∴ the distance of the point on the ground from the foot of the building is 55.5 m.

Big Idea

Diagrams

Diagrams help us visualise the relationships between the given mathematical objects. Here, with reference to the angle of 42°, the opposite side is 50 m. Assuming that the building is perpendicular to the ground, we can use the tangent ratio to find the length of the adjacent side.

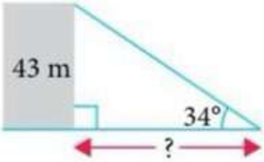
Practise Now 1

Similar and Further Questions

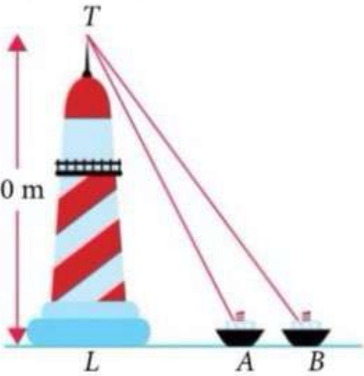
Exercise 7A

Questions 1-3, 7-10, 13

1. The angle of elevation of the top of an office tower of height 43 m from a point on level ground is 34°. Find the distance of the point on the ground from the foot of the tower.



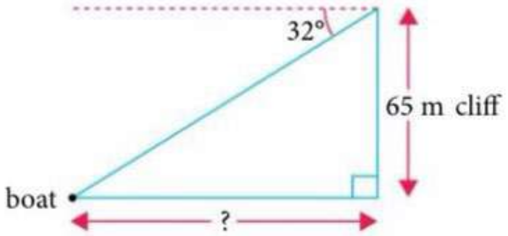
2. A lighthouse TL has a height of 50 m. The angles of elevation of the top of the lighthouse T from boat A and boat B are 48° and 38° respectively. Find the distance between boats A and B.



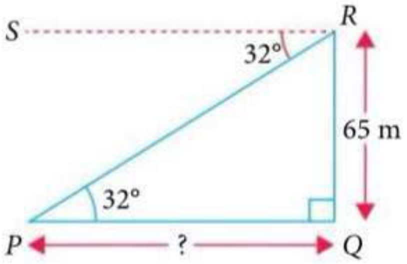
Worked Example 2

Solving problem involving angle of depression

A cliff is 65 m high. Given that the angle of depression from the top of the cliff to a boat is 32°, find the distance between the boat and the base of the cliff.



***Solution**



Problem-solving Tip

Label the vertices to represent the angles and lengths in the working.

Method 1:

$\angle RPQ = 32°$ (alt. $\angle$s, $SR \,//\, PQ$)

$$\tan 32° = \frac{65}{PQ}$$

$$PQ = \frac{65}{\tan 32°}$$

$$= 104 \text{ m (to 3 s.f.)}$$

$\therefore$ the distance between the boat and the base of the cliff is 104 m.

Method 2:

$\angle PRQ = 180° - 90° - 32°$ ($\angle$ sum of $\triangle$)

$$= 58°$$

$$\tan 58° = \frac{PQ}{65}$$

$$PQ = 65 \tan 58°$$

$$= 104 \text{ m (to 3 s.f.)}$$

$\therefore$ the distance between the boat and the base of the cliff is 104 m.

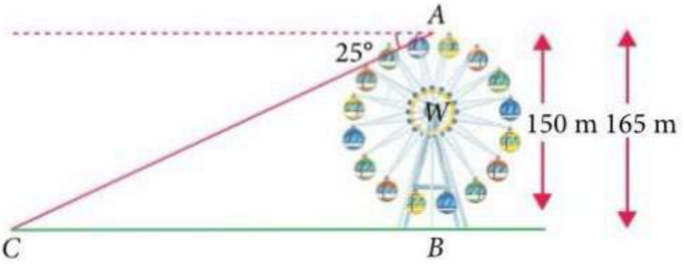
Reflection

Which method do you prefer? Why?

Practise Now 2

Similar and Further Questions
Exercise 7A
Questions 4–6, 11, 12, 14

1. The Singapore Flyer is an iconic giant observation wheel built on top of a terminal building. The diameter of the wheel is approximately 150 m and the highest point of the wheel is about 165 m above the ground. From point A at the top of wheel, Cheryl observes that the angle of depression to a sports car C on the ground is 25°.

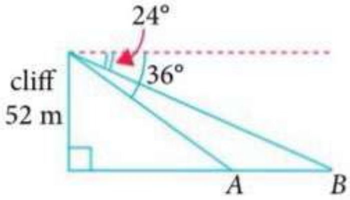


Find

(i) the distance of the sports car from point *B*, which is on ground level directly below *A*,

(ii) the angle of depression from the centre of the wheel, *W*, to the sports car *C*.

2. The angles of depression from the top of a cliff, 52 m high, to two ships *A* and *B* due east of it are 36° and 24° respectively.

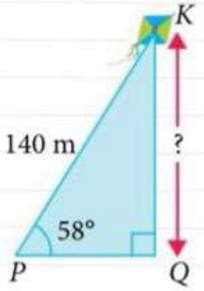


Calculate the distance between the two ships.

Basic Intermediate Advanced

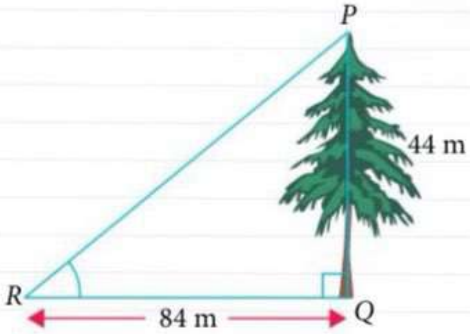
Exercise 7A

1. Li Ting, standing at *P*, is flying a kite attached to a string of length 140 m. The angle of elevation of the kite *K* from her hand is 58°. Assuming that the string is taut, determine the vertical distance between her hand and the kite.



2. Two buildings on level ground are 120 m and 85 m tall respectively. The angle of elevation of the top of the taller building from the top of the shorter building is 33.9°. Find the distance between the two buildings.

3. At a certain time in a day, a tree *PQ*, 44 m high, casts an 84-m long shadow *RQ*. Find the angle of elevation of *P* from the point *R*.

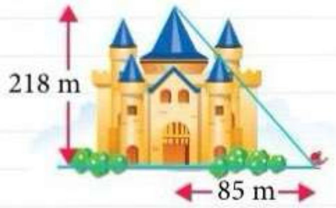


4. A building is 41 m high. The angle of depression of a fire hydrant from the top of the building is 33°. Find the distance between the fire hydrant and the foot of the building.

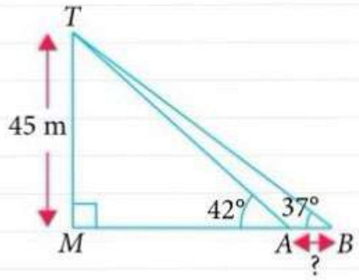
5. A boat is 65.7 m away from the base of the cliff. The angle of depression of the boat from the top of the cliff is 28.9°. Find the height of the cliff.

Exercise 7A

6. A castle has a height of 218 m. A bird stands on level ground 85 m away from the foot of the castle. Calculate the angle of depression of the bird from the top of the castle.



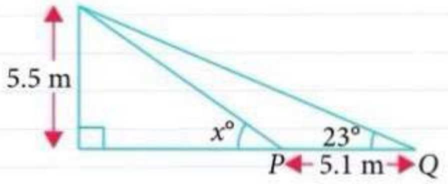
7. A clock tower, *MT*, has a height of 45 m. The angles of elevation of the top of the clock tower from two points *A* and *B* on the ground are 42° and 37° respectively.



Find the distance between *A* and *B*.

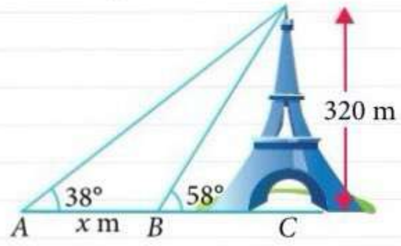
8. A castle is located on top of a mountain. Ali stands 55 m away from the foot of the mountain, on level ground. The angles of elevation of the top of the castle and the top of the mountain from Ali's line of sight are 60° and 45° respectively. Find the height of the castle.

9. An overhead bridge has a height of 5.5 m. The angles of elevation of the top of the bridge from two points *P* and *Q* on the ground are x° and 23° respectively.



Given that the distance between *P* and *Q* is 5.1 m, find the value of x.

10. The Eiffel Tower in Paris is 320 m tall. The angle of elevation from where Vasi stands at point *A* to the top of the tower is 38°. Vasi walks x metres to point *B* and observes that the angle of elevation from *B* to the top of the tower is now 58°.

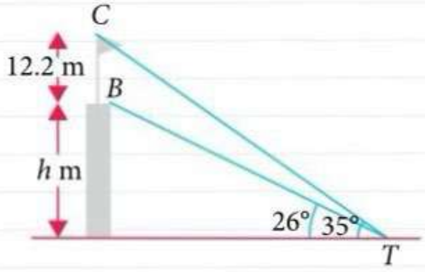


Find the value of x.

11. The angles of depression of two boats due west of a cliff from the top of the cliff are 23° and 18° respectively. Given that the cliff is 88 m high, calculate the distance between the two boats.

12. A satellite dish stands at the top of a cliff. The angle of depression of a ship, which is 80 m away from the base of the cliff, from the top of the satellite dish is 37°. The angle of depression of the same ship from the foot of the satellite dish is 32°. Find the height of the satellite dish.

13. A flagpole of height 12.2 m is placed on top of a building of height h metres. From a point *T* on level ground, the angle of elevation to the base of the flagpole *B* is 26° and the angle of elevation to the top of the flagpole *C* is 35°.



Find the value of h.

Exercise 7A

14. A tower with a height of 27 m stands at the top of a cliff. From the top of the tower, the angle of depression to a guard house is 56°. From the foot of the tower, the angle of depression to the same guard house is 49°. Find
(i) the distance between the base of the cliff and the guard house,
(ii) the height of the cliff.

7.2 Bearings

Imagine piloting a boat at sea. You have to turn 135 degrees to reach a port.
Where is 135 degrees? Should you turn clockwise or anticlockwise?

In mathematics, **bearings** are used to describe a precise position and direction of travel. A bearing is a **measure** of the amount of clockwise turn from the north direction. In other words, a bearing is an angle of a point in relation to a point in the north direction. It is always written as a *three-digit number*, if it is an integer.

Big Idea

Measures and Notations
To communicate the amount of clockwise turn in a concise and precise manner, bearings are always given as three figures, if it is an integer.
For example, if the clockwise turn from the north direction is 3°, then the bearing is written as 003°. This convention prevents the misinterpretation of the angle as 30° or 300°.
What does a bearing of 000° mean?
Can a bearing be negative? Why?

Fig. 7.5 shows the positions of four points A, B, C and D relative to an origin O.
N, **E**, **S** and **W** represent the directions north, east, south and west from O respectively.

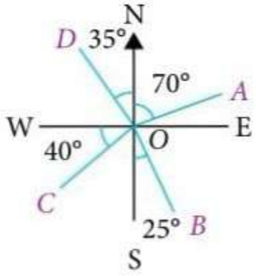
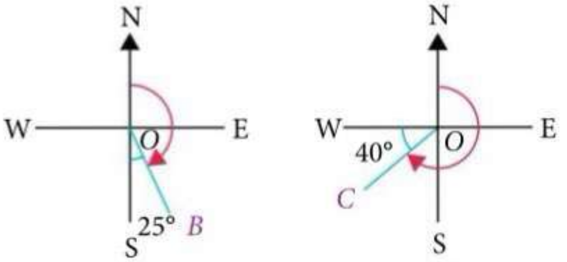


Fig. 7.5

The bearing of A from O is an angle measured from the north line at O to the line OA, in a clockwise direction. Hence, the bearing of A from O is 070°.



(a) **(b)**

Fig. 7.6

The bearing of B from O is equal to 180° – 25° (see Fig. 7.6 **(a)**). Hence, the bearing of B from O is 155°.

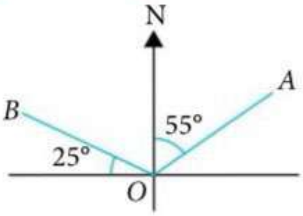
The bearing of C from O is equal to 270° – 40° (see Fig. 7.6 **(b)**). Hence, the bearing of C from O is 230°. What is the bearing of D from O?

The bearing of **N** from O is taken as 000° or 360°. The bearing of **E** from O is 090°. Similarly, the bearing of **S** from O is 180°. What is the bearing of **W** from O?

When reading compass bearings, directions are usually measured from either the north or the south. For example, 070° is written as **N70°E** and 210° is written as **S30°W**. When reading true bearings, directions are given in terms of the angles measured clockwise from the north.

Worked Example 3

Finding the bearing



Find the bearing of

(a) A from O,

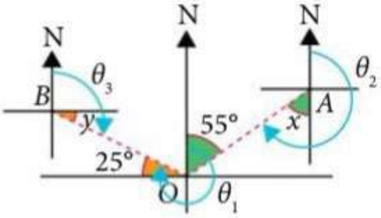
(b) B from O,

(c) O from A,

(d) O from B.

Solution

(a) The bearing of A from O is 055°.



(b) The bearing of B from O is given by the reflex angle θ_1, which is (270° + 25°).

$\therefore$ the bearing of B from O is 295°.

(c) The bearing of O from A is given by the reflex angle θ_2.

$\angle x = 55°$ (alt. $\angle$s)

$\theta_2 = 180° + 55°$

$= 235°$

$\therefore$ the bearing of O from A is 235°.

(d) The bearing of O from B is given by the obtuse angle θ_3.

$\angle y = 25°$ (alt. $\angle$s)

$\theta_3 = 90° + 25°$

$= 115°$

$\therefore$ the bearing of O from B is 115°.

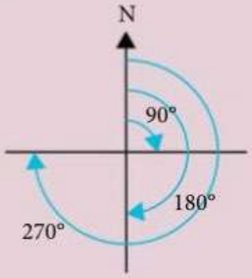
Big Idea

Diagrams

Using a diagram will help us to visualise the angles and see the relationships between the lines and the angles.

Problem-solving Tip

For **(b)**–**(d)**, note that the angle in each quadrant is 90°.



For **(c)**, follow the steps as shown.

Step 1: Draw the north line from A.

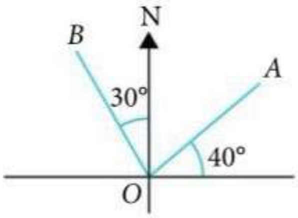
Step 2: Draw the angle clockwise from the north line to OA.

Step 3: Find the angle θ_2.

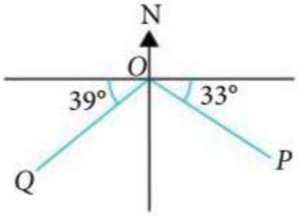
Practise Now 3

Similar and Further Questions
Exercise 7B
Questions 1(a)–(d), 2(a)–(f), 3(a)–(f), 6–8

1. Find the bearing of
   - **(a)** A from O,
   - **(b)** B from O,
   - **(c)** O from A,
   - **(d)** O from B.



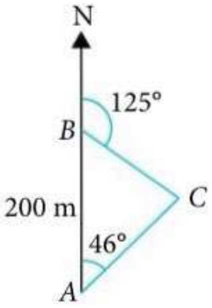
2. Find the bearing of
   - **(a)** P from O,
   - **(b)** Q from O,
   - **(c)** O from P,
   - **(d)** O from Q.



Worked Example 4

Solving problem involving bearings

Three points A, B and C are on level ground such that B is due north of A, the bearing of C from A is 046° and the bearing of C from B is 125°. Given that the distance between A and B is 200 m, find the distance of C from A.



Solution

Since the bearing of C from B is 125°,

$$\angle ACB = 125° - 46° \text{ (ext. } \angle \text{ of } \triangle)$$
$$= 79°$$

$$\angle ABC = 180° - 125° \text{ (adj. } \angle\text{s on a str. line)}$$
$$= 55°$$

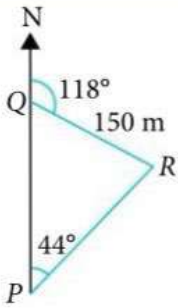
Using Sine Rule,

$$\frac{AC}{\sin 55°} = \frac{200}{\sin 79°}$$
$$AC = \frac{200 \sin 55°}{\sin 79°}$$
$$= 167 \text{ m (to 3 s.f.)}$$

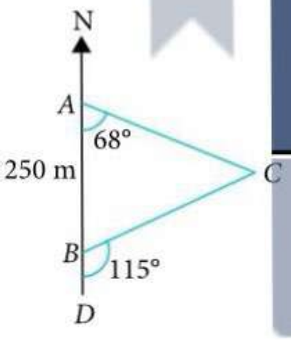
Practise Now 4

Similar and Further Questions
Exercise 7B
Questions 4, 5, 9, 14

1. Three points P, Q and R are on level ground such that P is due south of Q, the bearing of R from Q is 118° and the bearing of R from P is 044°. Given that the distance between Q and R is 150 m, find the distance of P from Q.



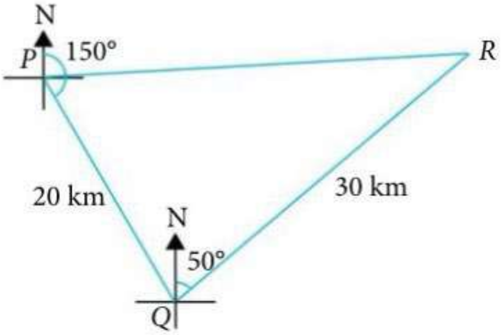
2. In the figure, A, B and C are positions in an amusement park on level ground, with A due north of B.
Given that $\angle BAC = 68°$, $AB = 250$ m and $\angle CBD = 115°$, find
   (i) the bearing of B from C,
   (ii) the length of AC and of BC.



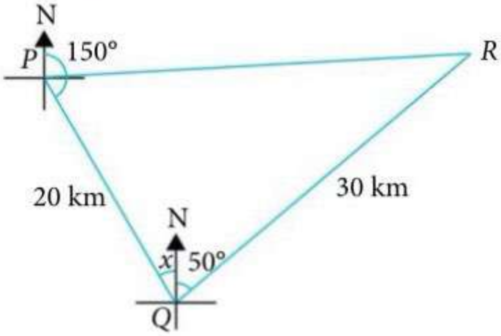
Worked Example 5

Solving problem involving bearings

A boat sailed 20 km from a point P to an island Q, on a bearing of 150°. It then sailed another 30 km on a bearing of 050° to a lighthouse R. Find the distance and the bearing of the lighthouse from P.



Solution



$\angle x = 180° - 150°$ (int. $\angle$s)
$= 30°$

$\angle PQR = 30° + 50°$
$= 80°$

Using Cosine Rule,
$PR^2 = 20^2 + 30^2 - 2 \times 20 \times 30 \times \cos 80°$
$= 1091.6$ (to 5 s.f.)
$PR = \sqrt{1091.6}$
$= 33.0$ km (to 3 s.f.)
$\therefore$ the lighthouse is 33.0 km away from P.

Using Sine Rule,

$$\frac{\sin \angle QPR}{30} = \frac{\sin 80°}{\sqrt{1091.6}}$$

$$\sin \angle QPR = \frac{30 \sin 80°}{\sqrt{1091.6}}$$

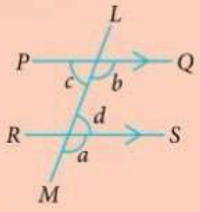
$= 0.894\ 21$ (to 5 s.f.)
$\angle QPR = \sin^{-1} 0.894\ 21$
$= 63.407°$ (to 3 d.p.)

$150° - 63.407° = 86.6°$ (to 1 d.p.)
$\therefore$ the bearing of the lighthouse from P is 086.6°.

Problem-solving Tip

To find distance PR using the Cosine Rule, we need to find $\angle PQR$ first.
The north line at P and the north line at Q are parallel.

Recall



If $PQ \parallel RS$,
- $\angle a = \angle b$ (corr. $\angle$s)
- $\angle c = \angle d$ (alt. $\angle$s)
- $\angle b + \angle d = 180°$ (int. $\angle$s)

Problem-solving Tip

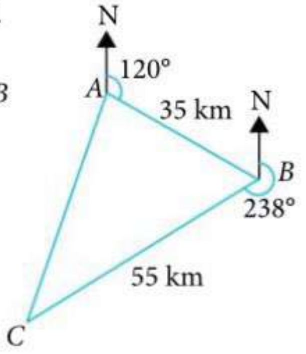
Bearing of R from $P = 150° - \angle QPR$. Hence, we need to find $\angle QPR$ first.

Practise Now 5

Similar and Further Questions
Exercise 7B
Questions 10, 11

The diagram shows the positions of three towns A, B and C. AB is 35 km and BC is 55 km.
The bearing of B from A is 120° and the bearing of C from B is 238°.

(i) Calculate AC.
(ii) Find the bearing of C from A.

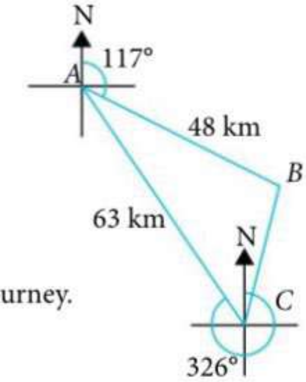


Worked Example 6

Solving problem involving bearings

Points A, B and C are at sea level.
B is 48 km from A on a bearing of 117°.
A is 63 km from C on a bearing of 326°.

(i) Calculate BC.
(ii) Calculate the bearing of B from C.
(iii) A boat, P, travels in a straight line from A to C.
Calculate the shortest distance of P from B during this journey.

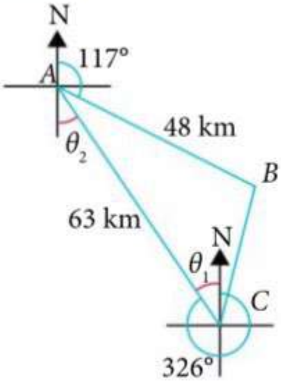


Problem-solving Tip

We are given two sides of the triangle (AB and AC). To find the third side (BC), we first have to find $\angle BAC$.

Solution

(i)



$$\theta_1 = 360° - 326° \text{ (}\angle\text{s at a point)}$$
$$= 34°$$

$$\theta_2 = \theta_1 \text{ (alt. }\angle\text{s)}$$
$$= 34°$$

$$\angle BAC = 180° - 117° - \theta_2 \text{ (adj. }\angle\text{s on a str. line)}$$
$$= 180° - 117° - 34°$$
$$= 29°$$

Using Cosine Rule,

$$BC^2 = 48^2 + 63^2 - 2 \times 48 \times 63 \times \cos 29°$$
$$= 983.30 \text{ (to 5 s.f.)}$$
$$BC = \sqrt{983.30}$$
$$= 31.4 \text{ km (to 3 s.f.)}$$

(ii) Using Sine Rule,

$$\frac{\sin \angle ACB}{48} = \frac{\sin 29^\circ}{\sqrt{983.30}}$$

$$\sin \angle ACB = \frac{48 \sin 29^\circ}{\sqrt{983.30}}$$

$$= 0.742\ 11 \text{ (to 5 s.f.)}$$

$$\angle ACB = \sin^{-1} 0.742\ 11$$

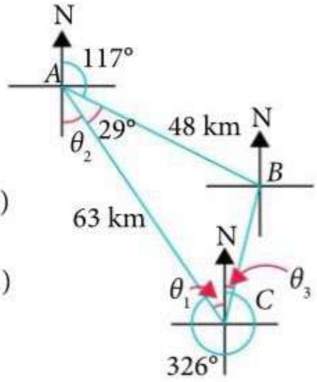
$$= 47.911^\circ \text{ (to 3 d.p.)}$$

$$\theta_3 = 47.911^\circ - \theta_1$$

$$= 47.911^\circ - 34^\circ$$

$$= 13.9^\circ \text{ (to 1 d.p.)}$$

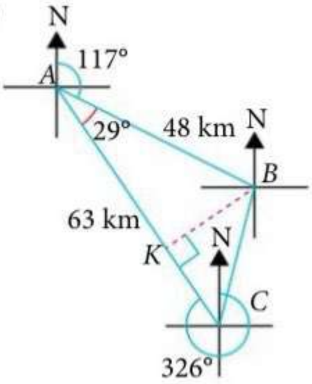
$\therefore$ the bearing of B from C is 013.9°.



Problem-solvi Tip

To find the bearing of B from C i.e. θ_3, which angles do we need to find?

(iii)



The shortest distance of P from B during the journey corresponds to BK, where BK is perpendicular to AC.

In $\triangle ABK$,

$$\sin \angle BAC = \frac{BK}{48}$$

$$BK = 48 \sin 29^\circ$$

$$= 23.3 \text{ km (to 3 s.f.)}$$

Problem-solving Tip

Since $\triangle ABK$ is a right-angled triangle,

$\sin \angle BAC = \frac{\text{opp}}{\text{hyp}} = \frac{BK}{AB}$.

From **(i)**, $\angle BAC = 29^\circ$.

Practise Now 6

Similar and Further Questions

Exercise 7B

Questions 12, 13, 15

The diagram shows a triangular field, ACD, which is crossed by a path BD.

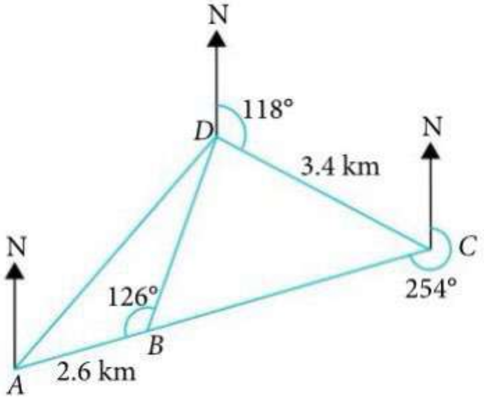
The bearing of C from D is 118° and the bearing of A from C is 254°.

$CD = 3.4$ km, $AB = 2.6$ km, $\angle ABD = 126^\circ$ and ABC is a straight line.

(a) Calculate

- **(i)** the bearing of B from D,
- **(ii)** BD,
- **(iii)** AD.

(b) A path of the shortest distance possible is to be constructed from B to AD. Find the length of the path.



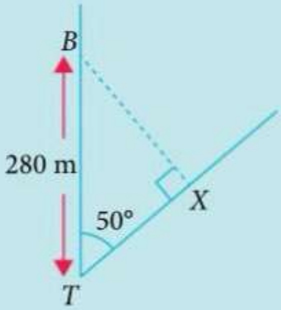
Introductory Problem Revisited

Now that we have learnt about bearings, can we draw a diagram to solve the Introductory Problem?

Let the bus stop be denoted by B and the taxi stand be denoted by T.

Nadia is closest to the bus stop when $\angle BXT = 90°$.

How can we find the distance TX?



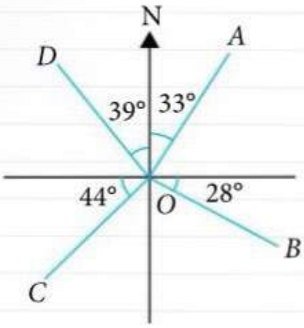
Reflection

1. What new knowledge have I gained in this section?
2. What have I previously learnt about angles and trigonometry that could have helped me better understand how to solve problems in this section?

Basic | Intermediate | Advanced

Exercise 7B

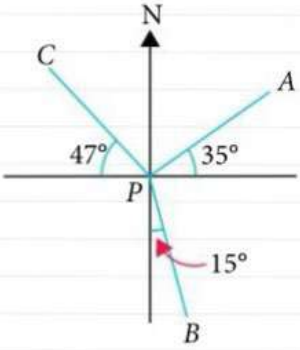
1. The figure shows the positions of O, A, B, C and D.



Find the bearing of

(a) A from O,
(b) B from O,
(c) C from O,
(d) D from O.

2. The figure shows the positions of P, A, B and C.

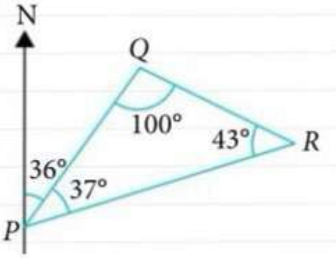


Find the bearing of

(a) A from P,
(b) B from P,
(c) C from P,
(d) P from A,
(e) P from B,
(f) P from C.

Exercise 7B

3. The figure shows the positions of *P*, *Q* and *R*.



Find the bearing of

(a) *Q* from *P*, **(b)** *P* from *Q*,
(c) *R* from *P*, **(d)** *P* from *R*,
(e) *Q* from *R*, **(f)** *R* from *Q*.

4. A point *Q* is 24 km from *P* and on a bearing of 072° from *P*. From *Q*, Waseem walks at a bearing of 320° to a point *R*, located directly north of *P*. Find

(i) the distance between *P* and *R*,
(ii) the distance between *Q* and *R*.

5. A petrol kiosk *P* is 12 km due north of another petrol kiosk *Q*. The bearing of a police station *R* from *P* is 135° and that from *Q* is 120°. Find the distance between *P* and *R*.

6. *A*, *B*, *C* and *D* are the four corners of a rectangular plot marked out on level ground. The bearing of *B* from *A* is 040° and the bearing of *C* from *A* is 090°. Find the bearing of

(i) *B* from *C*, **(ii)** *A* from *C*,
(iii) *D* from *C*.

7. *P*, *Q* and *R* are three points on level ground. Given that the bearing of *R* from *P* is 135°, $\angle PQR = 55°$ and $\angle PRQ = 48°$, find the bearing of

(i) *P* from *R*, **(ii)** *Q* from *R*,
(iii) *P* from *Q*.

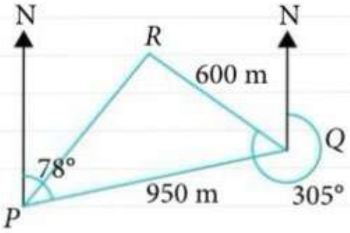
8. *A*, *B* and *C* are three points on level ground. Given that the bearing of *B* from *A* is 122°, $\angle CAB = 32°$ and $\angle ABC = 86°$, find the possible bearing(s) of *C* from *B*.

9. A bookshop is 250 m due north of a supermarket. Sara walks from the supermarket in the direction 055°. Calculate how far she has to walk before she is

(a) equidistant from the bookshop and the supermarket,
(b) due east of the bookshop.

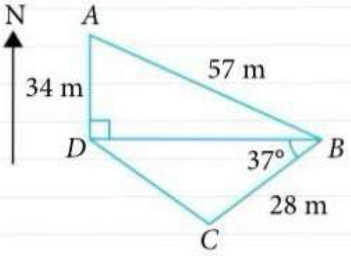
10. A helicopter flies 30 km from a point *P* to another point *Q* on a bearing of 128°. It then flies another 25 km to a point *R* on a bearing of 295°. Find the distance between *P* and *R*.

11. Points *P*, *Q* and *R* are on level ground.
R is 600 m from *Q* on a bearing of 305°.
Q is 950 m from *P* on a bearing of 078°.



(i) Calculate *PR*.
(ii) Find the bearing of *R* from *P*.

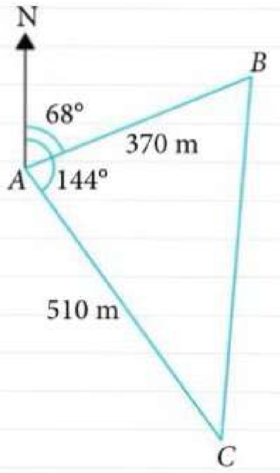
12. The diagram shows a park, *ABCD*, and a path *BD*. *A* is due north of *D*, *B* is due east of *D* and $\angle DBC = 37°$.
$AD = 34$ m, $AB = 57$ m and $BC = 28$ m.



(i) Calculate the bearing of *B* from *A*.
(ii) A path of the shortest distance possible is to be constructed from *C* to *BD*.
Find the length of the path.
(iii) Calculate the bearing of *D* from *C*.

Exercise 7B

13. *A*, *B* and *C* are three points on level ground. The bearing of *B* from *A* is 068° and the bearing of *C* from *A* is 144°. *AB* = 370 m and *AC* = 510 m.



Find

(i) the distance between *B* and *C*,
(ii) $\angle ACB$,
(iii) the bearing of *C* from *B*,
(iv) the shortest distance from *A* to *BC*.

14. Two cruise ships *P* and *Q* leave the port at the same time. *P* sails at 10 km/h on a bearing of 030° and *Q* sails at 12 km/h on a bearing of 300°. Find their distance apart and the bearing of *P* from *Q* after 2 hours.

15. *P*, *Q* and *R* represent three ports. *Q* is 35 km from *P* and on a bearing of 032° from *P*. *R* is 65 km from *P* and on a bearing of 108° from *P*.

(a) Calculate
  (i) *QR*,
  (ii) the bearing of *R* from *Q*.

A ship sets sail at 0930 hours from *P* directly to *R* at an average speed of 30 km/h and reaches a point *S* due south of *Q*.

(b) Find the time when it reaches *S*.

7.3 Three-dimensional problems

In Book 1, we learnt that a flat surface like the floor or the surface of a whiteboard has two dimensions (2D) – length and breadth. In mathematics, this flat surface is called a **plane**. A solid, on the other hand, has three dimensions (3D) – length, breadth and height (thickness or depth). Some examples of 3D solids that you have come across are cuboids, prisms, cylinders, pyramids, cones and spheres.

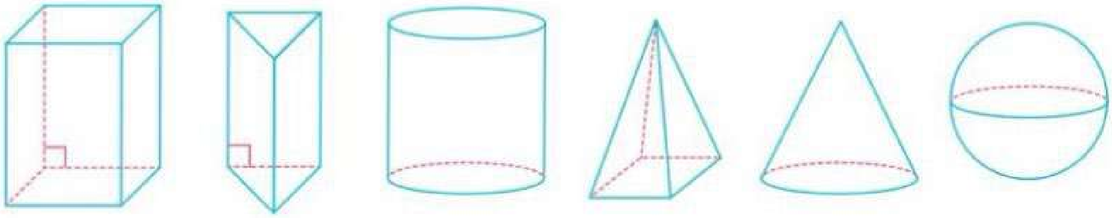

Cuboid Prism Cylinder Pyramid Cone Sphere

Fig. 7.7

Trigonometry can be applied to solve three-dimensional problems. To do so, we need to be able to visualise the appearance of 3D situations from 2D drawings and identify triangles within 3D figures or 3D situations.

Investigation Visualising right angles in 3D figures

1. Fig. 7.8(**a**) shows a plane with a few lines drawn on it.
Place a pencil perpendicular to the plane in Fig. 7.8(**a**), as shown in Fig. 7.8(**b**). Use a set square to check that the pencil is perpendicular to every line on the plane in Fig. 7.8(**a**).

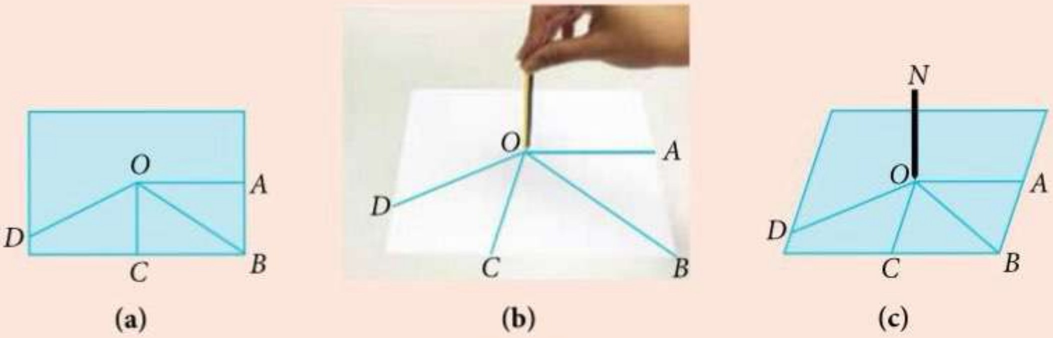


Fig. 7.8

Your pencil is called a **normal** to the plane since it is perpendicular to every line on the plane.

2. In Fig. 7.8(**c**), ON represents the pencil perpendicular to the plane, as shown in Fig. 7.8(**b**). $\angle NOA$ looks like it is a 90° angle, but $\angle NOB$ does not look like a 90° angle. Is $\angle NOB = 90°$? What about $\angle NOC$ and $\angle NOD$? Explain your answer.

Recall

Drawing a 3D solid or object on a flat surface may make a right angle look smaller or larger than 90°.

3. Fig. 7.9 shows a cuboid. Dotted lines represent lines that are hidden, i.e. you cannot see them from the front.

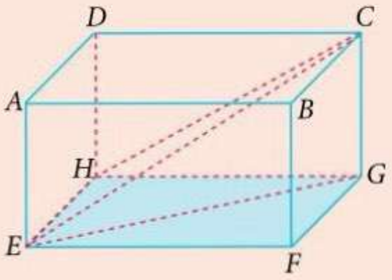


Fig. 7.9

A triangle can be formed between any three vertices of the cuboid. How can we tell if the triangles are right-angled?

We can use a geometry software template to visualise the cuboid in 3D. Go to www.sl-education.com/tmsoupp4/pg236 or scan the QR code on the right and open the geometry software template '3D Visualisation'.

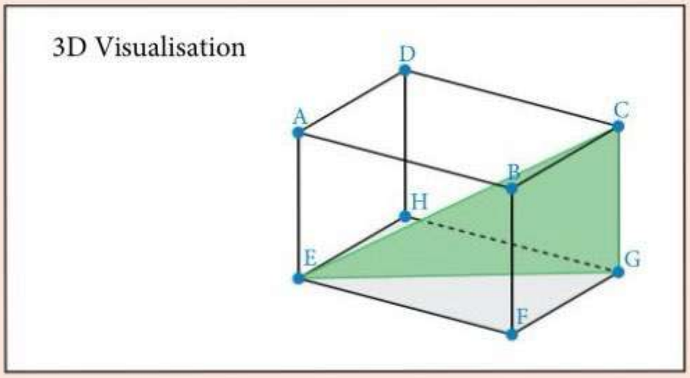


Fig. 7.10

There are two methods to determine whether a triangle in a cuboid is a right-angled triangle.

Method 1: Find a rectangle

To determine whether $\triangle EFG$ and $\triangle CGH$ are right-angled triangles:

- **(a)** Drag the vertices of the green triangle to form $\triangle EFG$. Note that $\triangle EFG$ lies on the plane *EFGH*. Click and drag the space around the cuboid to rotate it and observe the plane *EFGH*. Is the plane *EFGH* a rectangle?
- **(b)** Thus, is $\angle EFG = 90°$ and $\triangle EFG$ a right-angled triangle? Explain your answer.
- **(c)** Using the same method as above, determine whether $\triangle CGH$ is a right-angled triangle by identifying the appropriate rectangle and the right angle of the triangle.

Method 2: Find a normal to a plane

To determine whether $\triangle CGE$ and $\triangle CHE$ are right-angled triangles:

- **(a)** Drag the vertices of the green triangle to form $\triangle CGE$. Rotate the cuboid and observe the vertical line *CG* and the horizontal plane *EFGH*. Is the line *CG* a *normal* to the plane *EFGH*? Explain your answer.
- **(b)** Is the line *GE* a line on the plane *EFGH*?
- **(c)** Thus, is $\angle CGE = 90°$ and $\triangle CGE$ a right-angled triangle? Explain your answer.
- **(d)** Do the same and determine whether $\triangle CHE$ is a right-angled triangle. Identify the appropriate plane, the corresponding normal and the right angle of $\triangle CHE$, if applicable.

Problem-solving Tip

A normal to a plane is perpendicular to every line on the plane.

4. Fig. 7.11 shows a cuboid.

- **(a)** Which lines are perpendicular to *ABQP*?
- **(b)** Which lines are perpendicular to *ADSP*?

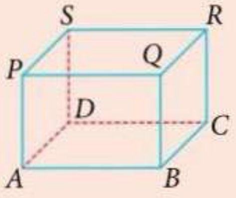


Fig. 7.11

5. Fig. 7.12 shows a square-based pyramid.

- **(a)** Identify all the right-angled triangles.
- **(b)** Which of these triangles are identical?

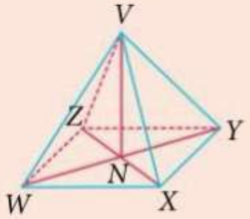


Fig. 7.12

From the Investigation on pages 235 and 236, we realise that a right-angled triangle in a 3D figure looks d er when the solid is represented as a 2D figure. Therefore, to solve three-dimensional problems, we need to be able to accurately identify the right angles, even if they may not appear as such in the 2D drawing. Then, how do we find th angle between a line and a plane?

We have seen that a line may intersect a plane at a point. If the line is not normal to the plane, it will be inclined at an angle to the plane. Fig. 7.13 shows the line *PO* intersecting a plane at the point *O*. How do we determine the angle between the line and the plane? Is it the same as the angle between the line and any other line of the plane passing through the point *O*?

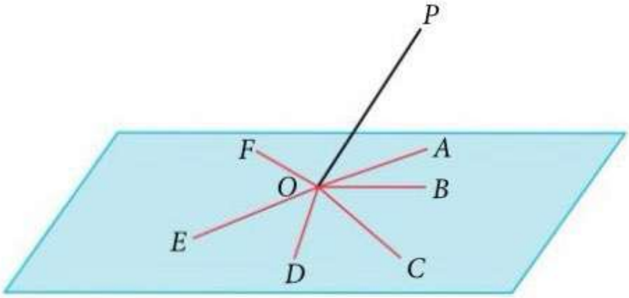


Fig. 7.13

On a piece of paper, draw the lines *OA*, *OB*, *OC* and so on as shown by the red lines in Fig. 7.13. Hold a pencil with one end on the paper to represent *OP*. Then, place another pencil on the plane in positions *OA*, *OB*, *OC* and so on, in turn. Are the angles between the two pencils the same?

To avoid ambiguity, the angle between *OP* and the plane is taken to be the *smallest angle* between the pencils. This angle can be found by dropping a normal from *P* to meet the plane at *Q* and then joining *OQ* as shown in Fig. 7.14. ∠*POQ* will be the angle between *OP* and the plane.

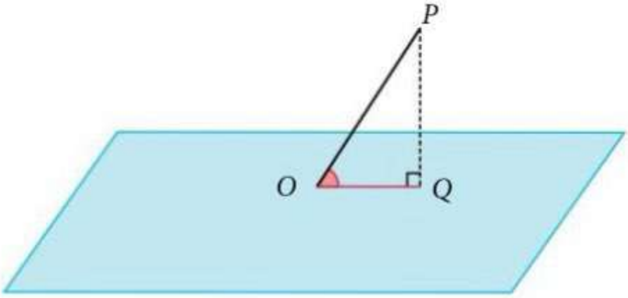


Fig. 7.14

Another way of looking at it is by using the idea of a 'shadow' or projection. Imagine that a lamp is vertically above *OP* as shown in Fig. 7.15. *OQ* will be the shadow of *OP* on a horizontal plane. This shadow *OQ* is known as the **projection** of *OP* onto the horizontal plane. Hence, the angle between *OP* and the plane is the angle between the line and its projection on the plane, which is ∠*POQ*.

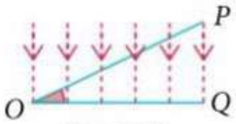


Fig. 7.15

The angle between a line and a plane is the angle between the line and its projection on the plane.

Similar and Further Questions
Exercise 7C
Questions 1(a)–(f), 2(a)–(d)

Worked Example 7

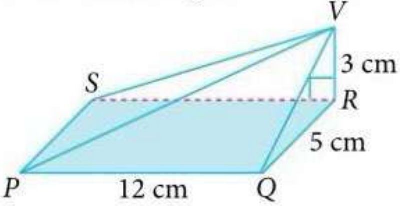
Solving three-dimensional problem involving pyramid

The figure shows a pyramid with a rectangular base *PQRS* and vertex *V* vertically above *R*.

Given that $PQ = 12$ cm, $QR = 5$ cm and $VR = 3$ cm, find

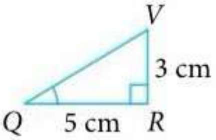
(i) $\angle VQR$,

(ii) the angle between *VP* and the base *PQRS*.



Solution

(i) In $\triangle VQR$,



$$\tan \angle VQR = \frac{3}{5}$$

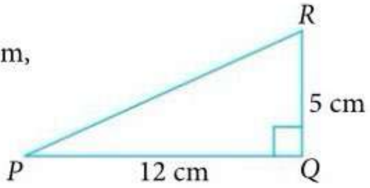
$$\angle VQR = \tan^{-1}\frac{3}{5}$$

$$= 31.0° \text{ (to 1 d.p.)}$$

(ii) The required angle is $\angle VPR$.

In $\triangle PQR$, $\angle PQR = 90°$.

Using Pythagoras' Theorem,



$$PR^2 = PQ^2 + QR^2$$

$$= 12^2 + 5^2$$

$$= 144 + 25$$

$$= 169$$

$$PR = \sqrt{169}$$

$$= 13 \text{ cm}$$

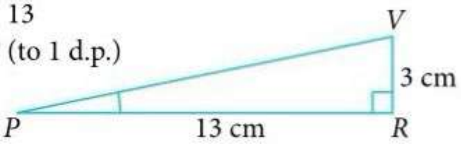
In $\triangle VRP$, $\angle VRP = 90°$.

VR is the normal to the plane PQRS

$$\tan \angle VPR = \frac{3}{13}$$

$$\angle VPR = \tan^{-1}\frac{3}{13}$$

$$= 13.0° \text{ (to 1 d.p.)}$$

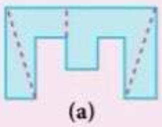


Problem-solving Tip

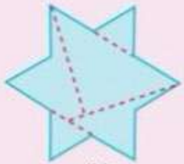
Reduce the three-dimensional to a problem in a plane by identifying the right-angled triangle that contains the side or the angle to be found.

Just For Fun

Two shapes are as shown.



(a)



(b)

Cut each shape along the dotted lines and use the pieces to form two squares.

Practise Now 7

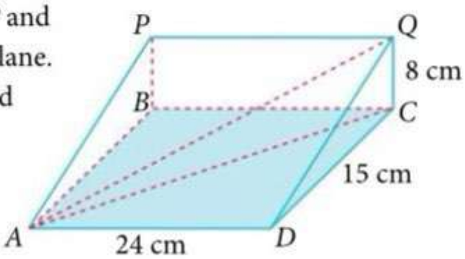
Similar and Further Questions
Exercise 7C
Questions 3, 8, 9, 16

1. The figure shows a wedge with a horizontal base *ABCD* and a vertical face *PQCB*. *APQD* is a rectangular sloping surface and $\triangle ABP$ and $\triangle DCQ$ are right-angled triangles in the vertical plane. Given that $CQ = BP = 8$ cm, $DC = AB = 15$ cm and $AD = BC = PQ = 24$ cm, find

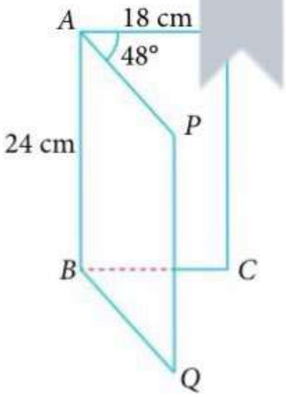
(i) $\angle BAC$,

(ii) $\angle CDQ$,

(iii) the angle between *AQ* and the plane *ABCD*.



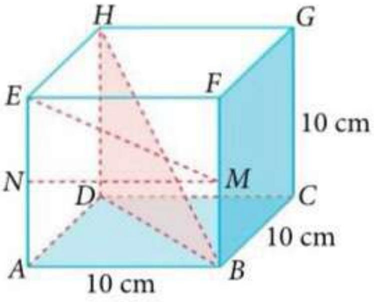
2. The diagram shows a photo frame which can be opened about AB. $ABCD$ and $ABQP$ are rectangles. The frame is opened through 48° as shown. Given that $AB = 24$ cm and $AP = AD = 18$ cm, find
   (i) the length of the straight line CQ,
   (ii) $\angle CAQ$.



Worked Example 8

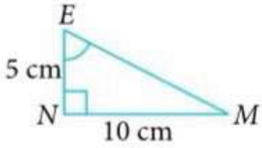
Solving three-dimensional problem involving cube

The figure shows a cube of length 10 cm.
M and N are the midpoints of BF and AE respectively.
Find
(i) $\angle MEN$, **(ii)** $\angle EMN$,
(iii) the angle between ME and the plane $EFGH$,
(iv) the angle between HB and the plane $ABCD$.



Solution

(i) In $\triangle MEN$,



$$\tan \angle MEN = \frac{10}{5}$$
$$= 2$$
$$\angle MEN = \tan^{-1} 2$$
$$= 63.4° \text{ (to 1 d.p.)}$$

(ii) **Method 1:**
In $\triangle MEN$,

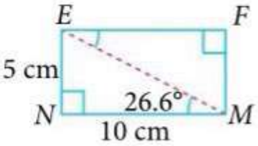
$$\tan \angle EMN = \frac{5}{10}$$
$$= \frac{1}{2}$$
$$\angle EMN = \tan^{-1} \frac{1}{2}$$
$$= 26.6° \text{ (to 1 d.p.)}$$

Method 2:

$$\angle EMN = 180° - \angle ENM - \angle MEN \text{ (}\angle\text{ sum of }\triangle\text{)}$$
$$= 180° - 90° - 63.435°$$
$$= 26.6° \text{ (to 1 d.p.)}$$

Reflection
(ii) Which method do you prefer? Why?

(iii) The required angle is $\angle MEF$.
$\angle MEF = \angle EMN$ (alt. $\angle$s, $EF \,//\, NM$)
$\therefore \angle MEF = 26.6°$



(iv) The required angle is $\angle HBD$.

In $\triangle BCD$, $\angle BCD = 90°$.

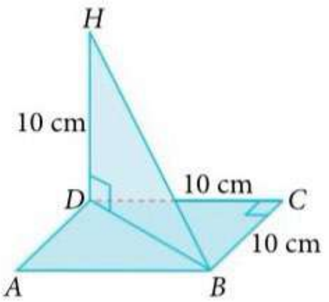
Using Pythagoras' Theorem,

$$\begin{aligned} DB^2 &= BC^2 + DC^2 \\ &= 10^2 + 10^2 \\ &= 100 + 100 \\ &= 200 \\ DB &= \sqrt{200} \end{aligned}$$

In $\triangle HBD$, $\angle HDB = 90°$.

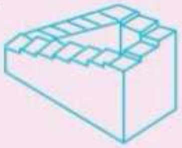
$$\tan \angle HBD = \frac{10}{\sqrt{200}}$$

$$\begin{aligned} \angle HBD &= \tan^{-1} \frac{10}{\sqrt{200}} \\ &= 35.3° \text{ (to 1 d.p.)} \end{aligned}$$



Just For Fun

Is it possible to construct the following three-dimensional objects from the solids that you know?



(a)

(b)

(c)

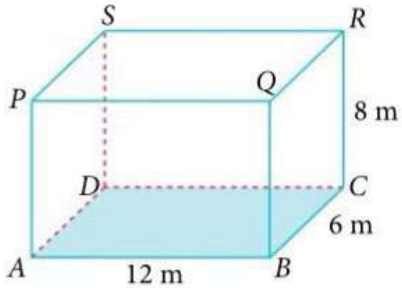
Practise Now 8

Similar and Further Questions

Exercise 7C

Questions 4, 5, 10, 11, 17

1. The diagram shows a cuboid where $AB = 12$ m, $BC = 6$ m and $CR = 8$ m.



Find

(i) $\angle ABP$,

(ii) $\angle BCQ$,

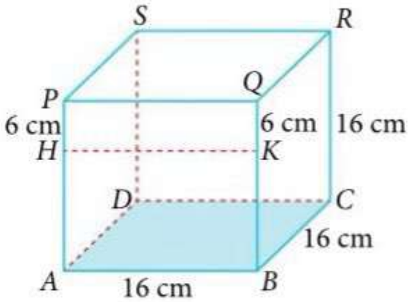
(iii) the angle between AR and the plane $ABCD$.

2. The figure shows a cube of length 16 cm.
Given that $PH = QK = 6$ cm,
find

(i) $\angle BCK$,

(ii) $\angle SBD$,

(iii) the angle between DK and the plane $BCRQ$.

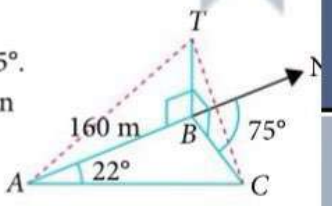


Worked Example 9

Finding angle of elevation in real-world context

Three points A, B and C are on level ground. B is due north of A, the bearing of C from A is 022° and the bearing of C from B is 075°.

(i) Given that A and B are 160 m apart, find the distance between B and C.



A vertical mast BT stands at B such that $\tan \angle TAB = \frac{3}{16}$.

(ii) Find the angle of elevation of T from C.

Solution

(i) $\angle BCA = 75° - 22°$ (ext. $\angle$ of $\triangle$)

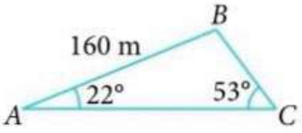
$= 53°$

Using Sine Rule,

$$\frac{BC}{\sin 22°} = \frac{160}{\sin 53°}$$

$$BC = \frac{160 \sin 22°}{\sin 53°}$$

$$= 75.0 \text{ m (to 3 s.f.)}$$



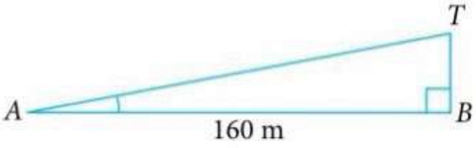
(ii) In $\triangle TAB$,

$$\tan \angle TAB = \frac{3}{16}$$

$$\frac{BT}{160} = \frac{3}{16}$$

$$BT = \frac{3}{16} \times 160$$

$$= 30 \text{ m}$$

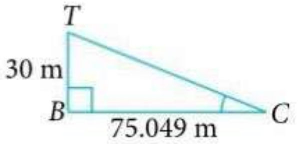


In $\triangle TCB$,

$$\tan \angle TCB = \frac{30}{75.049}$$

$$\angle TCB = \tan^{-1} \frac{30}{75.049}$$

$$= 21.8° \text{ (to 1 d.p.)}$$



$\therefore$ the angle of elevation of T from C is 21.8°.

Practise Now 9

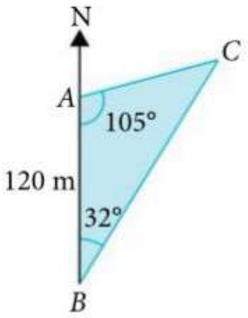
Similar and Further Questions
Exercise 7C
Questions 6, 7, 12, 13, 18

Three points A, B and C are on level ground. B is due south of A, the bearing of C from B is 032° and $\angle CAB = 105°$.

(i) Given that A and B are 120 m apart, find the distance between B and C.

A vertical mast CT of height 25 m stands at C.

(ii) Find the angle of elevation of T from B.

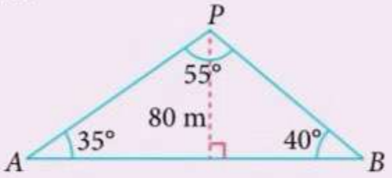


Thinking Time

Albert is given the following problem to solve.

> Shaha is in a cable car P at a height of 80 m above the ground. She observes a statue at A and a fountain at B. Given that the angles of depression of the statue and the fountain from P are 35° and 40° respectively and that $\angle APB = 55°$, find the distance between A and B.

In order to visualise the information in the problem, he drew the following diagram.
Is his diagram correct? Explain your answer.



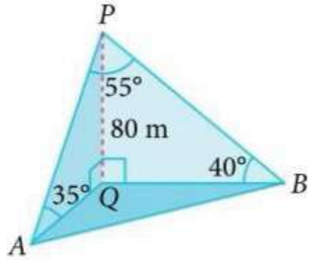
Worked Example 10

Solving problem involving angle of depression in real-world context

Shaha is in a cable car P at a height of 80 m above the ground. She observes a statue at A and a fountain at B. Given that the angles of depression of the statue and the fountain from P are 35° and 40° respectively and that $\angle APB = 55°$, find the distance between A and B.

*Solution

We will use **Pólya's Problem Solving Model** to guide us in solving this problem.



Stage 1: **Understand the problem**

We need to sketch a diagram to help us understand the problem.
A and B are on horizontal ground and P is above the ground.
Label the point on the ground that P is vertically above Q.
$\angle PAQ = 35°$, $\angle PBQ = 40°$ and $\angle APB = 55°$.

Stage 2: **Think of a plan**

Since $\angle APB$ is given, what can we use to find AB?
Observe that we can use the Cosine Rule, but we first have to find AP and BP.
Since $\triangle APQ$ and $\triangle BPQ$ are right-angled triangles, how can we find AP and BP?

Big Idea

Diagrams
Sketching a diagram can help us to visualise the relationships between the given information and discover other relationships.

Stage 3: **Carry out the plan**

In $\triangle APQ$, $\angle AQP = 90°$.

$$\sin 35° = \frac{80}{AP}$$

$$AP = \frac{80}{\sin 35°}$$

$$= 139.48 \text{ m (to 5 s.f.)}$$

In $\triangle BPQ$, $\angle BQP = 90°$.

$$\sin 40° = \frac{80}{BP}$$

$$BP = \frac{80}{\sin 40°}$$

$$= 124.46 \text{ m (to 5 s.f.)}$$

Using Cosine Rule,

$$AB^2 = AP^2 + BP^2 - 2 \times AP \times BP \cos 55°$$

$$= 139.48^2 + 124.46^2 - 2 \times 139.48 \times 124.46 \cos 55°$$

$$= 15\ 031 \text{ (to 5 s.f.)}$$

$$AB = \sqrt{15\ 031}$$

$$= 123 \text{ m (to 3 s.f.)}$$

Stage 4: Look back

What did we do to solve this problem?

How can we ensure that we draw an accurate diagram to visualise the information given?

Reflection

Is the diagram here different from the one Albert drew in the Thinking Time? Were you able to visualise the 3D figure correctly?

Practise Now 10

Similar and Further Questions
Exercise 7C
Questions 14, 15, 19

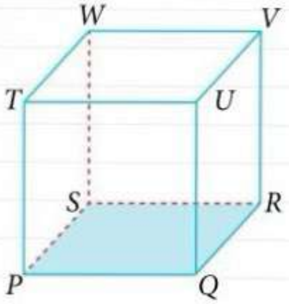
Imran is on the top T of an observation tower OT. The height of the tower is 54 m. He observes a car that has broken down at A, causing a traffic jam at B. Given that the angles of depression of A and B from T are 42° and 38° respectively and that $\angle ATB = 48°$, find the distance between A and B.

Reflection

1. Can I use my knowledge of angles of elevation and depression, and bearings, to draw diagrams correctly in order to visualise three-dimensional situations?
2. How can I consolidate all that I have learnt about triangles to solve three-dimensional problems?

Exercise 7C

1. With reference to the cube below, name, in each case, the angle between the line and the plane.

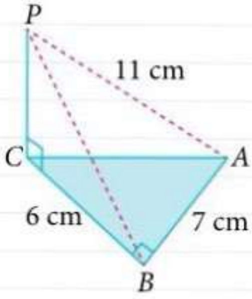


(a) PV and PQRS
(b) QV and PQRS
(c) SU and PQRS
(d) PV and SRVW
(e) TV and SRVW
(f) QW and PQUT

2. Given that the sides of the cube in Question 1 are of length 5 cm, calculate

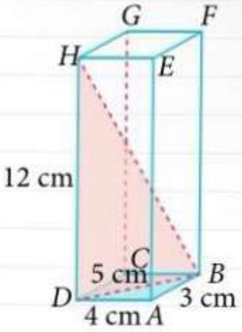
(a) the length of QV,
(b) the angle which QV makes with the plane PQRS,
(c) the length of PV,
(d) the angle which PV makes with the plane PQRS.

3. The figure shows △ABC, right-angled at B and lying on a horizontal plane. P is a point vertically above C.
Given that AB = 7 cm, BC = 6 cm and AP = 11 cm, find



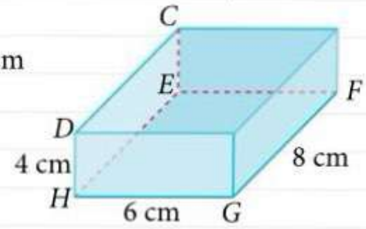
(i) AC,
(ii) PC,
(iii) ∠PAC,
(iv) the angle between PB and the plane ABC.

4. The diagram shows a rectangular box.
AB = 3 cm, AD = 4 cm, BD = 5 cm and DH = 12 cm.
Find



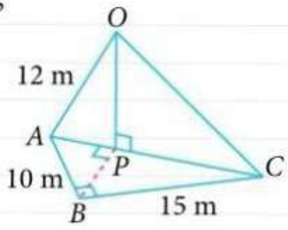
(i) the length of BH,
(ii) the angle between BD and the plane CDHG,
(iii) ∠HBD.

5. A rectangular block of sugar has a horizontal base EFGH. The corners C and D are vertically above E and H respectively.
DH = 4 cm, GH = 6 cm and FG = 8 cm.
Find



(i) ∠DGH,
(ii) HF,
(iii) the angle between DF and the plane EFGH.

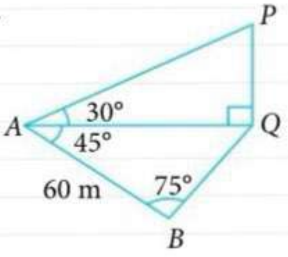
6. The diagram shows a triangular stage ABC on horizontal ground, where AB = 10 m, BC = 15 m and ∠ABC is a right angle. The base of a vertical pole, OP, is at point P along the edge AC of the stage. The pole is held in place by two cables, OA and OC. Given that ∠APB = 90° and OA = 12 m, find



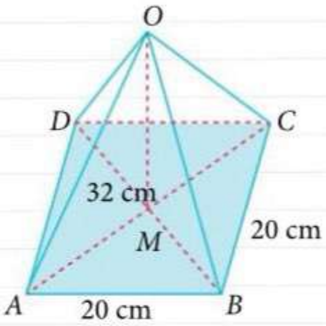
(i) ∠BAC,
(ii) the length of AP,
(iii) the length of OP,
(iv) the angle of elevation of O from B.

Exercise 7C

7. In the figure, the angle of elevation of the top of a vertical tower *PQ* from a point *A* is 30°. The foot of the tower *Q*, is on the same horizontal plane as *A* and *B*. Given that $AB = 60$ m, $\angle BAQ = 45°$ and $\angle ABQ = 75°$, find the height of the tower.



8. *OABCD* is a pyramid. The square base *ABCD* has sides of length 20 cm and lies on a horizontal plane. *M* is the point of intersection of the diagonals of the base and *O* is vertically above *M*.



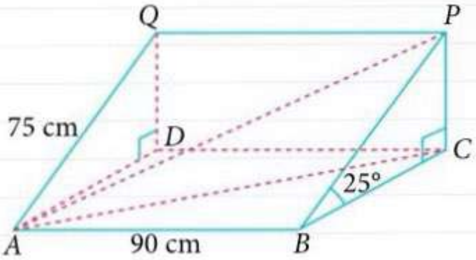
Given that $OA = 32$ cm, find

(i) the length of *AM*,

(ii) the height of the pyramid,

(iii) the angle between *OA* and the plane *ABCD*.

9. *ABPQ* is the rectangular sloping surface of a desk with *ABCD* lying on a horizontal plane. *Q* and *P* lie vertically above *D* and *C*.



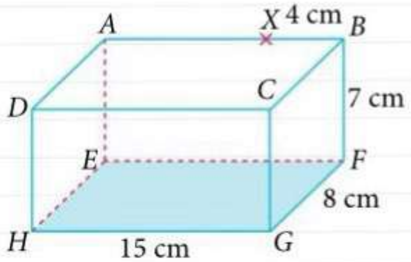
Given that $AB = PQ = 90$ cm, $AQ = BP = 75$ cm and $\angle PBC = \angle QAD = 25°$, find

(i) *AC*,

(ii) $\angle PAC$,

(iii) the angle which *BQ* makes with the plane *BCP*.

10. The diagram shows a cuboid with a horizontal base *EFGH* where $HG = 15$ cm, $GF = 8$ cm and $BF = 7$ cm.



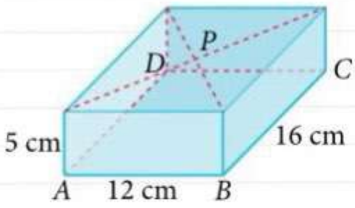
X is a point on *AB* such that $XB = 4$ cm.

Find

(i) the angle between *CE* and the plane *ABEF*,

(ii) $\angle GXF$.

11. *P* is the centre of the upper face of the rectangular block with *ABCD* as its base.



Find

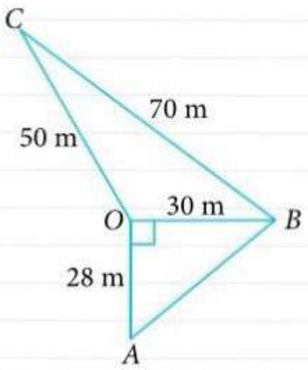
(i) $\angle PAC$,

(ii) $\angle PAB$.

Exercise 7C

12. Points P, Q and R are on level ground with Q due east of P and R due south of P.
A vertical mast PT stands at P and the angle of elevation of the top T from Q is 3.5°.
Given that $PQ = 1000$ m and $PR = 750$ m, find
- **(i)** the bearing of Q from R,
- **(ii)** the height of the mast,
- **(iii)** the angle of elevation of T from R.

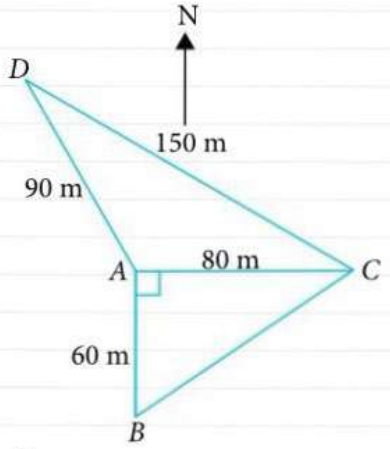
13. The diagram shows a campsite $OABC$ on horizontal ground. O is due north of A and B is due east of O.
$OA = 28$ m, $OB = 30$ m, $OC = 50$ m and $BC = 70$ m.



(a) Calculate
- **(i)** the bearing of A from B,
- **(ii)** $\angle COB$,
- **(iii)** the bearing of C from O.

(b) A bird is hovering vertically above B.
The angle of elevation of the bird from A is 29°.
Find the angle of elevation of the bird from C.

14. Points A, B, C and D are at sea level. B is 60 m due south of A and C is 80 m from A on a bearing of 090°. $AD = 90$ m and $CD = 150$ m.



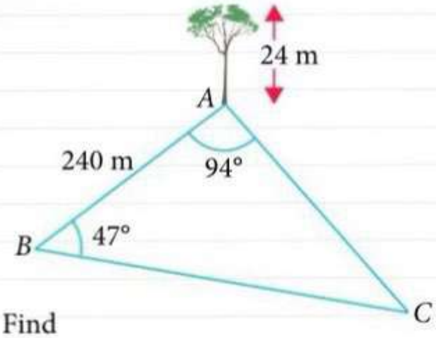
(a) Find
- **(i)** the bearing of C from B,
- **(ii)** the bearing of D from A.

A is a point at the base of a vertical cliff. T is a point on the top of the cliff vertically above A.
The angle of depression of C from T is 8.6°.

(b) Calculate
- **(i)** the height of the cliff,
- **(ii)** the angle of depression of D from T.

Exercise 7C

15. A tree of height 24 m stands vertically at *A* on the ground of an island. Two boats are at *B* and *C* such that ∠*BAC* = 94°, ∠*ABC* = 47° and *AB* = 240 m.



(a) Find

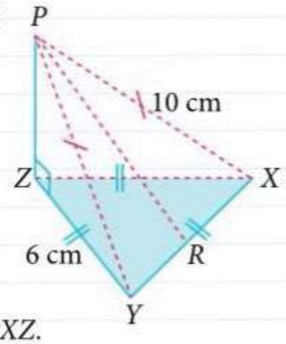
- **(i)** the distance between *B* and *C*,
- **(ii)** the area of △*ABC*,
- **(iii)** the shortest distance from *A* to *BC*.

The boat at *B* sails in a straight line towards *C*.

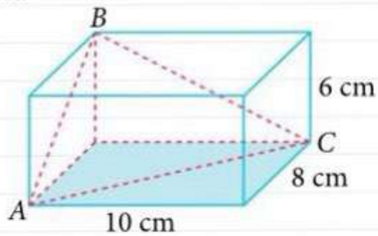
(b) Find the greatest angle of depression of the boat from the top of the tree.

16. In the diagram, *XYZ* is an equilateral triangle with sides of length of 6 cm lying on a horizontal plane. *P* lies vertically above *Z*, *R* is the midpoint of *XY* and *PX* = *PY* = 10 cm. Find

- **(i)** ∠*PYZ*,
- **(ii)** the angle which *RZ* makes with the plane *PXZ*.



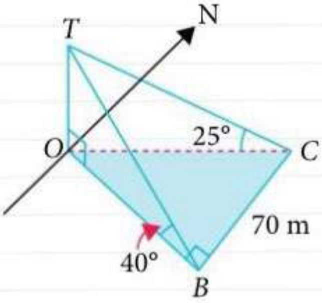
17. The figure shows a block of wood in the shape of a cuboid with dimensions 10 cm by 8 cm by 6 cm. Joyce cuts the block into two pieces such that the cutting tool passes through the points *A*, *B* and *C* as shown.



Given that the triangular surface *ACB* on one piece of the block is to be coated with varnish, find

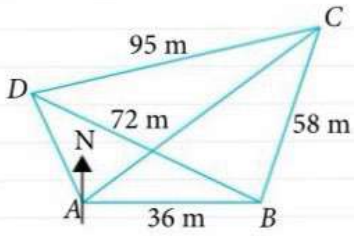
- **(i)** ∠*ABC*,
- **(ii)** the area of the surface that is to be coated with varnish.

18. Raju stands at a point *B*, due east of a vertical tower *OT*, and observes that the angle of elevation of the top of the tower *T* from *B* is 40°. He walks 70 m due north and finds that the angle of elevation of *T* from his new position at *C* is 25°.



Find the height of the tower and hence distance *OB*.

19. The diagram shows a plot of land *ABCD*. *B* is due east of *A* and the bearing of *C* from *A* is 048°. *AB* = 36 m, *BC* = 58 m, *BD* = 72 m and *CD* = 95 m.



- **(i)** Find the bearing of *C* from *B*.

A vertical control tower of height 35 m stands at *B*. Yasir cycles from *C* to *D* and reaches a point *P* where the angle of depression of *P* from the top of the tower is the greatest.

- **(ii)** Find the angle of depression of *P* from the top of the tower.

Looking Back

Our study of trigonometry began with the exploration of the **proportionality** of sides in similar right-angled triangles in Book 2. In this chapter, we conclude by seeing how trigonometry can be applied to solve many real-world problems. As described in the Chapter Opener, trigonometry examines the relationship between the angles and the lengths of the sides of a triangle.

Our knowledge of trigonometry provides a means for us to use triangles to **model** many real-world situations, including those related to angles of elevation and depression, as well as those involving bearings. To help us visualise the situations, we draw **diagrams** to represent the relationships between the angles and lengths of the sides of the triangles involved. These diagrams are crucial for us to apply the appropriate trigonometric ratios and formulae in order to find the required measurements. These measurements are often essential in the fields of engineering, surveying and navigation. There is another facet of trigonometry that you might uncover in more advanced levels of mathematics in the future — seeing the relationships between angles and sides of triangles as a **function**.

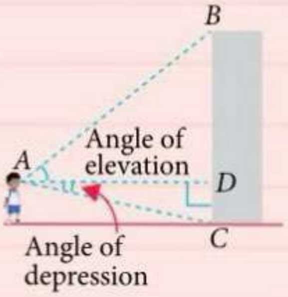
Summary

1. **Angles of elevation and depression**

   (a) When a person looks at the top of the wall, B, the angle between the horizontal AD and the line of sight AB, i.e. $\angle BAD$, is called the *angle of elevation*.

   (b) When a person looks at the bottom of the wall, C, the angle between the horizontal AD and the line of sight AC, i.e. $\angle CAD$, is called the *angle of depression*.

   - Think of a real-life problem that can be solved with the use of angle of elevation or angle of depression.

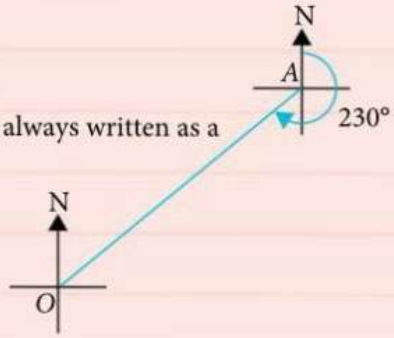


2. **Bearings**

   The bearing of a point O from another point A is an angle measured from the north line at A to the line OA, in a clockwise direction. It is always written as a *three-digit number*, if it is an integer.

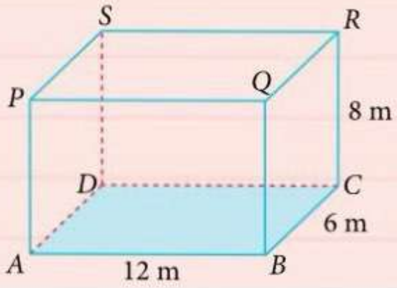
   The bearing of O from A is 230°.

   - Find the bearing of A from O.



3. The basic technique used to solve a three-dimensional problem is to reduce it to a problem in a plane.

   - Identify the right-angled triangle in the cuboid that can be used to find the angle of elevation of Q from D.



CHAPTER 8

Arc Length and Sector Area

The wheel can be considered one of the most important technological tools since ancient times. Wheels on wagons and other transportation vehicles help us move loads over long distances with less effort because of their circular shape. The circle is one of most fascinating geometric shapes. To the Greeks, it is a symbol of perfection in symmetry and balance.

All circles are similar because the ratio of a circle's circumference to its diameter is π (pi) for any circle. This idea of **proportionality** also applies to other elements of a circle including its arc length and sector area, allowing us to derive formulae to make these calculations, which we shall learn in this chapter.

Learning Outcomes

What will we learn in this chapter?

- What chord, arc, segment and sector of a circle are
- How to find the length of an arc, the area of a sector and the area of a segment of circle when given an angle in degrees

Introductory Problem

Fig. 8.1 shows a scale drawing of a room with a balcony.
The arc of the balcony is an arc of a circle with centre O and radius 1.6 m.
How do you find the length of the arc of the balcony?

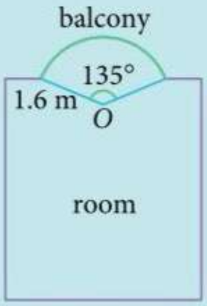


Fig. 8.1

In this chapter, we will learn to find the length of an arc and the area of a sector of a circle, in order to solve such problems.

8.1 Length of arc

Class Discussion

Understanding the parts of a circle

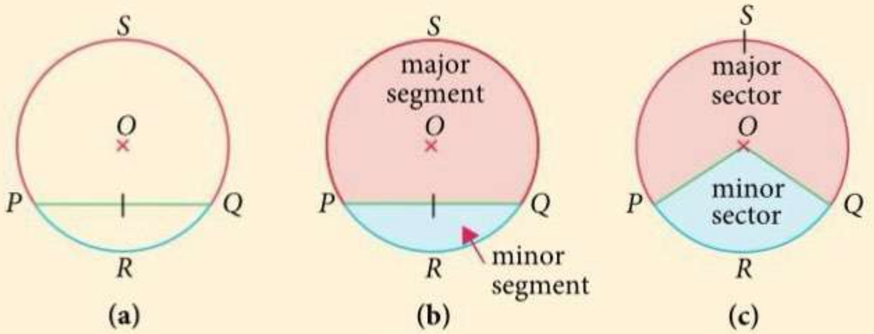


Fig. 8.2

Big Idea

Notations

A chord is a line segment described by the two endpoints, e.g. chord PQ. In contrast, we use three points to describe an arc to differentiate between the minor and major arcs e.g. arc PRQ and arc PSQ. Similarly, we use three points to describe a segment, e.g. segment PRQ and segment PSQ. Alternatively, we can refer to them as the minor or major segment.
We use four points to describe a sector e.g. sector $OPRQ$ or sector $OPSQ$. Alternatively, we can refer to them as the minor or major sector. Labelling points in a clockwise or anti-clockwise order helps us to easily identify the correct mathematical object.

Fig. 8.2(**a**) shows a circle with centre O. The line PQ is called a **chord**. PRQ and PSQ are called **arcs**, which are part of the *circumference*.
The arc PSQ is called a *major arc* and the arc PRQ is called a *minor arc*.

1. Why is the arc PRQ a minor arc?
2. Can a chord pass through the centre of a circle? If yes, what is the chord also called?
3. Is the *diameter* the longest chord? Explain.
4. If the chord PQ in Fig. 8.2(**a**) passes through the centre of the circle, will the arc PRQ still be a minor arc? Or will it be a major arc or neither?

In Fig. 8.2(**b**), the chord PQ divides the circle into two **segments**. The region enclosed by the chord PQ and the minor arc PRQ is called a *minor segment*. The region enclosed by the chord PQ and the major arc PSQ is called the *major segment*.

5. Why is the segment *PRQ* a minor segment?
6. Can a segment be a *semicircle*? When will this happen?

The part of a circle enclosed by any two radii of a circle and an arc is called a **sector**. In Fig. 8.2(**c**), the region enclosed by the radii *OP*, *OQ* and the minor arc *PRQ* is called a *minor sector* of the circle. The region enclosed by the radii *OP*, *OQ* and the major arc *PSQ* is called a *major sector* of the circle.

7. Why is the sector *OPRQ* a minor sector?
8. If the radii *OP* and *OQ* in Fig. 8.2(**c**) lie on the same straight line, will the sector *OPRQ* still be a minor sector? Or will it be a major sector or neither?
9. Can a sector be a *semicircle*? When will this happen?

Investigation

Discovering how to calculate arc length

In the first diagram in Table 8.1 below, a circle is divided into 2 equal parts (or sectors).

The circumference of the circle is also divided into 2 equal parts, so $\frac{\text{length of arc } APB}{\text{circumference}} = \frac{1}{2}$.

Similarly, the angle at the centre of the circle, which is 360°, is divided into 2 parts, so $\frac{\angle AOB}{360^\circ} = \frac{1}{2}$, where $\angle AOB$ refers to the *shaded* angle in the diagram.

Complete Table 8.1 by observing the number of equal parts that the circumference and the angle at the centre of the circle are divided into.

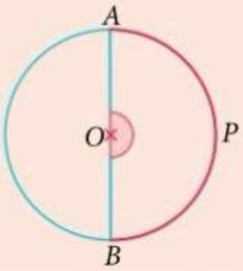
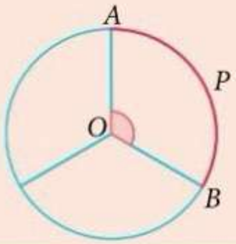
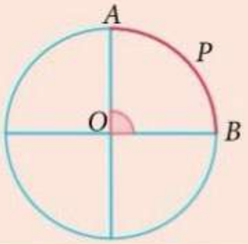
Diagram	**No. of equal parts**	$\frac{\text{length of arc } APB}{\text{circumference}}$	$\frac{\angle AOB}{360^\circ}$
	2	$\frac{1}{2}$	$\frac{1}{2}$
	3		
	4		

Table 8.1

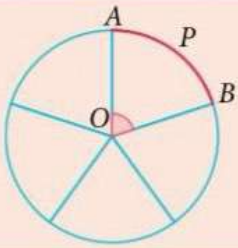
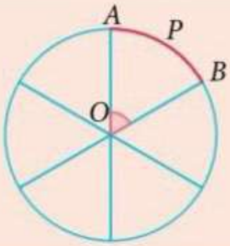
Diagram	No. of equal parts	$\frac{\text{length of arc } APB}{\text{circumference}}$	$\frac{\angle AOB}{360^\circ}$
	5		
	6		

Table 8.1

1. What do you observe about the values in the last two columns of Table 8.1?
2. If the circumference of a circle is divided into n equal arcs such that the arc APB is one of the n arcs, state the values of $\frac{\text{length of arc } APB}{\text{circumference}}$ and $\frac{\angle AOB}{360^\circ}$.
3. Write down a formula for finding the length of the arc APB in terms of $\angle AOB$.

Internet Resources

Go to www.sl-education.com/tmsoupp4/pg252 or scan the QR code and open the geometry template 'Arc Length'. Using the template, explore the relationship between the length of arc APB and $\angle AOB$, for any $\angle AOB$.

4. Recall that if $y = kx$, where $k \neq 0$, then y is **directly proportional** to x.

Fill in the blank. Since length of arc $APB = \frac{\angle AOB}{360^\circ} \times \text{circumference}$

$$= \frac{\text{circumference}}{360^\circ} \times \angle AOB,$$

where $\frac{\text{circumference}}{360^\circ}$ is a non-zero constant, then the length of arc APB is

to $\angle AOB$.

Big Idea

Proportionality

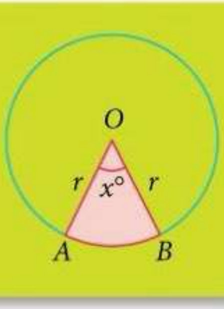
Since the arc length is directly proportional to the angle subtended by the arc, we can use the constant of proportionality $\frac{\text{circumference}}{360^\circ}$ to calculate the arc length.

From the above Investigation, we observe the following:

Arc length $= \frac{x^\circ}{360^\circ} \times$ **circumference**

$$= \frac{x^\circ}{360^\circ} \times 2\pi r,$$

where x° is the angle subtended by the arc at the centre of the circle of radius r.



Worked Example 1

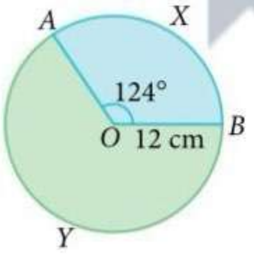
Calculating arc length

In the diagram, O is the centre of a circle of radius 12 cm and $\angle AOB = 124°$.

Find

(i) the length of the minor arc AXB,

(ii) the perimeter of the major sector $OAYB$.



Solution

(i) Length of minor arc $AXB = \frac{x°}{360°} \times 2\pi r$

$$= \frac{124°}{360°} \times 2\pi \times 12$$

$$= 26.0 \text{ cm (to 3 s.f.)}$$

(ii) **Method 1:**

Reflex $\angle AOB = 360° - 124°$ ($\angle$s at a point)

$$= 236°$$

Perimeter of major sector $OAYB$ = length of arc $AYB + OA + OB$

$$= \frac{236°}{360°} \times 2\pi \times 12 + 12 + 12$$

$$= 73.4 \text{ cm (to 3 s.f.)}$$

Method 2:

Length of arc AYB = circumference − length of arc AXB

$$= 2\pi \times 12 - 25.970$$

$$= 49.428 \text{ cm (to 5 s.f.)}$$

Perimeter of major sector $OAYB$

$= \text{length of arc } AYB + OA + OB$

$= 49.428 + 12 + 12$

$= 73.4$ cm (to 3 s.f.)

Problem-solving Tip

For the final answer to be accurate to 3 s.f., intermediate working should be correct to a higher number of accuracy, such as 4 or 5 s.f.

Reflection

(ii) Which method do you prefer? Why?

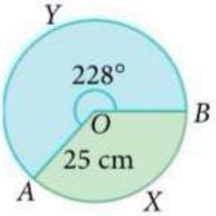
Practise Now 1

Similar and Further Questions
Exercise 8A
Questions 1–9

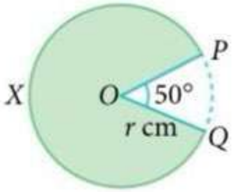
1. In the diagram, O is the centre of a circle of radius 25 cm and reflex $\angle AOB = 228°$. Find

(i) the length of the major arc AYB,

(ii) the perimeter of the minor sector $OAXB$.



2. The figure shows a logo design in which a sector has been removed from the circle, centre O and radius r cm. Given that the length of the major arc PXQ is 36 cm and $\angle POQ = 50°$, find the value of r.



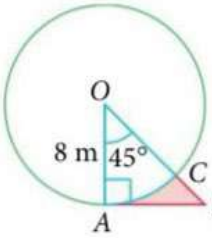
Introductory Problem Revisited

After learning how to calculate the length of an arc of a circle, can you solve the Introductory Problem?

Worked Example 2

Solving problem involving arc length

In the figure, O is the centre of a circle of radius 8 m.
The points A and C lie on the circumference of the circle and OCB is a straight line.
Given that $\angle AOB = 45°$ and OA is perpendicular to AB, find the perimeter of the shaded region ABC.



Problem-solving Tip

For greater accuracy, we can work in terms of π in the intermediate steps.

Solution

$$\begin{aligned}\text{Length of arc } AC &= \frac{x°}{360°} \times 2\pi r\\ &= \frac{45°}{360°} \times 2\pi \times 8\\ &= 2\pi \text{ m}\end{aligned}$$

$$\begin{aligned}\angle ABC &= 180° - 90° - 45° \ (\angle \text{ sum of } \triangle)\\ &= 45°\end{aligned}$$

$\therefore \triangle AOB$ is isosceles and so $AB = OA = 8$ m.

$$\begin{aligned}OB^2 &= 8^2 + 8^2 \quad \text{Pythagoras' Theorem}\\ &= 128\\ OB &= \sqrt{128}\\ &= 11.314 \text{ m (to 5 s.f.)}\end{aligned}$$

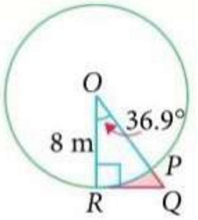
$$\begin{aligned}BC &= OB - OC\\ &= 11.314 - 8\\ &= 3.314 \text{ m}\end{aligned}$$

$$\begin{aligned}\therefore \text{perimeter of shaded region } ABC &= \text{length of arc } AC + AB + BC\\ &= 2\pi + 8 + 3.314\\ &= 17.6 \text{ m (to 3 s.f.)}\end{aligned}$$

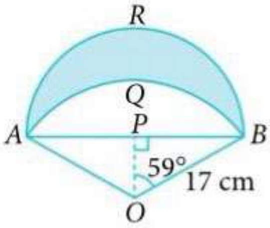
Practise Now 2

Similar and Further Questions
Exercise 8A
Questions 10–16

1. In the figure, O is the centre of a circle of radius 8 m. The points P and R lie on the circumference of the circle and OPQ is a straight line. Given that $\angle QOR = 36.9°$ and OR is perpendicular to QR, find the perimeter of the shaded region PQR.



2. In the diagram, AQB is the minor arc of a circle with centre O and radius 17 cm. ARB is a semicircle with AB as its diameter and P as its centre. $\triangle OPB$ is a right-angled triangle with $\angle OPB = 90°$ and $\angle POB = 59°$. Find
   (i) the length of AB,
   (ii) the perimeter of the shaded region.



Reflection

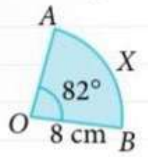
1. How has my knowledge of circumference and proportionality helped me to obtain the formula for calculating arc length?
2. What have I learnt in this section that I am still unclear of?

Advanced Intermediate Basic

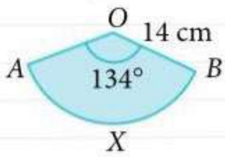
Exercise 8A

1. Find the length of each of the following arcs AXB.

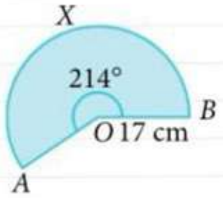
(a)



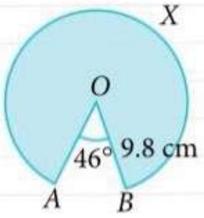
(b)



(c)



(d)

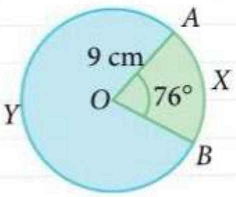


2. For each of the following circles, find

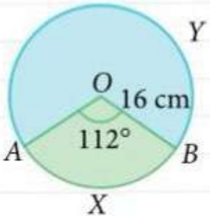
(i) the length of the minor arc AXB,

(ii) the perimeter of the major sector $OAYB$.

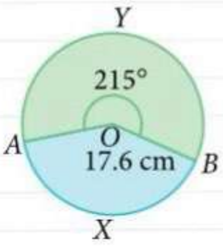
(a)



(b)

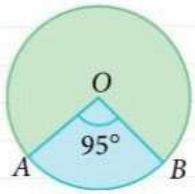


(c)



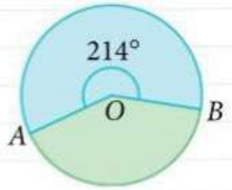
3. Find the radius of each of the following circles.

(a)



Length of minor arc = 26.53 cm

(b)



Length of major arc = 104.6 cm

4. The radius of a circle is 14 m. Find the angle at the centre of the circle subtended by an arc of length

(a) 12 m, **(b)** 19.5 m,

(c) 64.2 m, **(d)** 84.6 m,

giving your answers correct to the nearest degree.

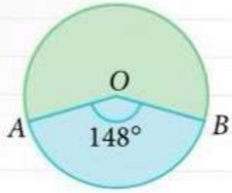
5. The hour hand of a large clock mounted on a clock tower travels through an angle of 45°. If the hour hand is 1.5 m long, how far does the tip of the hour hand travel?

6. A piece of wire 32 cm long is bent to form a sector of a circle of radius 6 cm. Find the angle subtended by the wire at the centre of the circle.

Exercise 8A

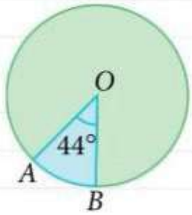
7. Find the radius of each of the following circles.

(a)



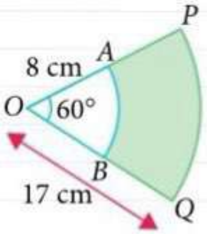
Perimeter of minor sector = 77.91 cm

(b)



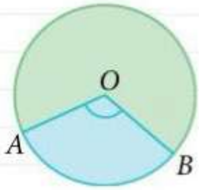
Perimeter of major sector = 278.1 cm

8. The figure shows two sectors OAB and OPQ with O as the common centre, and OAP and OBQ are straight lines. The lengths of OA and OQ are 8 cm and 17 cm respectively.



Given that $\angle AOB = \angle POQ = 60°$, find the perimeter of the shaded region, giving your answer in the form $a + b\pi$, where a and b are rational numbers.

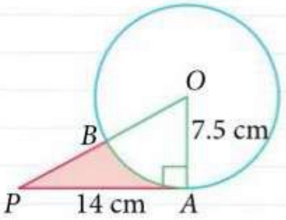
9. In the diagram, the length of the minor arc is $\frac{7}{24}$ of the circumference of the circle.



(i) Find $\angle AOB$.

(ii) Given that the diameter of the circle is 14 cm, find the length of the minor arc.

10. In the figure, O is the centre of a circle of radius 7.5 cm. The points A and B lie on the circumference of the circle and OBP is a straight line.

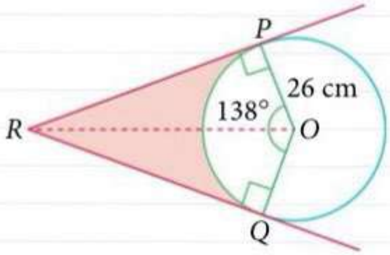


Given that $PA = 14$ cm and OA is perpendicular to AP, find

(i) $\angle POA$,

(ii) the perimeter of the shaded region PBA.

11. The diagram shows a circle with centre O and radius 26 cm. Triangles OPR and OQR are congruent and $\angle OPR = \angle OQR = 90°$.

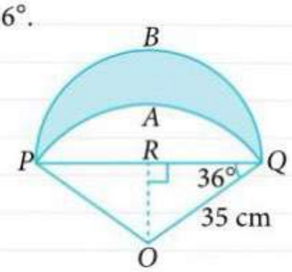


Given that $\angle POQ = 138°$, find

(i) the length of QR,

(ii) the perimeter of the shaded region.

12. In the diagram, PAQ is the minor arc of a circle with centre O and radius 35 cm. PBQ is a semicircle with PQ as its diameter and R as its centre. $\triangle ORQ$ is a right-angled triangle with $\angle ORQ = 90°$ and $\angle RQO = 36°$.



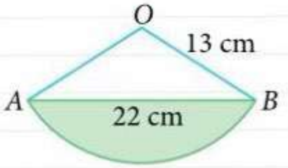
Find

(i) the length of PQ,

(ii) the perimeter of the shaded region.

Exercise 8A

13. The figure shows a sector of a circle with centre O and radius 13 cm.

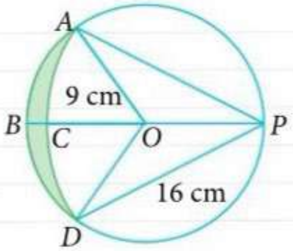


Given that the length of the chord $AB = 22$ cm,

(i) show that $\angle AOB$ is approximately 115.6°,

(ii) find the perimeter of the shaded region.

14. In the figure, ABD is the minor arc of a circle with centre O and radius 9 cm. ACD is the minor arc of a circle with centre P and radius 16 cm.



Given that $BCOP$ is the diameter of circle $ABDP$, with centre O and radius 9 cm, find

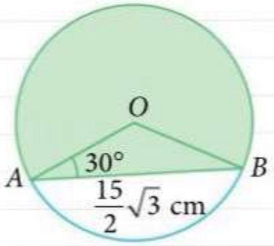
(i) $\angle APB$,

(ii) $\angle AOB$,

(iii) the perimeter of the shaded region.

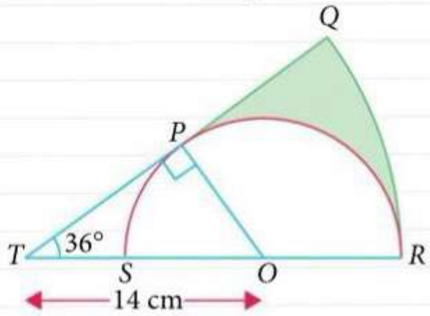
15. In the diagram, O is the centre of a circle. The points A and B lie on the circumference of the circle such that $AB = \frac{15}{2}\sqrt{3}$ cm and $\angle OAB = 30°$.

Find the perimeter of the shaded region.



16. The figure shows a semicircle with centre O and diameter SR. QR is an arc of another circle with centre T and T lies on RS produced. P is a point on the semicircle and QT such that OP is perpendicular to PT.

Given that $TO = 14$ cm and $\angle OTP = 36°$, find the perimeter of the shaded region.



8.2 Area of sector

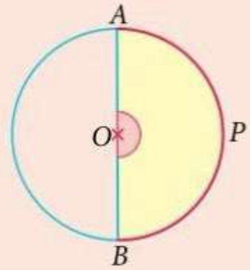
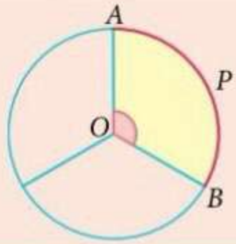
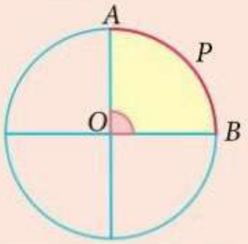
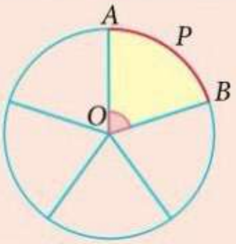
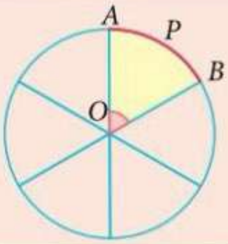
Investigation — Discovering how to calculate sector area

In the first diagram in Table 8.2 below, a circle is divided into 2 equal parts (or sectors).

The area of the circle is divided into 2 equal parts, so $\frac{\text{area of sector } OAPB}{\text{area of circle}} = \frac{1}{2}$.

The angle at the centre of the circle, which is 360°, is divided into 2 parts, so $\frac{\angle AOB}{360°} = \frac{1}{2}$, where $\angle AOB$ refers to the *shaded* angle in the diagram.

Complete Table 8.2 by observing the number of equal parts that the area and the angle at the centre of the circle are divided into.

Diagram	**No. of equal parts**	$\frac{\textbf{area of sector } OAPB}{\textbf{area of circle}}$	$\frac{\angle AOB}{360^\circ}$
	2	$\frac{1}{2}$	$\frac{1}{2}$
	3		
	4		
	5		
	6		

1. What do you observe about the values in the last two columns of Table 8.2?
2. If a circle is divided into n equal sectors such that the sector $OAPB$ is one of the n sectors, state the values of $\frac{\text{area of sector } OAPB}{\text{area of circle}}$ and $\frac{\angle AOB}{360^\circ}$.

3. Write down a formula for finding the area of the sector $OAPB$ in terms of $\angle AOB$.

4. Recall that if $y = kx$, where $k \neq 0$, then y is **directly proportional** to x.

Fill in the blank. Since area of sector $OAPB = \frac{\angle AOB}{360°} \times$ area of circle

$$= \frac{\text{area of circle}}{360°} \times \angle AOB,$$

where $\frac{\text{area of circle}}{360°}$ is a non-zero constant, then the area of sector $OAPB$ is ______________ to $\angle AOB$.

5. Is the area of sector $OABP$ directly proportional to the length of arc APB? Explain.

Internet Resources

Go to www.sl-education.com/tmsoupp4/pg259 or scan the QR code and open the geometry template 'Sector Area'. Using the template, explore the relationship between the area of sector $OAPB$ and $\angle AOB$, for any $\angle AOB$.

Big Idea

Proportionality

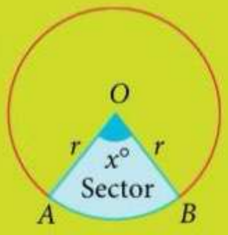
Since the sector area is directly proportional to the sector angle, we can use the constant of proportionality $\frac{\text{area of circle}}{360°}$ to calculate the sector area.

From the above Investigation, we observe the following:

Sector area $= \frac{x°}{360°} \times$ **area of circle**

$$= \frac{x°}{360°} \times \pi r^2,$$

where $x°$ is the angle of the sector subtended at the centre of the circle of radius r.



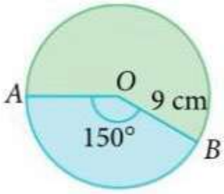
Worked Example 3

Calculating sector area

In the figure, O is the centre of a circle of radius 9 cm and $\angle AOB = 150°$. Find the area of

(i) the minor sector, **(ii)** the major sector.

Leave your answers in terms of π.



Solution

(i) Area of minor sector $= \frac{x°}{360°} \times \pi r^2$

$$= \frac{150°}{360°} \times \pi \times 9^2$$

$$= 33.75\pi \text{ cm}^2$$

(ii) **Method 1:**

Reflex $\angle AOB = 360° - 150°$ ($\angle$s at a point)

$$= 210°$$

Area of major sector $= \frac{x°}{360°} \times \pi r^2$

$$= \frac{210°}{360°} \times \pi \times 9^2$$

$$= 47.25\pi \text{ cm}^2$$

Method 2:

Area of major sector

= area of circle − area of minor sector

$= \pi \times 9^2 - 33.75\pi$

$= 47.25\pi \text{ cm}^2$

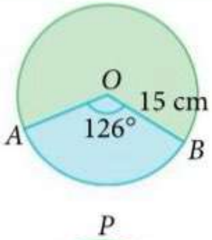
Practise Now 3

Similar and Further Questions

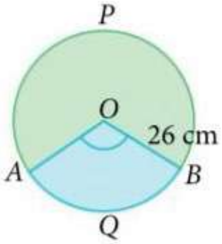
Exercise 8B

Questions 1–9

1. In the figure, O is the centre of a circle of radius 15 cm and $\angle AOB = 126°$.
   Find the area of
   (i) the minor sector,
   (ii) the major sector.



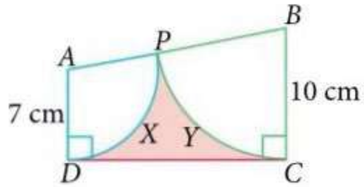
2. In the diagram, O is the centre of a circle of radius 26 cm. The length of minor arc AQB is 52 cm.
   (i) Show that $\angle AOB$ is approximately 114.6°.
   (ii) Hence, find the area of the major sector $OAPB$.



Worked Example 4

Solving problem involving sector area

In the diagram, $ABCD$ is a trapezium in which AD is parallel to BC, $AD = 7$ cm, $BC = 10$ cm and angle ADC = angle BCD = 90°. P is a point on AB and PXD is an arc of a circle with centre A while PYC is an arc of a circle with centre B.



(i) Show that angle BAD is equal to 100.2°, correct to one decimal place.

(ii) Hence, find the area of the shaded region.

***Solution**

We will use **Pólya's Problem Solving Model** to guide us in solving this problem.

Stage 1: Understand the problem

What information is given and what is not?

- What is given: $ABCD$ is a trapezium in which two of its angles are 90°; radii of the two sectors
- What is not given: angles of the two sectors; area of the two sectors; arc lengths DXP and CYP

What are we supposed to find?

- $B\hat{A}D$, which is the angle of sector $ADXP$; area of shaded region

Stage 2: Think of a plan

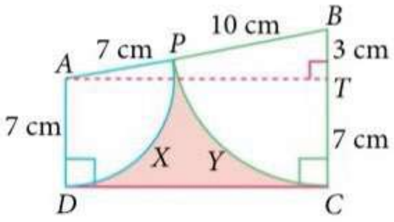
What formula involving angle can we use? Can we use the sector area formula to find $B\hat{A}D$?
We cannot find the sector angle without the arc length DXP and the sector area $ADXP$. Therefore, we have to think of another way.

What given information have we not used?

- $ABCD$ is a trapezium in which two of its angles are 90°
- The radii of the two sectors

What other formula involving angle can we use?
Since the trapezium has two right angles, can we identify a right-angled triangle to use the definitions of trigonometric ratios?
Yes, we can. Draw a line AT perpendicular to BC such that T lies on BC (see diagram below).

Stage 3: Carry out the plan



(i) $BT = BC - TC$
$= 10 - 7$
$= 3$ cm

$AB = AP + PB$
$= 7 + 10$
$= 17$ cm

$\sin B\hat{A}T = \dfrac{BT}{AB}$
$= \dfrac{3}{17}$ refer to $\triangle ABT$

$B\hat{A}T = \sin^{-1}\left(\dfrac{3}{17}\right)$
$= 10.164°$ (to 3 d.p.)

$B\hat{A}D = 90° + 10.164°$
$= 100.2°$ (to 1 d.p.) (shown)

(ii) $AT^2 = 17^2 - 3^2$ Pythagoras' Theorem
$= 280$
$AT = \sqrt{280}$
$= 16.733$ cm (to 5 s.f.)

$A\hat{B}T = 180° - 90° - 10.164°$ ($\angle$ sum of $\triangle$)
$= 79.836°$ (to 3 d.p.)

Area of shaded region
= Area of trapezium $ABCD$ – area of sector ADP – area of sector BCP
$= \dfrac{1}{2} \times (7 + 10)(16.733) - \dfrac{100.164°}{360°} \times \pi \times 7^2 - \dfrac{79.836°}{360°} \times \pi \times 10^2$
$= 29.7$ cm^2 (to 3 s.f.)

Stage 4: Look back

How can we check whether the answers are correct?

Can we work backwards by finding the sum of the area of the shaded region and the areas of the two sectors to see if it is equal to the area of the trapezium?

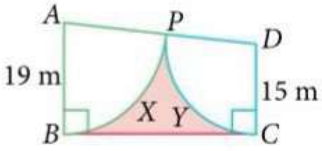
What can we learn from this problem?

We can try to divide a figure into different components, e.g. right-angled triangles, which can allow us to apply concepts we have learnt in trigonometry to solve for unknowns.

Practise Now 4

Similar and Further Questions
Exercise 8B
Questions 10, 11, 14, 15

In the diagram, $ABCD$ is a trapezium in which AB is parallel to DC, $AB = 19$ m, $DC = 15$ m and angle ABC = angle $DCB = 90°$.
P is a point on AD and PXB is an arc of a circle with centre A while PYC is an arc of a circle with centre D.



(i) Show that angle ADC is equal to 96.8°, correct to one decimal place.

(ii) Hence, find the area of the shaded region.

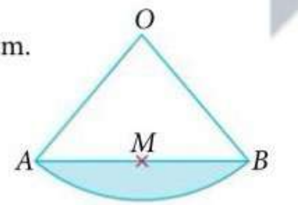
Worked Example 5

Finding area of segment

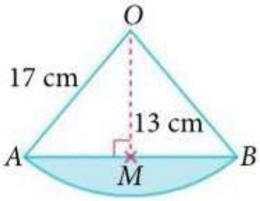
The diagram shows a sector of a circle, centre O, with radius 17 cm.
M is the midpoint of the chord AB and $OM = 13$ cm.

(i) Show that angle $AOB = 80.2°$, correct to 1 decimal place.

(ii) Calculate the shaded area.



Solution



(i)
$$\begin{aligned}\cos \angle AOM &= \frac{OM}{OA}\\ \angle AOM &= \cos^{-1}\left(\frac{13}{17}\right)\\ &= 40.119° \text{ (to 3 d.p.)}\end{aligned}$$

$$\begin{aligned}\angle AOB &= 2 \times \angle AOM\\ &= 2 \times 40.119°\\ &= 80.238°\\ &= 80.2° \text{ (to 1 d.p.) (shown)}\end{aligned}$$

(ii)
$$\begin{aligned}\text{Area of sector} &= \frac{80.238°}{360°} \times \pi\left(17^2\right)\\ &= 202.36 \text{ cm}^2 \text{ (to 5 s.f.)}\end{aligned}$$

$$\begin{aligned}\text{Area of } \triangle AOB &= \frac{1}{2} \times OA \times OB \times \sin \angle AOB\\ &= \frac{1}{2} \times 17 \times 17 \times \sin 80.238°\\ &= 142.41 \text{ cm}^2 \text{ (to 5 s.f.)}\end{aligned}$$

$$\begin{aligned}\text{Shaded area} &= \text{area of sector} - \text{area of } \triangle AOB\\ &= 202.36 - 142.41\\ &= 60.0 \text{ cm}^2 \text{ (to 3 s.f.)}\end{aligned}$$

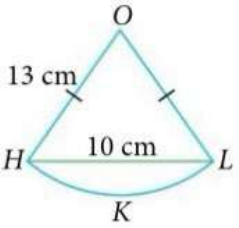
Recall

Area of triangle = $\frac{1}{2}ab \sin C$

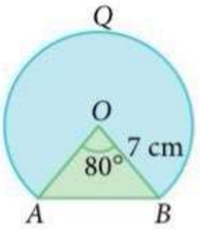
Practise Now 5

Similar and Further Questions
Exercise 8B
Questions 12, 13, 16

1. $HKLO$ is a sector of a circle, centre O and radius 13 cm.
   (i) Show that angle $HOL = 45.2°$, correct to 1 decimal place.
   (ii) Calculate the area of the minor segment HKL.



2. The surface of a glass commemorative plaque can be modelled by segment AQB of a circle, centre O and radius 7 cm.
   A piece of metal in the shape of triangle AOB, where angle $AOB = 80°$, is attached to the front.
   Express the area of the metal portion as a percentage of the glass surface, segment AQB.



Reflection

1. How has my knowledge of area of circle and proportionality helped me to obtain the formula for calculating sector area?
2. What have I learnt in this section that I am still unclear of?

Advanced Intermediate Basic

Exercise 8B

1. Copy and complete the table for each of the sectors of a circle.

	Radius	Angle at centre	Arc length	Area	Perimeter
(a)	7 cm	72°			
(b)	35 mm				136 mm
(c)		270°		1848 mm^2	
(d)		150°	220 cm		
(e)	14 m		55 m		
(f)		75°		154 cm^2	

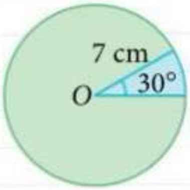
2. For each of the following circles with centre O, find

(i) the perimeter,

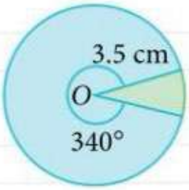
(ii) the area,

of the minor sector.

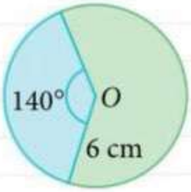
(a)



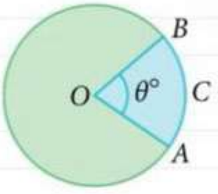
(b)



(c)



3. The figure shows a circle with centre O and $\angle AOB = \theta°$. The circumference of the circle is 88 cm.



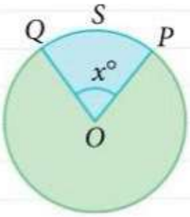
Find the length of arc ACB and the area of sector $OACB$ for each of the following values of θ.

(a) 60 **(b)** 99

(c) 126 **(d)** 216

4. The diagram shows a circle with centre O and $\angle POQ = x°$.

The area of the circle is 3850 cm^2.



Find the area of sector $OPSQ$ and the length of arc PSQ for each of the following values of x.

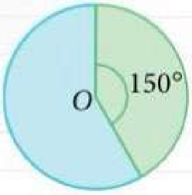
(a) 36 **(b)** 84

(c) 108 **(d)** 198

Exercise 8B

5. Find the radius of each of the following circles.

(a)



Area of minor sector = 114 cm²

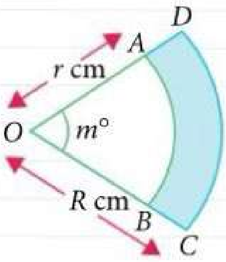
(b)

Area of major sector = 369 cm²

6. The diameter of a circle is 18 cm. Find the angle subtended by the arc of a sector with each of the following areas.

(a) 42.6 cm²
(b) 117.2 cm²
(c) 214.5 cm²
(d) 18.9 cm²

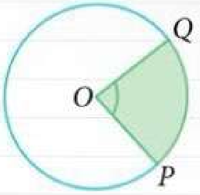
7. The figure shows two sectors *OAB* and *ODC* with *O* as the common centre.



Given that *OAD* and *OBC* are straight lines, *OA* = *r* cm, *OC* = *R* cm and ∠*AOB* = *m*°, find the perimeter and the area of the shaded region *ABCD* for each of the following cases.

(a) $r = 10$, $R = 20$, $m = 45$
(b) $r = 5$, $R = 8$, $m = 120$
(c) $r = 35$, $R = 49$, $m = 160$

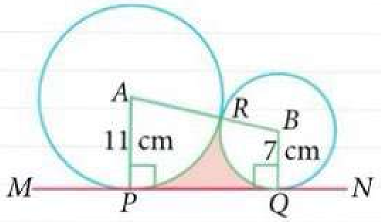
8. In the figure, the area of the shaded sector *POQ* is $\frac{5}{18}$ of the area of the whole circle.



(i) Find ∠*POQ*.
(ii) Given that the area of the shaded sector is 385 cm², find the diameter of the circle.

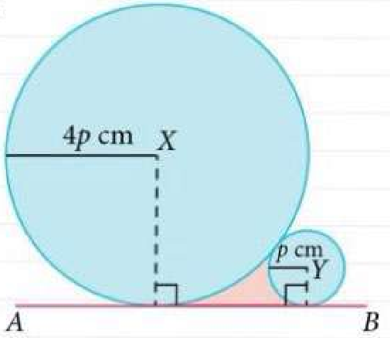
9. David forms a sector of a circle of radius 12 cm. Find the area of the paper that he used, given that the perimeter of the sector formed is 38 cm.

10. The diagram shows two circular discs with centres *A* and *B*, of radii 11 cm and 7 cm respectively. They lie on a straight line *MPQN* and touch each other at *R* such that *ARB* is a straight line.



(i) Show that ∠*PAB* is approximately 77.2°.
(ii) Hence, find the area of the shaded region.

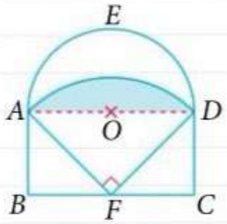
11. Two circular discs of radii 4*p* cm and *p* cm touch each other externally and lie on a straight line *AB* as shown.



Find an expression, in terms of *p*, for the area enclosed by the two discs and the line *AB*.

Exercise 8B

12. ABCD is a rectangle, where $BC = 2AB$. AODE is a semicircle, centre O, and arc AD has centre F. Angle $AFD = 90°$ and F is the midpoint of BC.



Given that $AB = x$ units, find the fraction of the figure that is shaded.

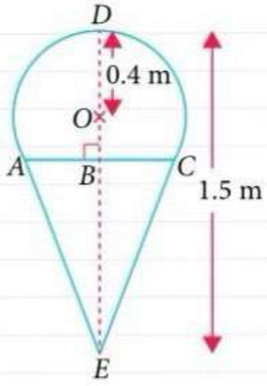
(Take $\pi = \frac{22}{7}$.)

13. An ice cream seller makes a sign in the shape of an ice cream cone, with a major segment ABCD of a circle, centre O and radius 0.4 m, on top of a triangle ABCE, such that EBOD is a straight line, as shown in the diagram.

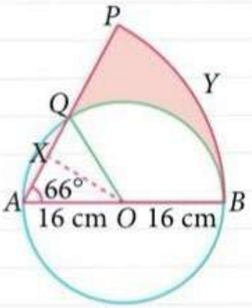
The sign is symmetrical about DE, which is perpendicular to AC.

The height of the sign DE is 1.5 m and $DB = \frac{2}{3}BE$.

Find the area of the sign.

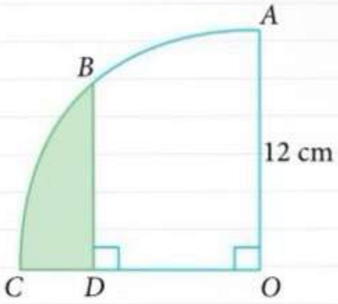


14. In the figure, O is the centre of a circle with radius 16 cm. APYB, a sector of a circle with centre A and radius 32 cm, intersects the circle at A, B and Q. OX divides △OAQ into two congruent triangles. Given that ∠OAQ = 66°, find



- **(i)** ∠BOQ,
- **(ii)** the length of AQ,
- **(iii)** the perimeter of the shaded region,
- **(iv)** the area of the shaded region.

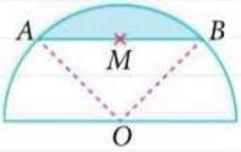
15. The figure shows a quadrant of a circle of radius 12 cm.



Given that B is the midpoint of the arc AC, find

- **(i)** the length of BD,
- **(ii)** the perimeter of the shaded region,
- **(iii)** the area of the shaded region.

16. The diagram shows a semicircle, centre O, radius 7.5 cm. M is the midpoint of the chord AB and $OM = 5$ cm.



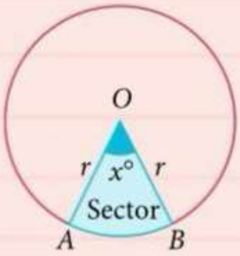
- **(a)** Show that angle $AOB = 96.4°$, correct to 3 significant figures.
- **(b)** Calculate the shaded area.
- **(c)** The semicircle is the cross section of a 32-cm long cake that has the shape of half a cylinder. The shaded area represents the chocolate mousse layer of the cake.
  - **(i)** Calculate the volume of the chocolate mousse layer.
  - **(ii)** Dark chocolate comprises 45% of the volume of the chocolate mousse. Calculate the number of such cakes that can be made with 10 litres of dark chocolate.

Looking Back

In this chapter, we used the idea of **proportionality** to extend our knowledge of circles. We derived formulae to calculate the arc length and sector area of a given circle based on concepts that we have previously learnt. It is important to be familiar with the terms used for the different parts of a circle, e.g. arc, sector and segment so that we can use these **notations** carefully to communicate ideas clearly. As you learn more in geometry, you would better appreciate real-life applications involving an arc, a sector or a segment of a circle.

Summary

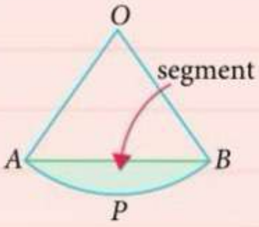
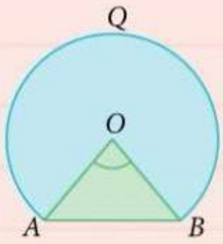
1. **Angle x° subtended by arc AB at centre of circle of radius r**



Length of arc $AB = \dfrac{x^\circ}{360^\circ} \times 2\pi r$

Area of sector $OAB = \dfrac{x^\circ}{360^\circ} \times \pi r^2$

2.

Minor segment	Major segment
	
Area of minor segment = area of minor sector $OAPB$ – area of $\triangle OAB$	Area of major segment = area of major sector $OAQB$ + area of $\triangle OAB$

CHAPTER 9

Geometrical Properties of Circles

Circles are fascinating shapes. The ratio of the circumference of a circle to its diameter is **invariant** no matter how big or small the circle is. Many people think the circle is a perfect and the most symmetrical shape that can be used in many artistic and practical ways. For example, the famous Stonehenge in the Wiltshire, UK, comprises of many standing stones arranged in almost a perfect circle! Not too far away from the Stonehenge, large geometric patterns, mainly circles, are found in fields of corn, barley and other crops. These are crop circles, fabricated using ropes and planks. Closer to home, we see balconies on some buildings that are shaped like part of a circle, or an arc. In this chapter, we will rediscover some of the symmetric and angle properties of circles that make them so fascinating.

Learning Outcomes

What will we learn in this chapter?

- What symmetric and angle properties circles have
- How to apply symmetric and angle properties of circles to solve problems

Introductory Problem

Fig. 9.1 shows a scale drawing of a room with a balcony. The arc of the balcony is an arc of a circle. Construct another scale drawing of the room using 2 cm to represent 1 m.

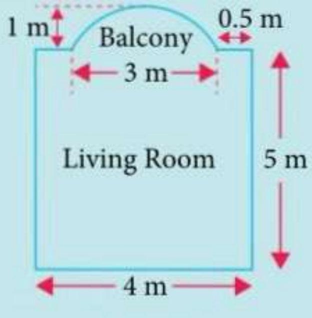


Fig. 9.1

In this chapter, we will learn some symmetric and angle properties of circles. One of these properties can help us construct an arc of a circle, given the dimensions of the balcony as stated in the **Introductory Problem**.

9.1 Symmetric properties of circles

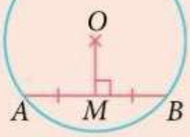
In this section, we will learn four symmetric properties of circles – two on chords and another two on tangents.

A. Perpendicular bisector of a chord

Investigation Discovering circle symmetric property 1

There are three conditions:

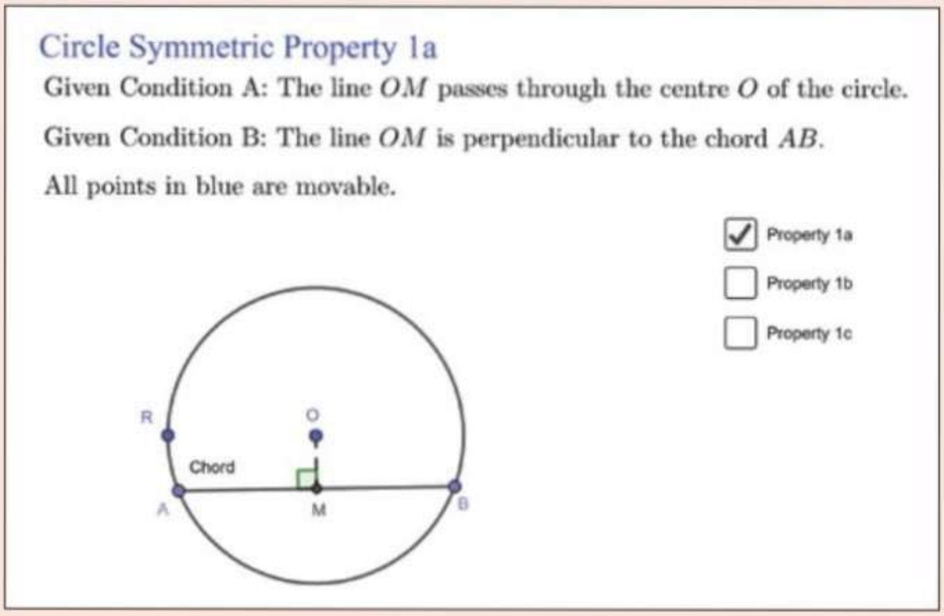
Condition A: The line *OM* passes through the centre *O* of the circle.
Condition B: The line *OM* is perpendicular to the chord *AB*.
Condition C: The line *OM* *bisects* the chord *AB* (i.e. divides the chord into two equal parts).
Note that the chord *AB* must not be the diameter of the circle.
In this Investigation, you will learn that **any two** of the above three conditions will imply the third one.



Go to www.sl-education.com/tmsoupp4/pg268 or scan the QR code on the right and open the geometry software template 'Circle Symmetric Property 1'.

Part 1

1. The template 'Circle Symmetric Property 1a' shows a circle with centre *O* and the line *OM* perpendicular to the chord *AB* (see Fig. 9.2). Which two of the three conditions *A*, *B* and *C* are given?



Circle Symmetric Property 1a

Given Condition A: The line OM passes through the centre O of the circle.

Given Condition B: The line OM is perpendicular to the chord AB.

All points in blue are movable.

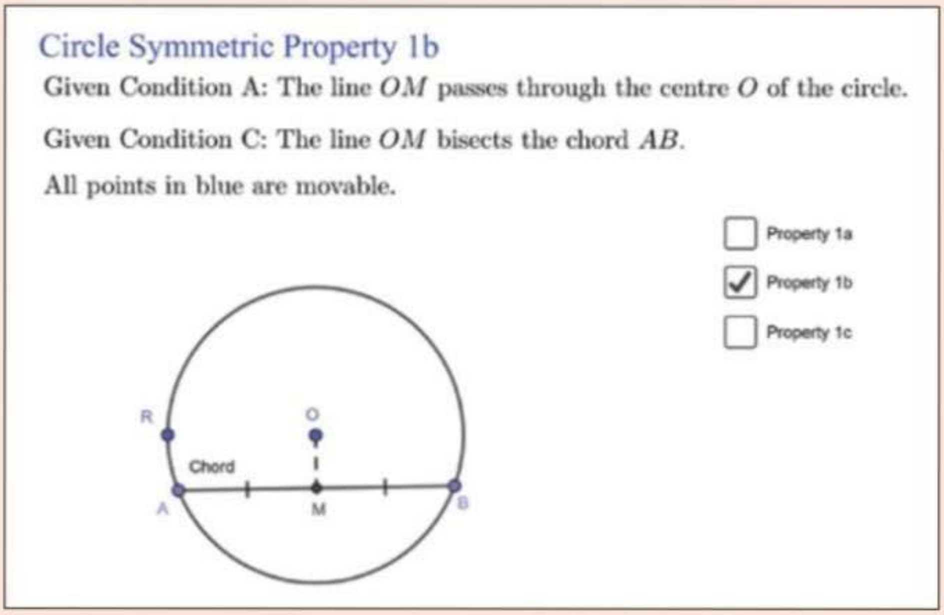
☑ Property 1a
☐ Property 1b
☐ Property 1c

Fig. 9.2

2. Click and drag point *A* or *B* to change the length of the chord.
   Click and drag point *R* to change the size of the circle.
   (a) What do you notice about the length of *AM* and *MB*?
   Hint: Use the 'Distance or Length' tool to measure the length.
   (b) What do you call point *M*?

Part 2

3. The template 'Circle Symmetric Property 1b' shows a circle with centre *O*, and the line *OM* bisecting the chord *AB* (see Fig. 9.3). Which two of the three conditions *A*, *B* and *C* are given?



Circle Symmetric Property 1b

Given Condition A: The line OM passes through the centre O of the circle.

Given Condition C: The line OM bisects the chord AB.

All points in blue are movable.

☐ Property 1a
☑ Property 1b
☐ Property 1c

Fig. 9.3

4. Click and drag point A or B to change the length of the chord.
   Click and drag point R to change the size of the circle.
   What do you notice about the size of $\angle AMO$ and $\angle BMO$?
   Hint: Use the 'Angle' tool ∡ to measure the angle.

Part 3

Use a sheet of paper to draw and cut out a circle.

Information

You can also investigate **Part 3** using the third page of the template.

5. To find the centre of a circle, fold the circle into two equal halves, and then again into two equal halves as shown in Fig. 9.4. Unfold the paper. The dotted lines in the last diagram indicate the fold lines obtained.

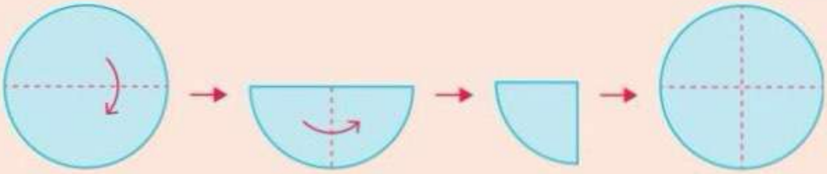


Fig. 9.4

   Where is the centre of the circle? Explain.
   Mark the centre of the circle with a cross and label it O.

6. Using the same cut-out circle, fold along a chord AB that is not a diameter of the circle and then fold into two equal halves as shown in Fig. 9.5. Unfold the paper. The dotted lines in the last diagram indicate the fold lines obtained.

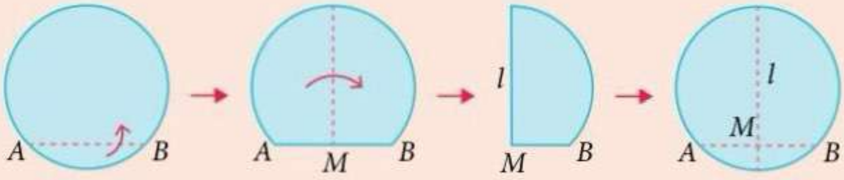
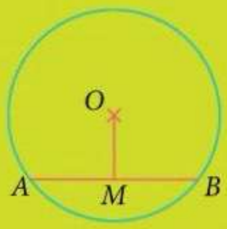


Fig. 9.5

   As the paper is folded into two equal halves, the line l bisects the chord AB and $\angle AMB$. Since $\angle AMB = 180°$ and the line l bisects $\angle AMB$, then l is perpendicular to the chord AB.
   (a) Which two of the three conditions A, B and C are given?
   (b) Does the line l pass through the centre O of the circle?

From the Investigation on page 268, we can observe three variations of the following property (where any o the three conditions will imply the third one):

Circle Symmetric Property 1: Perpendicular Bisector of Chord (abbreviation: ⊥ bisector of chord)



Any two of the three conditions imply the third:

(i) If a line *l passes through the centre* of a circle and is *perpendicular to a chord AB* of the circle, then the line *l bisects the chord AB*, i.e. if $OM \perp AB$, then OM bisects the chord AB.

(ii) If a line *l passes through the centre* of a circle and *bisects a chord AB* (which is not the diameter) of the circle, then the line *l* is *perpendicular to the chord AB*, i.e. if OM bisects the chord AB (which is not the diameter), then $OM \perp AB$.

(iii) If a line *l* bisects the chord *AB* of a circle and is perpendicular to the chord *AB*, then the line *l passes through the centre* of the circle, i.e. the **perpendicular bisector** of a chord will pass through the centre of the circle.

Information

Circle symmetric property 1**(i)** and **(iii)** are true even if the chord is the diameter of the circle.

Reflection

Why is it called circle *symmetric* property 1? Where is the line of symmetry?

Big Idea

Invariance

If a line *l* is the perpendicular bisector of a chord, then it will pass through the centre of the circle, regardless of how we translate, rotate, reflect or change the size of the circle. We say that this circle symmetric property 1**(iii)** is **invariant**. Invariance refers to a property of a mathematical object which remains unchanged when it undergoes some form of transformation. Are circle symmetric properties 1**(i)** and 1**(ii)** invariant as well? Explain.

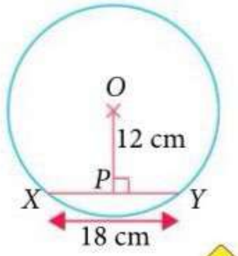
Thinking Time

1. A line *l* passes through the centre of a circle and is perpendicular to a chord *AB* (which is not the diameter) of the circle. Using congruent triangles, prove that the line *l* bisects the chord *AB*.
2. A line *l* passes through the centre of a circle and bisects a chord *AB* (which is not the diameter) of the circle. Using congruent triangles, prove that the line *l* is perpendicular to the chord *AB*.

Worked Example 1

Applying circle symmetric property 1

In the figure below, *XY* is a chord of the circle with centre *O*. The point *P* lies on *XY* such that *OP* is perpendicular to *XY*. Given that *XY* = 18 cm and *OP* = 12 cm, find the radius of the circle.



Solution

$XP = PY$ (⊥ bisector of chord) $OP \perp XY$, so OP bisects XY

$$= \frac{18}{2}$$

$$= 9 \text{ cm}$$

Problem-solving Tip

What are the given conditions? Should you use circle symmetric property 1**(i)**, **(ii)** or **(iii)**?

$OX^2 = XP^2 + OP^2$ Pythagoras' Theorem

$= 9^2 + 12^2$

$= 225$

$OX = \sqrt{225}$ (since length $OX > 0$)

$= 15$

$\therefore$ radius = 15 cm

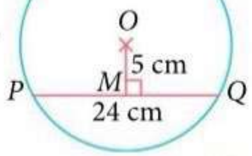
Practise Now 1

Similar and Further Questions

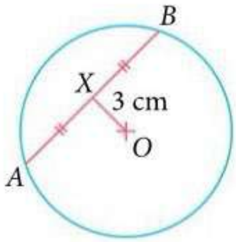
Exercise 9A

Questions 1–5, 15–17, 28

1. In the figure below, PQ is a chord of the circle with centre O. Point M lies on PQ such that OM is perpendicular to PQ. Given that $PQ = 24$ cm and $OM = 5$ cm, find the radius of the circle.



2. In the figure below, AB is a chord of the circle with centre O. Point X lies on AB such that $AX = XB$. Given that the radius of the circle is 7 cm and $OX = 3$ cm, find the length of the chord AB.



Problem-solving Tip

For Question 2, where is the radius in the figure? Should you use circle symmetric property 1(**i**) or (**ii**)?

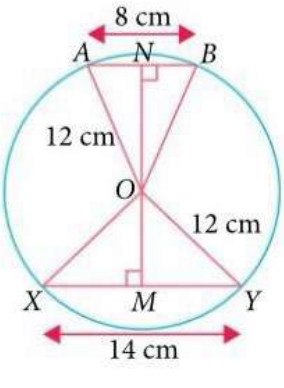
Worked Example 2

Solving problem using circle symmetric property 1

The lengths of two parallel chords of a circle of radius 12 cm are 8 cm and 14 cm respectively. Find the two possible distances between the chords.

Solution

Case 1: The chords are on opposite sides of the centre O.



$AN = NB$ ($\perp$ bisector of chord) $ON \perp AB$, so ON bisects AB

$= \frac{AB}{2}$

$= \frac{8}{2}$

$= 4$ cm

$ON^2 = 12^2 - 4^2$ Pythagoras' Theorem

$= 128$

$ON = \sqrt{128}$

$= 11.314$ cm (to 5 s.f.)

$XM = MY$ ($\perp$ bisector of chord) $OM \perp XY$, so OM bisects XY

$= \frac{XY}{2}$

$= \frac{14}{2}$

$= 7$ cm

$$OM^2 = 12^2 - 7^2$$ Pythagoras' Theorem

$$= 95$$

$$OM = \sqrt{95}$$

$$= 9.7468 \text{ cm (to 5 s.f.)}$$

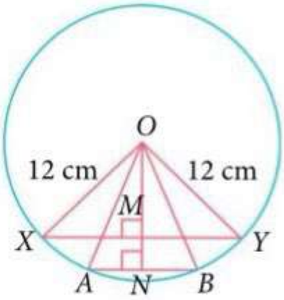
Shortest distance between the chords $= MN$

$= NO + OM$

$= 11.314 + 9.7468$

$= 21.1$ cm (to 3 s.f.)

Case 2: The chords are on the same side of the centre *O*.



Shortest distance between the chords $= MN$

$= ON - OM$

$= 11.314 - 9.7468$

$= 1.57$ cm (to 3 s.f.)

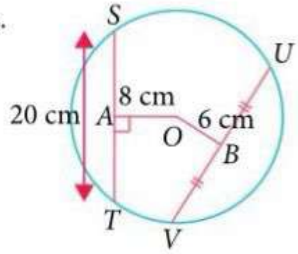
$\therefore$ the two possible distances between the chords are 21.1 cm and 1.57 cm.

Practise Now 2

Similar and Further Questions
Exercise 9A
Questions 6(a)–(c), 18–20, 29

1. The lengths of two parallel chords of a circle of radius 20 cm are 10 cm and 30 cm respectively. Find the two possible distances between the chords.

2. The figure shows a circle with centre *O* and two chords *ST* and *UV*. Point *A* lies on *ST* such that *OA* is perpendicular to *ST* and point *B* lies on *UV* such that *UB* = *BV*. Given that *ST* = 20 cm, *OA* = 8 cm and *OB* = 6 cm, find the length of the chord *UV*.



Introductory Problem Revisited

Before we solve the Introductory Problem, let us learn how to construct a circle passing through three given points.

1. Using a pair of compasses, construct a circle that passes through the three given points *A*, *B* and *C*.
   Hint: Apply circle symmetric property 1**(iii)** to chords *AB* and *BC* to locate the centre of the circle.

2. Solve the Introductory Problem by reconstructing the arc of the balcony in the scale drawing.

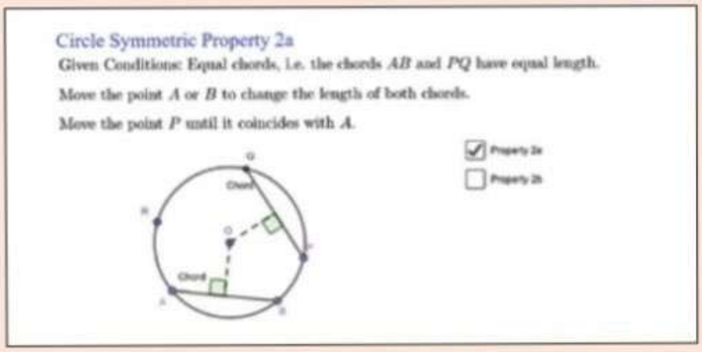
B. Equal chords

Investigation Discovering circle symmetric property 2

Go to www.sl-education.com/tmsoupp4/pg274 or scan the QR code on the right and open the geometry template 'Circle Symmetric Property 2'.

Part 1

1. The template 'Circle Symmetric Property 2a' shows a circle with centre *O* and two equal chords (see Fig. 9.6).



Circle Symmetric Property 2a

Given Conditions: Equal chords, i.e. the chords *AB* and *PQ* have equal length.

Move the point *A* or *B* to change the length of both chords.

Move the point *P* until it coincides with *A*.

Fig. 9.6

Attention

Two chords are equal if they have the same length.

2. Click and drag point *A* or *B* to change the lengths of both chords.
   Click and drag point *R* to change the size of the circle.
   Click and drag point *P* until it coincides with the point *A*.
   What do you notice about the distances of both chords from the centre *O*?
   Hint: You can use the 'Distance or Length' tool to measure the distance.

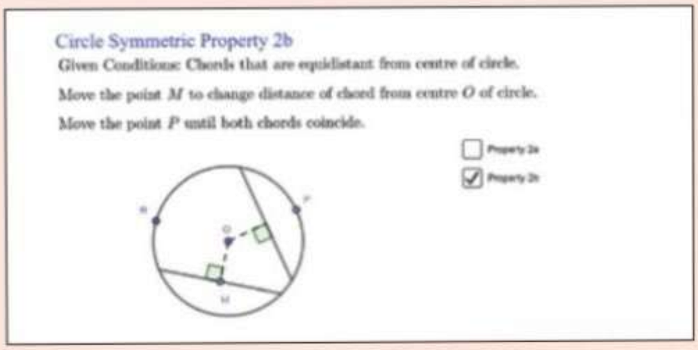
Attention

The 'distance of a point from a line' refers to the *perpendicular distance* of the point from the line. This distance is also the shortest distance from the point to the line.

3. Complete the following sentence.
   In general, equal chords of a circle are from the centre of the circle.

Part 2

4. The template 'Circle Symmetric Property 2b' shows two chords of a circle that are equidistant from its centre *O* (see Fig. 9.7).



Circle Symmetric Property 2b

Given Conditions: Chords that are equidistant from centre of circle.

Move the point *M* to change distance of chord from centre *O* of circle.

Move the point *P* until both chords coincide.

Fig. 9.7

5. Click and drag point *M* to change the distance of both chords from the centre *O*.
   Click and drag point *R* to change the size of the circle.
   Click and drag point *P* until both chords coincide.

What do you notice about the lengths of both chords?

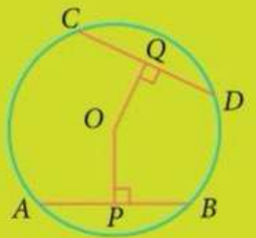
Hint: Use the 'Distance or Length' tool to measure the length.

6. Complete the following sentence.

In general, chords that are equidistant from the centre of a circle are ______ (in length).

From the above Investigation, we can observe the following property:

Circle Symmetric Property 2: Equal Chords (abbreviation: equal chords)



(i) Equal chords of a circle are *equidistant* from the centre of the circle, i.e. if $AB = CD$, then $OP = OQ$.

(ii) Chords that are equidistant from the centre of a circle are *equal* (in length), i.e. if $OP = OQ$, then $AB = CD$.

Reflection

Why is it called circle *symmetric* property 2? Where is the line of symmetry?

Big Idea

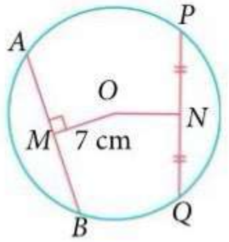
Invariance

Are circle symmetric properties 2(**i**) and 2(**ii**) invariant? Explain.

Worked Example 3

Applying circle symmetric property 2

In the figure, AB and PQ are equal chords of the circle with centre O. Point M lies on AB such that $\angle OMA = 90°$ and point N is the midpoint of PQ. Given that $OM = 7$ cm, find the length of ON.



Solution

$\angle ONP = 90°$ (⊥ bisector of chord) N is midpoint of PQ (i.e. ON bisects PQ), so $ON \perp PQ$

$\therefore ON$ = distance of PQ from O

= distance of AB from O (equal chords) equal chords are equidistant from centre of circle

$= OM$

$= 7$ cm

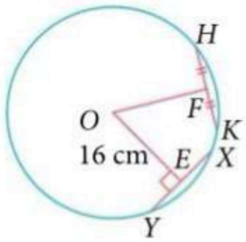
Practise Now 3

Similar and Further Questions

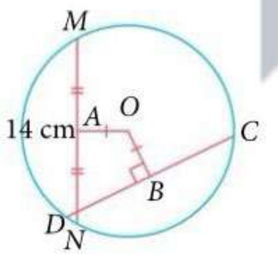
Exercise 9A

Questions 7, 8, 9(a), (b)

1. In the figure, XY and HK are chords of the circle with centre O. Point E lies on XY such that $\angle YEO = 90°$ and point F is the midpoint of HK. Given that $HK = XY$ and $OE = 16$ cm, find the length of OF.

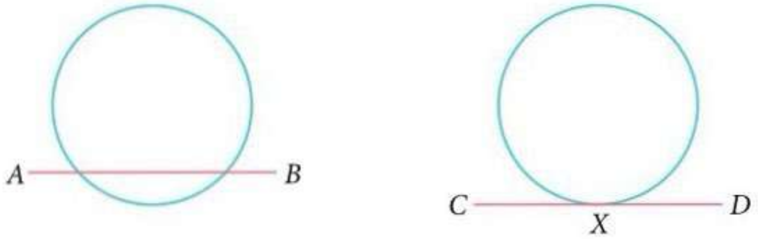


2. In the figure, MN and CD are chords of the circle with centre O. Point A lies on MN such that $MA = AN$ and point B lies on CD such that $\angle OBD = 90°$. Given that $OA = OB$ and $MN = 14$ cm, find the length of CD.



C. Tangent perpendicular to radius

Fig. 9.8(**a**) shows a straight line AB that cuts a circle at two distinct points. The line AB is called a *secant*. Fig. 9.8(**b**) shows a straight line CD that touches a circle at one point, X. The line CD is called a **tangent** and X is called the *point of contact* between the tangent and the circle.



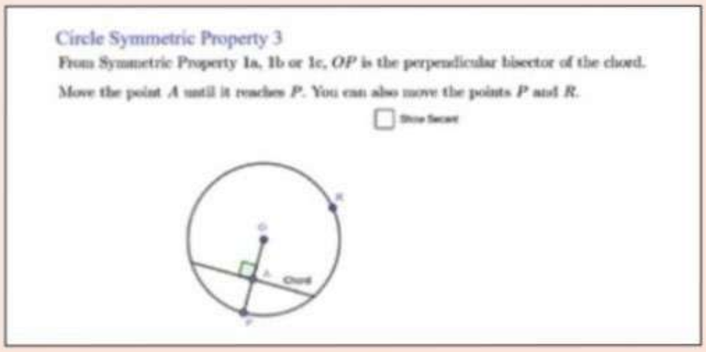
(a) (b)

Fig. 9.8

Investigation Discovering circle symmetric property 3

Go to www.sl-education.com/tmsoupp4/pg276 or scan the QR code on the right and open the geometry template 'Circle Symmetric Property 3'.

1. The template below shows a circle with centre O and radius OP, which is perpendicular to the chord at A (see Fig. 9.9).



Circle Symmetric Property 3

From Symmetric Property 1a, 1b or 1c, OP is the perpendicular bisector of the chord.

Move the point A until it reaches P. You can also move the points P and R.

☐ Show Secant

Fig. 9.9

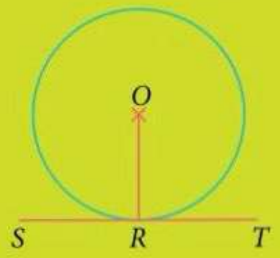
2. Check the box 'Show Secant' to reveal a secant that coincides with the chord, i.e. the secant is also perpendicular to the radius OP. (A chord is a line *segment* with two endpoints on a circle while a secant is a line without any endpoints, that cuts the circle at two different points.)

3. Click and drag point P to move the radius OP and the secant around the circle.
   Click and drag point R to change the size of the circle.
   Click and drag point A until it coincides with the point P.
   (a) What do you notice about the secant? What has it become?
   (b) What is the angle between the tangent at the point of contact P and the radius of the circle?
   Hint: Use the 'Angle' tool to measure the angle.

4. Complete the following sentence.
   In general, the tangent at the point of contact is ______ to the radius of the circle.

From the above Investigation, we can observe the following property:

Circle Symmetric Property 3: Tangent Perpendicular to Radius
(abbreviation: tangent ⊥ radius)



The tangent to a circle is *perpendicular* to its radius at the point of contact, i.e. $ST \perp OR$.

Reflection

Why is it called circle *symmetric* property 3? Where is the line of symmetry?

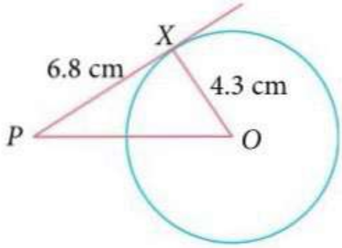
Big Idea

Invariance
Is circle symmetric property 3 invariant? Explain.

Worked Example 4

Applying circle symmetric property 3

In this figure, PX is a tangent to the circle with centre O.
Given that $PX = 6.8$ cm and $OX = 4.3$ cm, find
(i) $O\hat{P}X$,
(ii) the length of OP,
(iii) the area of $\triangle OPX$.



Solution

(i) $O\hat{X}P = 90°$ (tangent ⊥ radius)

$$\tan O\hat{P}X = \frac{\text{opp}}{\text{adj}} = \frac{OX}{PX} = \frac{4.3}{6.8}$$

$$O\hat{P}X = \tan^{-1}\frac{4.3}{6.8} = 32.307° \text{ (to 3 d.p.)} = 32.3° \text{ (to 1 d.p.)}$$

(ii) **Method 1:**

$$OP^2 = 6.8^2 + 4.3^2 \quad \text{Pythagoras' Theorem}$$
$$= 64.73$$
$$OP = \sqrt{64.73}$$
$$= 8.05 \text{ cm (to 3 s.f.)}$$

Method 2:

$$\sin O\hat{P}X = \frac{\text{opp}}{\text{hyp}}$$
$$= \frac{OX}{OP}$$
$$\sin 32.307° = \frac{4.3}{OP}$$
$$OP = \frac{4.3}{\sin 32.307°}$$
$$= 8.05 \text{ cm (to 3 s.f.)}$$

(iii) Area of $\triangle OPX = \frac{1}{2} \times \text{base} \times \text{height}$

$$= \frac{1}{2} \times PX \times OX$$
$$= \frac{1}{2} \times 6.8 \times 4.3$$
$$= 14.62 \text{ cm}^2$$

Attention

(iii) The answer 14.62 cm^2 is exact. You can either leave it as it is, or you can still correct it to 3 s.f.

Practise Now 4

Similar and Further Questions

Exercise 9A

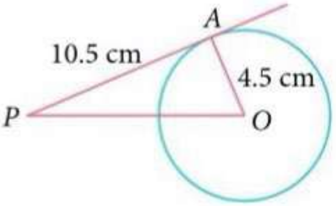
Questions 10–12, 21–25, 30

1. In this figure, PA is a tangent to the circle with centre O. Given that $PA = 10.5$ cm and $OA = 4.5$ cm, find

(i) $\angle OPA$,

(ii) the length of OP,

(iii) the area of $\triangle OPA$.



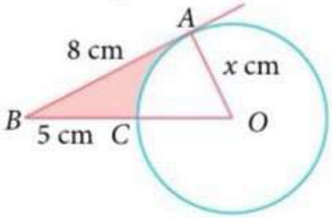
2. In this figure, AB is a tangent to the circle with centre O and the line segment BO intersects the circumference of the circle at C.

Given that $AB = 8$ cm, $BC = 5$ cm and $OA = x$ cm, find

(i) the value of x,

(ii) $\angle AOB$,

(iii) the area bounded by AB, BC and the minor arc AC.



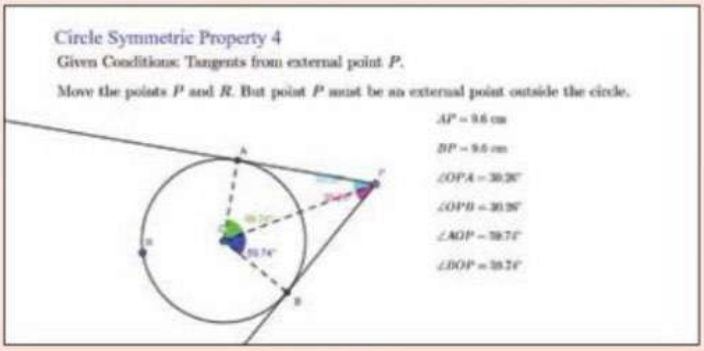
D. Tangents from external point

Investigation

Discovering circle symmetric property 4

Go to www.sl-education.com/tmsoupp4/pg278 or scan the QR code on the right and open the geometry template 'Circle Symmetric Property 4'.

1. The template below shows a circle with centre O and two tangents from an external point P that touches the circle at A and B respectively (see Fig. 9.10).
   Although a tangent is a line (i.e. without any endpoints) whose length is infinite, the *length of a tangent from an external point* is the distance between the external point and the point of contact with the circle. In this case, the lengths of the two tangents are AP and BP.



Circle Symmetric Property 4

Given Conditions: Tangents from external point P.

Move the points P and R. But point P must be an external point outside the circle.

$AP = 9.6$ cm

$BP = 9.6$ cm

$\angle OPA = 30.26°$

$\angle OPB = 30.26°$

$\angle AOP = 59.74°$

$\angle BOP = 59.74°$

Fig. 9.10

2. Click and drag point P to change the position of the external point, making sure that P remains outside the circle.
   Click and drag point R to change the size of the circle.
   (a) What do you notice about the length of AP and of BP?
   (b) What do you notice about $\angle OPA$ and $\angle OPB$?
   (c) What do you notice about $\angle AOP$ and $\angle BOP$?

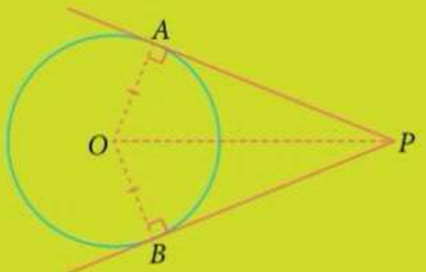
Reflection

Why must the point P remain outside the circle?

3. Complete the following sentences.
   In general,
   (a) tangents from an external point are __________ (in length);
   (b) the line from the centre of a circle to an external point __________ the angle between the two tangents;
   (c) the line from the centre of a circle to an external point __________ the angle between the radii OA and OB, where A and B are the points of contact between the two tangents and the circle.
4. Prove the three results in Question 3.
   Hint: How are $\triangle OAP$ and $\triangle OBP$ related?

From the above Investigation, we can observe the following property:

Circle Symmetric Property 4: Tangents from External Point (abbreviation: tangents from ext. pt.)



(i) Tangents from an external point are *equal* (in length), i.e. $AP = BP$.
(ii) The line from the centre of a circle to an external point *bisects* the angle between the two tangents from the external point, i.e. OP bisects $\angle APB$.
(iii) The line from the centre of a circle to an external point *bisects* the angle between the radii OA and OB, i.e. OP bisects $\angle AOB$ (where A and B are the points of contact of the two tangents with the circle).

Reflection

Why is it called circle *symmetric* property 4? Where is the line of symmetry?

Big Idea

Invariance
Is circle symmetric property 4 invariant? Explain.

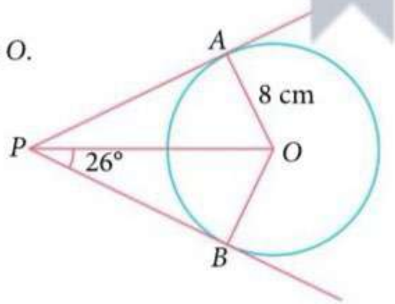
Worked Example 5

Applying circle symmetric property 4

In the figure, PA and PB are tangents to the circle with centre O. Given that $OA = 8$ cm and $\angle OPB = 26°$, find

(i) $\angle AOB$, **(ii)** the length of AP,

(iii) the area of the quadrilateral $OAPB$.



Solution

(i) $\angle OBP = 90°$ (tangent $\perp$ radius)

$\angle BOP = 180° - 90° - 26°$ ($\angle$ sum of $\triangle$)

$= 64°$

$\angle AOB = 64° \times 2$ (tangents from ext. pt.)

$= 128°$

$AP = BP$, so OP bisects $\angle AOB$

(ii) $\angle APO = \angle BPO$ (tangents from ext. pt.)

$= 26°$

$$\tan \angle APO = \frac{\text{opp}}{\text{adj}}$$

$$= \frac{OA}{AP}$$

$$\tan 26° = \frac{8}{AP}$$

$$AP = \frac{8}{\tan 26°}$$

$= 16.402$ cm (to 5 s.f.)

$= 16.4$ cm (to 3 s.f.)

(iii) Area of $\triangle OAP = \frac{1}{2} \times \text{base} \times \text{height}$

$$= \frac{1}{2} \times AP \times OA$$

$$= \frac{1}{2} \times 16.402 \times 8$$

$= 65.608$ cm^2

Area of quadrilateral $OAPB$

$= 2 \times$ area of $\triangle OAP$

$= 2 \times 65.608$

$= 131$ cm^2 (to 3 s.f.)

$AP = BP$, so $\triangle OAP$ and $\triangle OBP$ are congruent

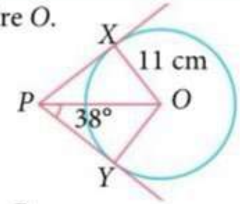
Practise Now 5

Similar and Further Questions
Exercise 9A
Questions 13(a)–(f), 14, 26, 27

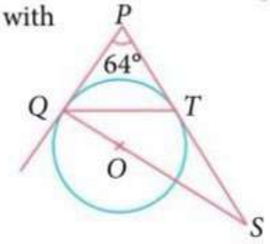
1. In the figure, PX and PY are tangents to the circle with centre O. Given that $OX = 11$ cm and $\angle OPY = 38°$, find

   (i) $\angle XOY$, **(ii)** the length of PX,

   (iii) the area of the quadrilateral $OXPY$.



2. In this figure, PQ and PT are tangents to the circle with centre O, at the points Q and T respectively. PT produced meets QO produced, at S. Given that $\angle QPT = 64°$, find $\angle SQT$.



Reflection

1. Table 9.1 summarises the four symmetric properties of circles. Fill in the blanks.

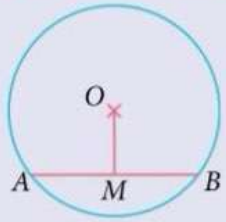
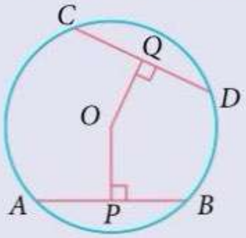
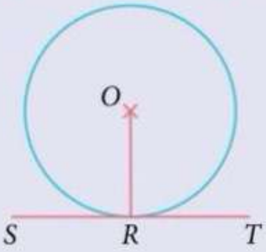
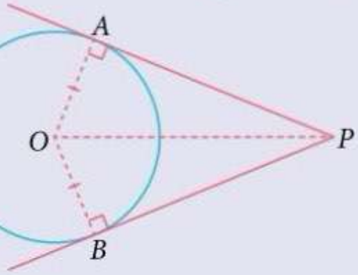
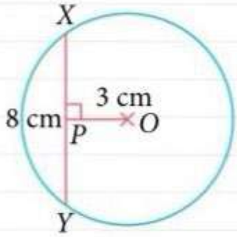
	Perpendicular properties	Equal length properties
Properties of chords	**Property 1: Perpendicular Bisector of Chord (abbreviation: ⊥ bisector of chord)**  Any two of the three conditions imply the third: **(i)** If $OM \perp AB$, then OM ______ the chord AB. **(ii)** If OM bisects the chord AB (which is not the diameter), then OM ______ AB. **(iii)** The perpendicular bisector of a chord will pass through the ______ of the circle.	**Property 2: Equal Chords (abbreviation: equal chords)**  **(i)** Equal chords are ______ from the centre of the circle, i.e. if $AB = CD$, then $OP = OQ$. **(ii)** If two chords are equidistant from the centre of the circle, then they are ______ (in length), i.e. if $OP = OQ$, then $AB = CD$.
Properties of tangents	**Property 3: Tangent Perpendicular to Radius (abbreviation: tangent ⊥ radius)**  The tangent to a circle is ______ to its radius at the point of contact, i.e. $ST \perp OR$.	**Property 4: Tangents from External Point (abbreviation: tangents from ext. pt.)**  **(i)** Tangents from an external point are ______ (in length). **(ii)** OP ______ $\angle APB$. **(iii)** OP ______ $\angle AOB$.

Table 9.1

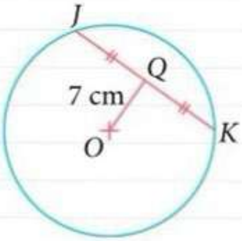
2. What are some similarities and differences among the four symmetric properties of circles?
3. For each of the properties, how can I prove it for the general case? What concepts from other topics can I use to prove them?
4. Given a line that passes through the centre of a circle and bisects a chord, which of the above properties could I apply?
5. Given a tangent to a circle, which of the above properties could I apply?
6. What have I learnt in this section that I am still unclear of?

Exercise 9A

1. In the figure below, XY is a chord of the circle with centre O. Point P lies on XY such that OP is perpendicular to XY. Given that $XY = 8$ cm and $OP = 3$ cm, find the radius of the circle.



2. In the figure below, JK is a chord of the circle with centre O. Point Q lies on JK such that $JQ = QK$. Given that the radius of the circle is 16 cm and $OQ = 7$ cm, find the length of the chord JK.



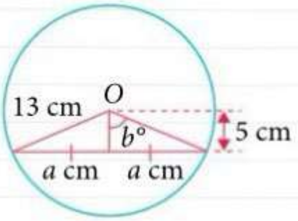
3. AB is a chord of a circle with centre O and radius 17 cm. Given that $AB = 16$ cm, find the perpendicular distance from O to AB.

4. A chord of length 24 m is at a distance of 5 m from the centre of a circle. Find the radius of the circle.

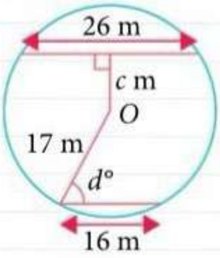
5. A chord of a circle of radius 8.5 cm is 5 cm from the centre. Find the length of the chord.

6. Given that O is the centre of each of the following circles, find the values of the unknowns.

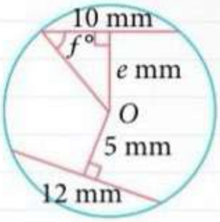
(a)



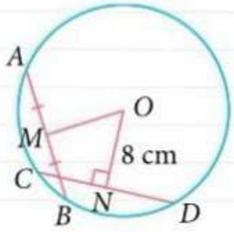
(b)



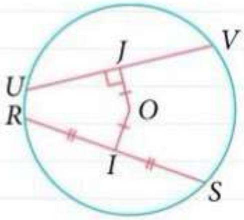
(c)



7. In the figure below, AB and CD are chords of the circle with centre O. Point M is the midpoint of AB and point N lies on CD such that $\angle ONC = 90°$. Given that $AB = CD$ and $ON = 8$ cm, find the length of OM.

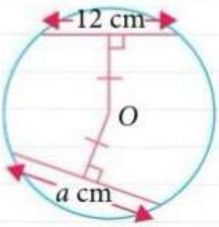


8. In the figure below, RS and UV are chords of the circle with centre O. Point I is the midpoint of RS and point J lies on UV such that $\angle OJU = 90°$. Given that $OI = JO$ and $UV = 11$ cm, find the length of RS.

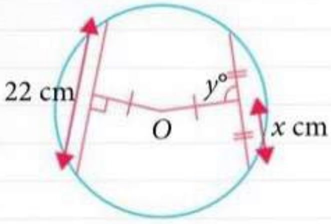


9. Given that O is the centre of each of the following circles, find the values of the unknowns.

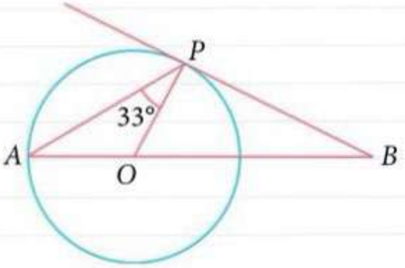
(a)



(b)



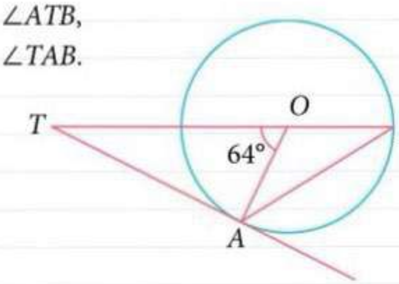
10. In the figure below, O is the centre of the circle passing through the points A and P. AO produced meets the tangent at P at B. Given that $\angle APO = 33°$, find $\angle PBA$.



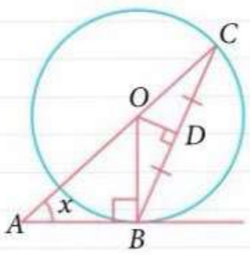
11. In the figure below, O is the centre of the circle passing through the points A and B. TA is a tangent to the circle at A and TOB is a straight line. Given that $\angle AOT = 64°$, find

(i) $\angle ATB$,

(ii) $\angle TAB$.

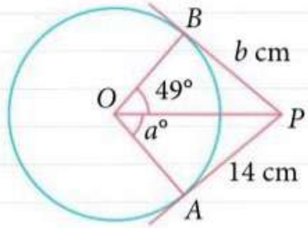


12. In the figure below, AB is a tangent to the circle with centre O. D is the midpoint of the chord BC. Given that $\angle BAC = x$, find $\angle COD$ in terms of x.

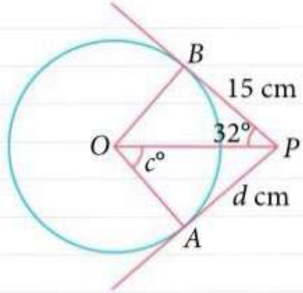


13. Given that PA and PB are tangents to each of the following circles with centre O, find the values of the unknowns.

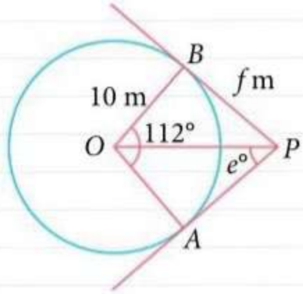
(a)



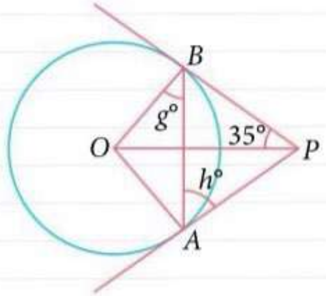
(b)



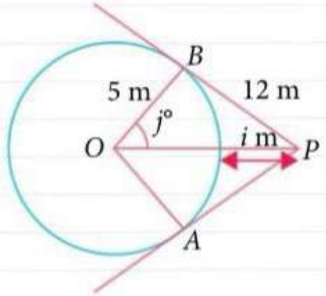
(c)



(d)

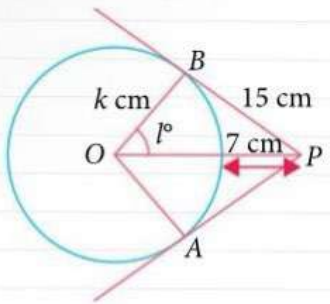


(e)



Exercise 9A

(f)



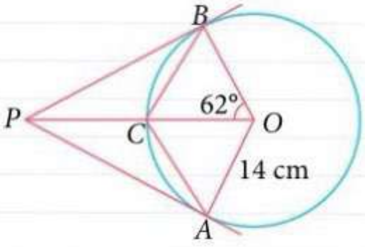
14. In the figure below, O is the centre of the circle passing through the points A, C and B. PA and PB are tangents to the circle. Given that $OA = 14$ cm and $\angle BOP = 62°$, find

(i) $\angle OPB$,

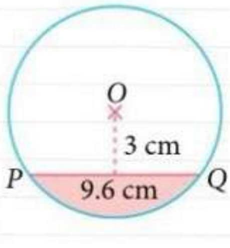
(ii) $\angle OAC$,

(iii) the length of BP,

(iv) the area of the quadrilateral $OAPB$.



15. The figure below shows the cross section of a circular water pipe. The shaded region shows the water flowing through the pipe. Given that $PQ = 9.6$ cm and that the surface of the water is 3 cm below the centre O of the circle, find the cross-sectional area of the water in the pipe.

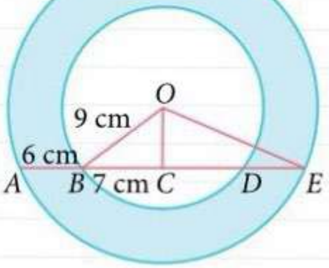


16. The perpendicular bisector of a chord XY cuts XY at N and the circle at P. Given that $XY = 16$ cm and $NP = 2$ cm, calculate the radius of the circle.

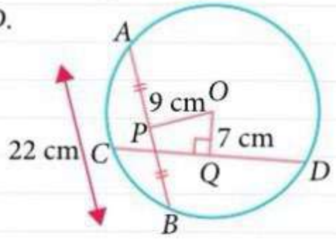
17. The figure below shows two concentric circles with centre O. The points A and E lie on the circumference of the larger circle while the points B and D lie on the circumference of the smaller circle. Given that $ABCDE$ is a straight line, $OB = 9$ cm, $AB = 6$ cm, $BC = 7$ cm and $AC = CE$, find

(i) the length of OE,

(ii) the shaded area.



18. In the figure below, AB and CD are chords of the circle with centre O. The point P lies on AB such that $AP = BP$ and the point Q lies on CD such that OQ is perpendicular to CD. Given that $AB = 22$ cm, $OP = 9$ cm and $OQ = 7$ cm, find the length of the chord CD.



19. The lengths of two parallel chords of a circle of radius 5 cm are 6 cm and 8 cm respectively. Find the two possible distances between the chords.

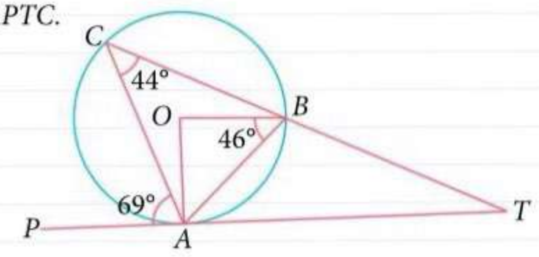
20. Two parallel chords PQ and MN are 3 cm apart on the same side of a circle where $PQ = 7$ cm and $MN = 14$ cm. Calculate the radius of the circle.

Exercise 9A

21. In the figure below, PAT is a tangent to the circle with centre O, at A. B and C are points on the circle such that TBC is a straight line and $\angle ACB = 44^\circ$. Given that $\angle OBA = 46^\circ$ and $\angle PAC = 69^\circ$, find

(i) $\angle BAT$,

(ii) $\angle PTC$.

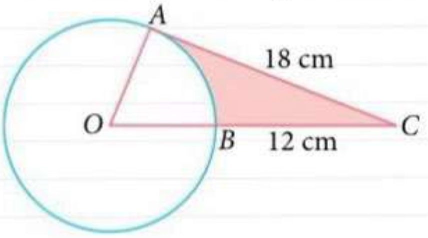


22. The figure shows a circle with centre O passing through points A and B. AC is a tangent to the circle at A and OBC is a straight line.
Given that $AC = 18$ cm and $BC = 12$ cm, find

(i) the radius of the circle,

(ii) $\angle AOB$,

(iii) the area of the shaded region.

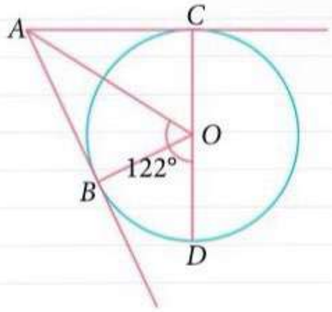


23. PQ is a chord of a circle with centre O. Given that $\angle POQ = 84^\circ$, find the obtuse angle between PQ and the tangent at P.

24. The tangent from a point P touches a circle at N. Given that the radius of the circle is 5.6 cm and that P is 10.6 cm away from the centre, find the length of the tangent PN.

25. A point T is 9.1 m away from the centre of a circle. The tangent from T to the point of tangency is 8.4 m. Find the diameter of the circle.

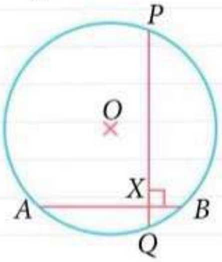
26. In the figure below, AB and AC are tangents to the circle at B and C respectively. O is the centre of the circle, CD is a diameter of the circle and $\angle AOD = 122^\circ$.
Find $\angle BAC$.



27. The tangents from a point T touch a circle with centre O, at the points A and B. Given that $\angle AOT = 51^\circ$, find $\angle BAT$.

28. The radius of a circle is 17 cm. A chord XY lies 9 cm from the centre and divides the circle into two segments. Find the perimeter of the minor segment.

29. The figure shows a circle with centre O and radius 7 cm. The chords AB and PQ have lengths 11 cm and 13 cm respectively, and intersect at right angles at X. Find the length of OX.



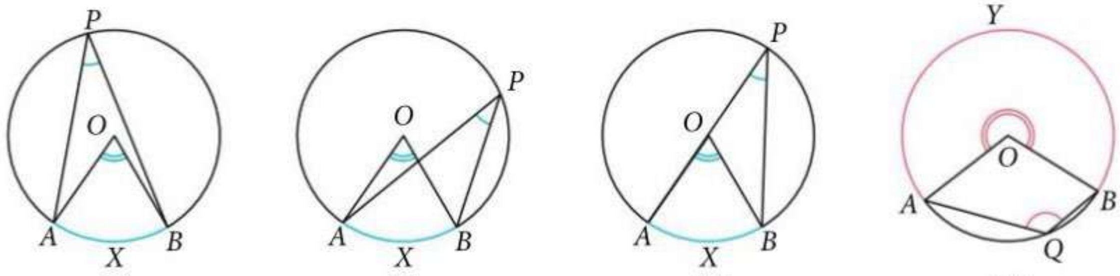
30. Two concentric circles have radii 12 cm and 25.5 cm respectively. A tangent to the inner circle cuts the outer circle at the points H and K. Find the length of HK.

9.2 Angle properties of circles

In this section, we will learn four angle properties of circles. We will first have to learn some new terms.
A triangle has 3 interior angles and a quadrilateral has 4 interior angles.
Although a circle does not seem to have any angles, we can *define* two types of angles: one subtended at the centre and one subtended at the circumference.

A. Angles subtended at centre and at circumference

Fig. 9.11**(a)** shows a circle with centre O.
$\angle AOB$ is an angle **subtended at the centre** of the circle by the (blue) *minor arc* AXB.
$\angle APB$ is an angle **subtended at the circumference** of the circle by the same *minor arc* AXB.



(a) **(b)** **(c)** **(d)**

Fig. 9.11

Consider Fig. 9.11**(b)** and **(c)**. Identify the angle subtended at the centre of the circle and the angle subtended at the circumference by the minor arc AXB.

Fig. 9.11**(d)** shows another circle with centre O.
Reflex $\angle AOB$ is an angle subtended at the centre of the circle by the (red) *major arc* AYB. $\angle AQB$ is an angle subtended at the circumference of the circle by the same *major arc* AYB.

Attention

In Fig. 9.11**(d)**, there are two $\angle AOB$. One of them is acute/obtuse and the other is reflex. So whenever we refer to the reflex angle, we have to specify *reflex* $\angle AOB$.

One way to identify the angle subtended by an arc is to look at the shape of the arc. For example, the shape of the blue arc indicating $\angle APB$ in Fig. 9.11**(a)** is the same shape as that of the blue minor arc AXB which subtends the angle; and the shape of the red arc indicating reflex $\angle AOB$ in Fig. 9.11**(d)** is the same shape as that of the red major arc AYB which subtends the angle.

Class Discussion Identifying angles at centre and at circumference

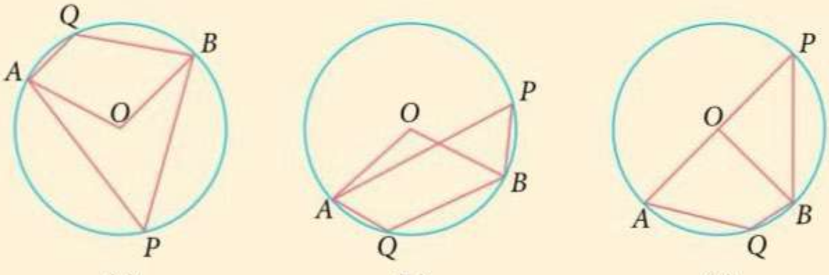
Identify each of the following by drawing the angle in each circle, using a different coloured pencil or pen.

(i) Angle at the centre subtended by the minor arc *AQB*

(ii) Angle at the circumference subtended by the minor arc *AQB*

(iii) Angle at the centre subtended by the major arc *APB*

(iv) Angle at the circumference subtended by the major arc *APB*



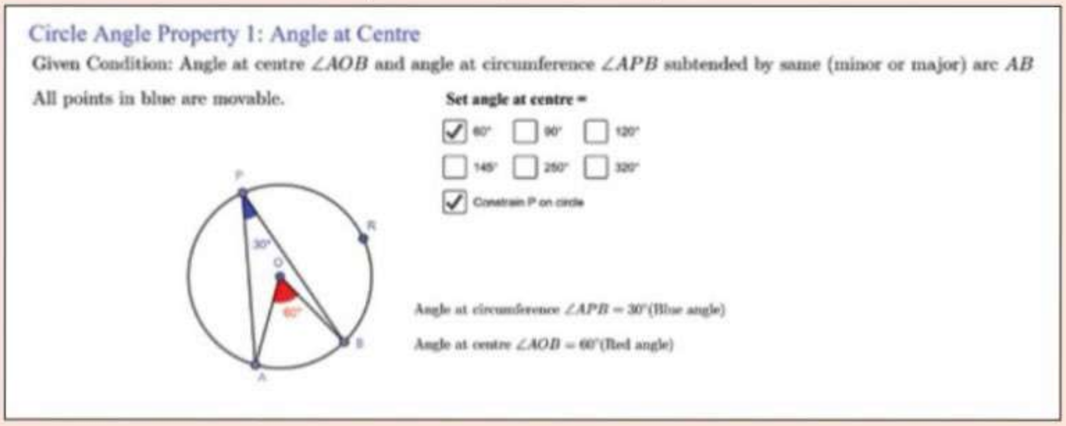
(a) **(b)** **(c)**

Fig. 9.12

Investigation Discovering circle angle property 1

Go to www.sl-education.com/tmsoupp4/pg287 or scan the QR code on the right and open the geometry template 'Circle Angle Property 1'.

1. The template shows a circle with centre O (see Fig. 9.13). $\angle AOB$ is an angle at the centre while $\angle APB$ is an angle at the circumference subtended by the same (minor or major) arc AB.



Circle Angle Property 1: Angle at Centre

Given Condition: Angle at centre $\angle AOB$ and angle at circumference $\angle APB$ subtended by same (minor or major) arc AB

All points in blue are movable.

Set angle at centre =

☑ 60° ☐ 90° ☐ 120°

☐ 145° ☐ 250° ☐ 320°

☑ Constrain P on circle

Angle at circumference $\angle APB = 30°$ (Blue angle)

Angle at centre $\angle AOB = 60°$ (Red angle)

Fig. 9.13

2. Set $\angle AOB$ to the values shown in Table 9.2. You can also move the point R to change the size of the circle. Complete Table 9.2.

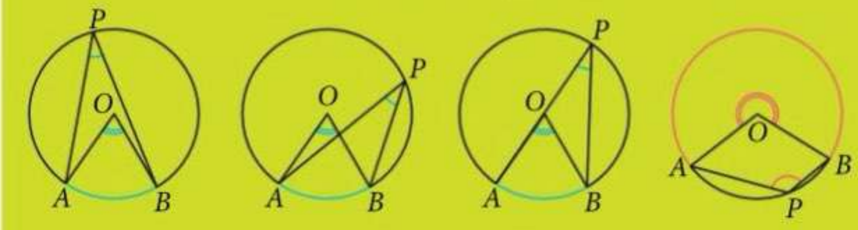
$\angle AOB$	60°	90°	120°	145°	250°	320°
$\angle APB$						
$\dfrac{\angle AOB}{\angle APB}$						

Table 9.2

3. What is the relationship between $\angle AOB$ and $\angle APB$?
4. What if the point P does not lie on the circumference of the circle, i.e. if P lies inside the circle or outside the circle? Will $\angle AOB = 2 \times \angle APB$?
5. Complete the following sentence.
   In general, an angle subtended at the centre of a circle is ______ that of any angle subtended at the circumference by the same arc.

From the above Investigation, we can observe the following property:

Circle Angle Property 1: Angle at Centre
(abbreviation: $\angle$ at centre $= 2\angle$ at $\odot^{ce}$)



An angle subtended at the centre of a circle is **twice** that of any angle subtended at the circumference by the *same arc*, i.e. $\angle AOB = 2 \times \angle APB$.

Class Discussion Finding unknown angle using circle angle property 1

For each of the following diagrams, O is the centre of the circle passing through points A, B and P. Find the unknown $\angle x$ in each of the following diagram. State your reason clearly.

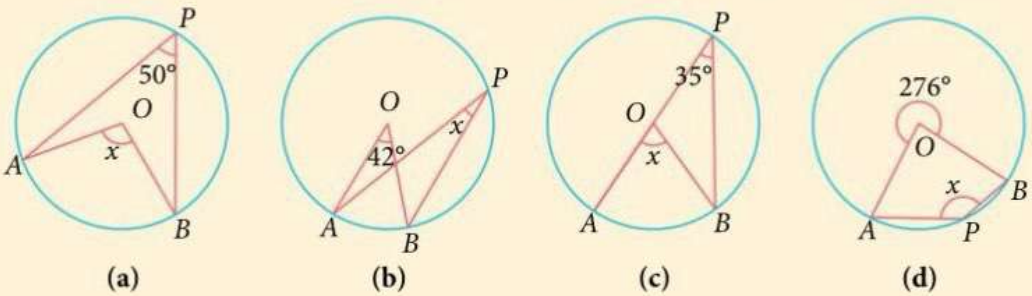


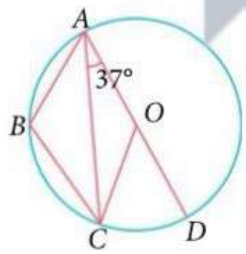
Fig. 9.14

Worked Example 6

Applying circle angle property 1

A, B, C and D are four points on a circle with centre O. Given that AOD is a diameter of the circle and $\angle CAD = 37°$, find

(i) $\angle COD$, (ii) $\angle ABC$.



***Solution**

(i) $\angle COD = 2 \times \angle CAD$ ($\angle$ at centre = 2 $\angle$ at $\odot^{ce}$)

$$= 2 \times 37°$$

$$= 74°$$

(ii) Reflex $\angle AOC = 180° + \angle COD$

$$= 180° + 74°$$

$$= 254°$$

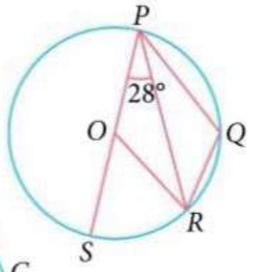
$\therefore \angle ABC = \dfrac{254°}{2}$ ($\angle$ at centre = 2 $\angle$ at $\odot^{ce}$)

$$= 127°$$

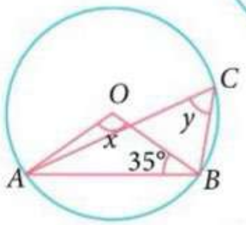
Practise Now 6

Similar and Further Questions
Exercise 9B
Questions 1(a)–(h), 10, 11

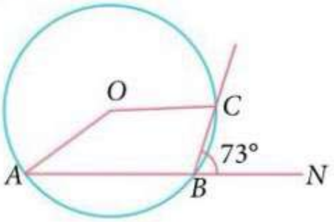
1. P, Q, R and S are four points on a circle with centre O. Given that POS is a diameter of the circle and $\angle OPR = 28°$, find
   (i) $\angle SOR$,
   (ii) $\angle PQR$.



2. A, B and C are points on a circle with centre O. Given that $\angle ABO = 35°$, find the angles marked x and y.



3. In this figure, O is the centre of the circle and A and B lie on the circumference such that ABN is a straight line. Given that C lies on the circumference such that $\angle NBC = 73°$, find $\angle AOC$.

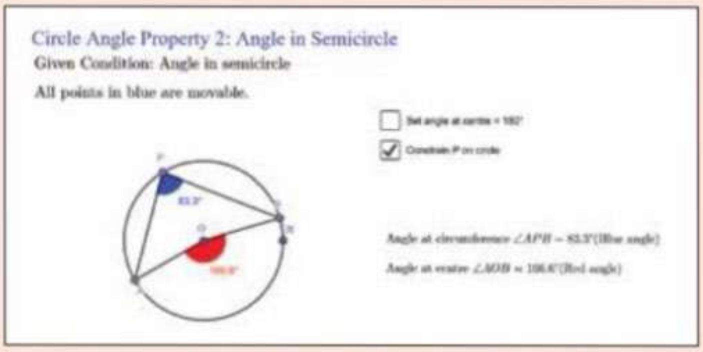


B. Angle in a semicircle

Investigation Discovering circle angle property 2

Go to www.sl-education.com/tmsoupp4/pg289 or scan the QR code on the right and open the geometry template 'Circle Angle Property 2'.

1. The template below shows a circle with centre O (see Fig. 9.15). $\angle AOB$ is an angle at the centre while $\angle APB$ is an angle at the circumference subtended by the same arc AB.


Circle Angle Property 2: Angle in Semicircle

Given Condition: Angle in semicircle

All points in blue are movable.

☐ Set angle at centre = 180°

☑ Constrain P on circle

Angle at circumference ∠APB = 83.3° (Blue angle)

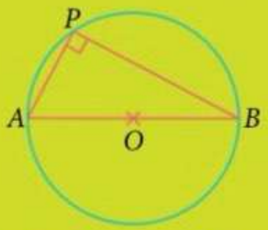
Angle at centre ∠AOB = 166.6° (Red angle)

Fig. 9.15

2. Set $\angle AOB = 180°$. You can also move the point R to change the size of the circle, and the point P to change $\angle APB$.
   - **(a)** What is $\angle APB$ equal to?
   - **(b)** What is the special name given to the sector APB when $\angle AOB = 180°$?
3. What if the point P does not lie on the circumference of the circle, i.e. if P lies inside the circle or outside the circle? Will $\angle APB$ still be 90°?
4. Complete the following sentence.
   In general, an angle in a semicircle is always equal to ______°.
5. Prove the angle property in Question 4.

From the above Investigation, we observe the following property:

Circle Angle Property 2: Angle in Semicircle (abbreviation: rt. ∠ in semicircle)



An angle in a *semicircle* is always a right angle, i.e. $\angle APB = 90°$.

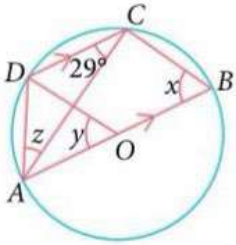
Attention

This property is a special case of circle angle property 1.

Worked Example 7

Applying circle angle property 2

A, B, C and D are four points on a circle with centre O. Given that AOB is a diameter of the circle, DC is parallel to AB and $\angle DCA = 29°$, find $\angle x$, $\angle y$ and $\angle z$.



Solution

$\angle ACB = 90°$ (rt. ∠ in semicircle)

$\angle CAB = 29°$ (alt. ∠s, $AB // DC$)

$\therefore \angle x = 180° - 90° - 29°$ (∠ sum of △)

$= 61°$

$\angle y = 2 \times \angle ACD$ ($\angle$ at centre = 2 $\angle$ at $\odot^{ce}$)

$= 2 \times 29°$

$= 58°$

$\angle DAO = \angle ADO$ (base $\angle$ of isos. $\triangle$)

$= \frac{180° - 58°}{2}$

$= 61°$

$\therefore \angle z = \angle DAO - \angle CAB$

$= 61° - 29°$

$= 32°$

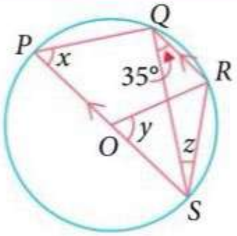
Practise Now 7

Similar and Further Questions

Exercise 9B

Questions 2(a)–(d), 12, 13

1. P, Q, R and S are four points on a circle with centre O. Given that POS is a diameter of the circle, RQ is parallel to SP and $\angle SQR = 35°$, find $\angle x$, $\angle y$ and $\angle z$.

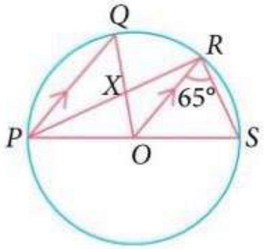


2. P, Q, R and S are four points on a circle with centre O. Given that POS is a diameter of the circle, PQ is parallel to OR and $\angle ORS = 65°$, find

(i) $\angle OPR$,

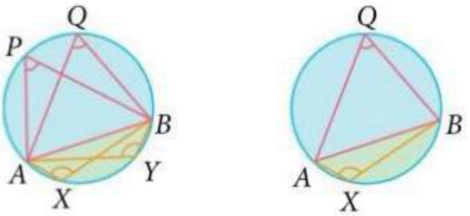
(ii) $\angle QOR$,

(iii) $\angle PXQ$.



C. Angles in same segment or in opposite segments

Fig. 9.16**(a)** shows a circle with a chord AB that divides the circle into two segments. The larger segment AQB is called the *major segment* (shaded blue) while the smaller segment AXB is called the *minor segment* (shaded green). $\angle APB$ and $\angle AQB$ are angles subtended at the circumference of the circle by the same minor arc AXB. Since $\angle APB$ and $\angle AQB$ lie in the same segment AQB, they are called **angles in the same segment**.
$\angle AXB$ and $\angle AYB$ are angles subtended at the circumference of the circle by the same major arc APB. Since $\angle AXB$ and $\angle AYB$ lie in the same segment AXB, they are also called angles in the same segment.



(a) **(b)**

Fig. 9.16

Attention

Opposite segments must be formed by the *same chord*.

Fig. 9.16**(b)** shows a circle with a chord AB that divides the circle into two segments. The segment AQB and the segment AXB are called *opposite segments* (not different segments).

$\angle AQB$ and $\angle AXB$ are angles subtended at the circumference of the circle by the minor arc AXB and by the major arc AQB respectively. Since $\angle AQB$ and $\angle AXB$ lie in opposite segments, they are called **angles in opposite segments**.

Class Discussion

Angles in same segment or in opposite segments

Fig. 9.17 shows a circle passing through points *A*, *B*, *C*, *D*, *E* and *F* with four angles labelled *w*, *x*, *y* and *z*.

1. Work in pairs to identify which pairs of the four angles are in the same segment and which pairs of the four angles are in opposite segments. For each case, specify the chord that forms the segment(s).
2. Are ∠*w* and ∠*y* angles in opposite segments? Explain.

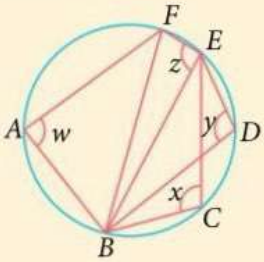


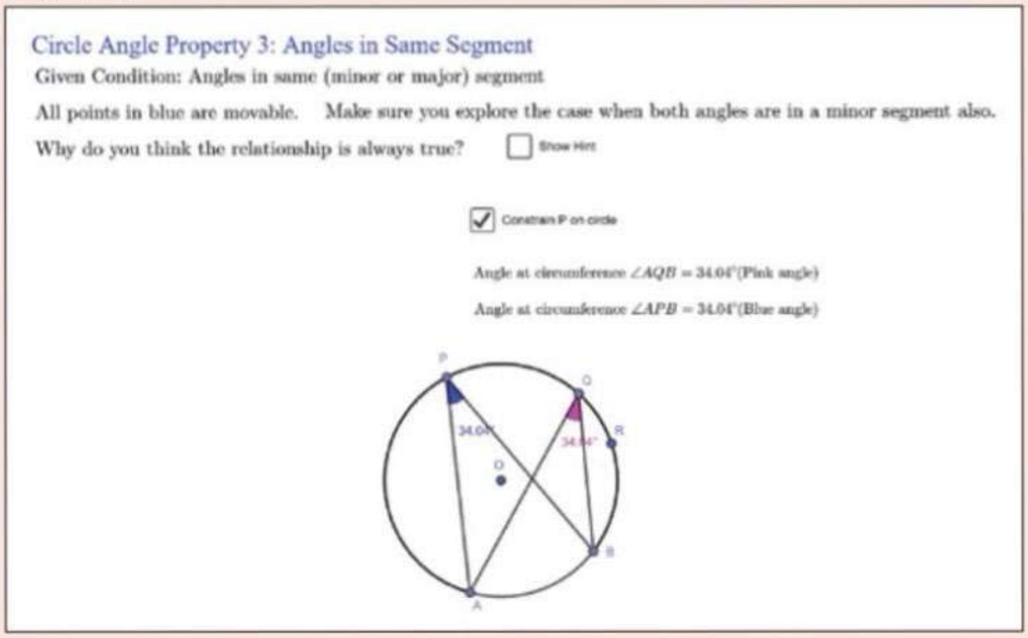
Fig. 9.17

Investigation

Discovering circle angle property 3

Go to www.sl-education.com/tmsoupp4/pg292 or scan the QR code on the right and open the geometry template 'Circle Angle Property 3'.

1. The template below shows a circle with centre *O* (see Fig. 9.18). ∠*APB* and ∠*AQB* are angles in the same (minor or major) segment.



Circle Angle Property 3: Angles in Same Segment

Given Condition: Angles in same (minor or major) segment

All points in blue are movable. Make sure you explore the case when both angles are in a minor segment also.

Why do you think the relationship is always true? ☐ Show Hint

☑ Constrain P on circle

Angle at circumference ∠AQB = 34.04°(Pink angle)

Angle at circumference ∠APB = 34.04°(Blue angle)

Fig. 9.18

2. Click and drag point *A* or *B* to change the size of ∠*APB* and of ∠*AQB*.
Click and drag point *R* to change the size of the circle.
Click and drag point *P* or *Q* to change the position of ∠*APB* and of ∠*AQB*.
What do you notice about ∠*APB* and ∠*AQB*?

Attention

To adjust ∠*APB* and ∠*AQB* until they are in the same minor segment, click and drag point *A* or *B* until arc *APB* is a minor arc.

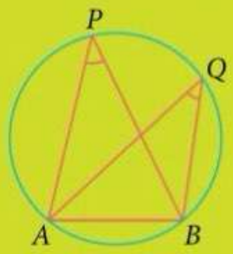
3. Complete the following sentence.

In general, angles in the same segment are ________.

4. Prove the angle property in Question 3. You can also click on the button 'Show Hint' in the template.

From the above Investigation, we observe the following property:

Circle Angle Property 3: Angles in Same Segment
(abbreviation: ∠s in same segment)



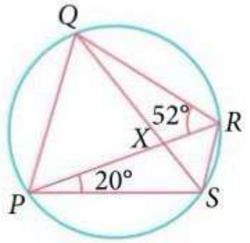
Angles in the **same segment** are *equal*, i.e. $\angle APB = \angle AQB$.

Worked Example 8

Applying circle angle property 3

In this figure, P, Q, R and S are points on the circumference of a circle.
Given that PR and QS intersect at the point X, $\angle RPS = 20°$ and $\angle PRQ = 52°$, find

(i) $\angle SQR$, **(ii)** $\angle XSP$, **(iii)** $\angle PXQ$.



Solution

(i) $\angle SQR = \angle RPS$ (∠s in same segment)
$= 20°$

(ii) $\angle XSP = \angle PRQ$ (∠s in same segment)
$= 52°$

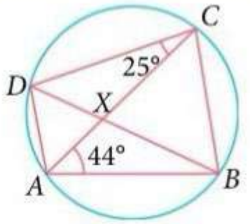
$\angle XSP = \angle QSP$

(iii) $\angle PXQ = \angle SQR + \angle PRQ$ (ext. ∠ of $\triangle QRX$)
$= 20° + 52°$
$= 72°$

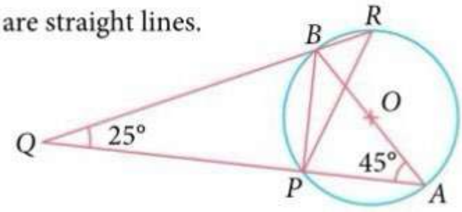
Practise Now 8

Similar and Further Questions
Exercise 9B
Questions 3(a), (b), 4, 5, 14–16, 27, 28

1. In this figure, A, B, C and D are points on the circumference of a circle. Given that AC and BD intersect at the point X, $\angle BAC = 44°$ and $\angle ACD = 25°$, find
(i) $\angle CDB$, **(ii)** $\angle ABX$,
(iii) $\angle CXB$.



2. In the figure, A, P, B and R are points on the circumference of a circle with centre O. AB is a diameter of the circle. APQ and RBQ are straight lines.
Find $\angle BPR$.

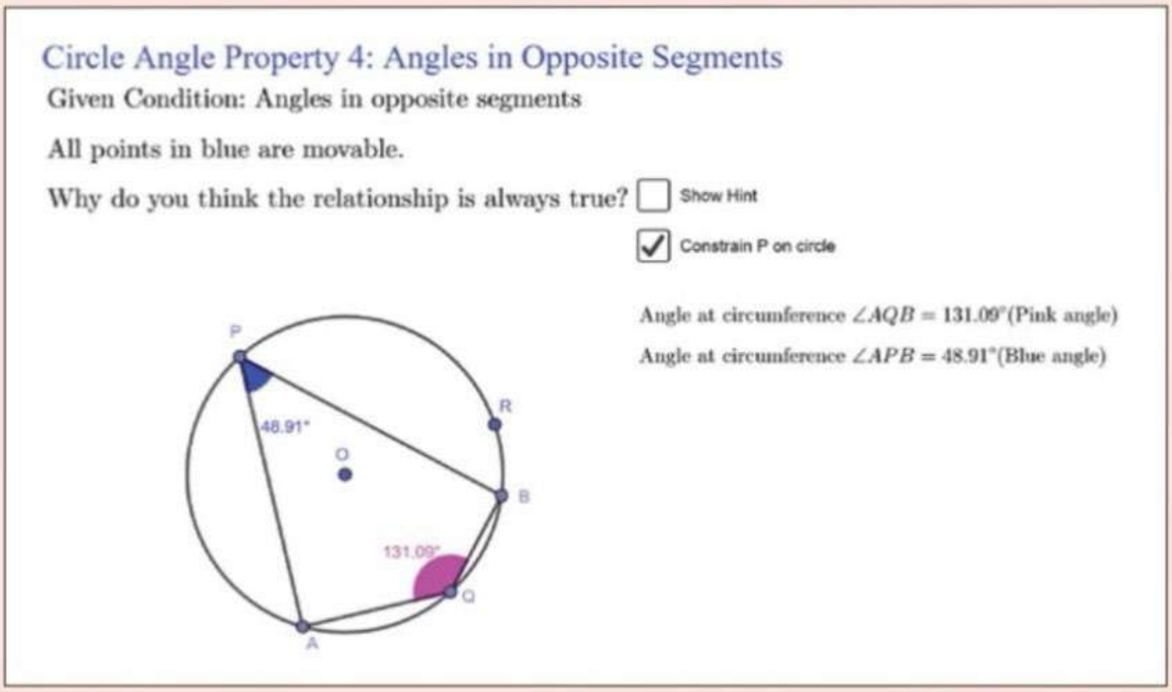


Investigation

Discovering circle angle property 4

Go to www.sl-education.com/tmsoupp4/pg294 or scan the QR code on the right and open the geometry template 'Circle Angle Property 4'.

1. The template below shows a circle with centre O (see Fig. 9.19). $\angle APB$ and $\angle AQB$ are angles in opposite (minor or major) segments.



Circle Angle Property 4: Angles in Opposite Segments

Given Condition: Angles in opposite segments

All points in blue are movable.

Why do you think the relationship is always true? ☐ Show Hint

☑ Constrain P on circle

Angle at circumference $\angle AQB = 131.09°$(Pink angle)

Angle at circumference $\angle APB = 48.91°$(Blue angle)

Fig. 9.19

2. Click and drag the point A or B to change the size of $\angle APB$ and of $\angle AQB$.
   Click and drag the point R to change the size of the circle.
   Click and drag the point P or Q to change the position of $\angle APB$ and of $\angle AQB$.
   What do you notice about the sum of $\angle APB$ and $\angle AQB$?

3. Complete the following sentence.
   In general, angles in opposite segments are *supplementary*, i.e. they add up to ____°.

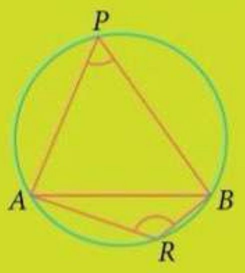
4. Prove the angle property in Question 3. You can also click on the button 'Show Hint' in the template.

Attention

Another way of viewing this angle property is to look at the quadrilateral $APBQ$ in Fig. 9.19. First, we notice that all the vertices of the quadrilateral lie on the circumference of the circle and we call this kind of quadrilateral a *cyclic quadrilateral*. Second, we notice that $\angle APB$ and $\angle AQB$, which are angles in opposite segments, are also opposite angles of the cyclic quadrilateral $APBQ$. In other words, whenever we see a cyclic quadrilateral, we know the relationship between its opposite angles.

From the Investigation on page 294, we observe the following property:

Circle Angle Property 4: Angles in Opposite Segments
(abbreviation: $\angle$s in opp. segments)



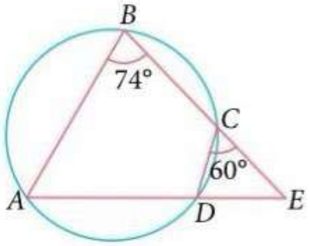
Angles in **opposite segments** are *supplementary*, i.e. $\angle APB + \angle ARB = 180^\circ$.

Worked Example 9

Applying circle angle property 4

This figure shows a circle with points A, B, C and D on its circumference. BCE and ADE are straight lines.
Given that $\angle ABC = 74^\circ$ and $\angle DCE = 60^\circ$, find

(i) $\angle ADC$, **(ii)** $\angle CED$.



Solution

(i) $\angle ADC$

$= 180^\circ - 74^\circ$ ($\angle$s in opp. segments)

chord AC divides circle into opp. segments

$= 106^\circ$

(ii) $\angle CED = 106^\circ - 60^\circ$ (ext. $\angle$ of $\triangle$)

$= 46^\circ$

Problem-solving Tip

From the previous Attention, whenever we see a cyclic quadrilateral (in this case, quadrilateral $ABCD$), its opposite angles are actually angles in opposite segments (separated by chord AC in this case).

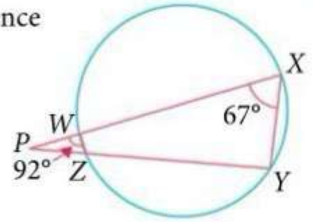
Practise Now 9

Similar and Further Questions
Exercise 9B
Questions 6, 7(a)–(d), 8, 9, 17–22

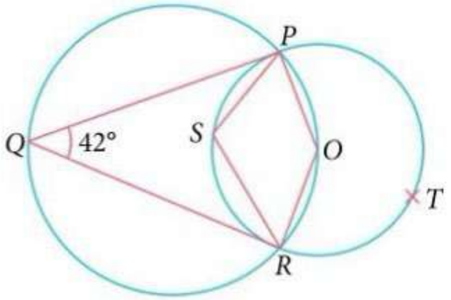
1. In this figure, the points W, X, Y and Z lie on the circumference of a circle. PWX and PZY are straight lines.
If $\angle PXY = 67^\circ$ and $\angle PWZ = 92^\circ$, calculate

(i) $\angle WZY$,

(ii) $\angle WPZ$.



2. In this figure, O is the centre of the smaller circle passing through the points P, S, R and T.
The points P, Q, R and O lie on the larger circle.
Given that $\angle PQR = 42^\circ$, find $\angle PSR$.



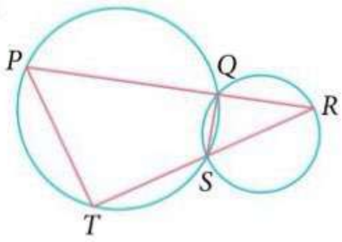
Worked Example 10

Solving problem involving circle angle property 4

In this figure, P, Q, S and T are points on the circumference of the larger circle while points Q, R and S lie on the circumference of the smaller circle. PQR and RST are straight lines.

(i) Show that $\triangle PRT$ and $\triangle SRQ$ are similar.

(ii) If $PQ = 4$ cm, $QR = 2$ cm and $SR = 2.4$ cm, find the length of ST.



Solution

We will use **Pólya's Problem Solving Model** to guide us in solving this problem.

Stage 1: Understand the problem

What information is given and what is not?

- Given information: two circles with some points on their circumferences; PQR and RST are straight lines; for part **(ii)**, the lengths of PQ, QR and SR are also given.
- Missing information: any angles in the circles.

What are we supposed to find?

- For part **(i)**, show that $\triangle PRT$ and $\triangle SRQ$ are similar; for part **(ii)**, find the length of ST.

Stage 2: Think of a plan

For part ***(i)****, what have we learnt about proving that can help us prove that the two triangles are similar? Which similarity test can we use? AA, SAS or SSS Similarity Tests?*

At first glance, since no angles are given, we should use SSS Similarity Test. But the lengths of some sides are given only in part **(ii)**, so we cannot use those lengths.

Think of another plan

What have we learnt in this chapter about angles in a circle?

When we see a cyclic quadrilateral PQST (i.e. the points P, Q, S and T lie on the circumference), what do we know about its opposite angles (see Attention on page 294)?

We observe that its opposite angles are angles in opposite segments. Let us also recall what we have learnt about circle angle property 4.

Since no angles are given, we have to let one of the angles of the quadrilateral be x. In this case, it can be any of the four angles. Then we use AA Similarity Test.

Stage 3: Carry out the plan

(i) Let $\angle PQS = x$.

Then $\angle PTS = 180° - x$ ($\angle$s in opp. segments)

and $\angle SQR = 180° - x$ (adj. $\angle$s on a str. line)

$\therefore \angle PTR = \angle SQR$ $\quad$ $\angle PTR$ and $\angle PTS$ are the same angle

$= 180° - x$

$\angle PRT = \angle SRQ$ (common $\angle$)

$\therefore \triangle PRT$ and $\triangle SRQ$ are similar (AA Similarity Test).

Problem-solving Tip

We cannot assume the vertices of $\triangle PRT$ and $\triangle SRQ$ are given in order. We have to identify the corresponding vertices of the two triangles by comparing the size of their angles:

$R \leftrightarrow R$ (common $\angle$)

$T \leftrightarrow Q$ (angle $= 180° - x$)

$P \leftrightarrow S$ (remaining angles)

Stage 4: Look back

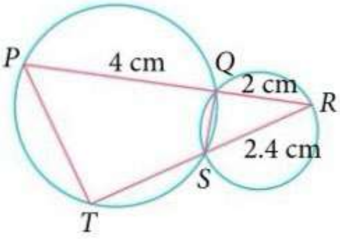
Reflect on solution for part (i)

What if we let $\angle PTS = x$? Would it work? What is the difference?
What if we let $\angle QPT = x$ or $\angle QST = x$? Is there any difference?

Stage 2: Think of a plan (continued)

Since we have already proven that the two triangles are similar in part **(i)***, what properties of similar triangles can we use to find the length of the unknown side in part* **(ii)***?*

Stage 3: Carry out the plan (continued)



(ii) Let $ST = y$.

$$\text{Then } \frac{RT}{RQ} = \frac{RP}{RS} \qquad R \leftrightarrow R,\ T \leftrightarrow Q,\ P \leftrightarrow S$$

$$\frac{y+2.4}{2} = \frac{4+2}{2.4}$$

$$y + 2.4 = \frac{12}{2.4}$$

$$= 5$$

$$y = 5 - 2.4$$

$$= 2.6 \text{ cm}$$

$$\therefore ST = 2.6 \text{ cm}$$

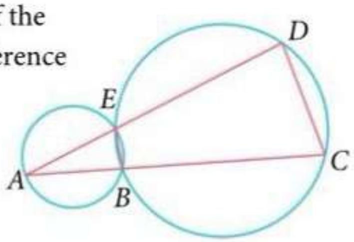
Stage 4: Look back (continued)

How can we check whether the answers are correct? Can we work backwards?

$$\frac{RT}{RQ} = \frac{y+2.4}{2} = \frac{2.6+2.4}{2} = 2.5; \frac{RP}{RS} = \frac{6}{2.4} = 2.5 = \frac{RT}{RQ}$$

Practise Now 10

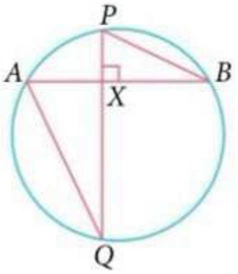
Similar and Further Questions
Exercise 9B
Questions 23–25, 29



1. In this figure, A, B and E are points on the circumference of the smaller circle while points B, C, D and E lie on the circumference of the larger circle. ABC and AED are straight lines.

(i) Show that $\triangle ABE$ is similar to $\triangle ADC$.

(ii) Given that $AB = 3.8$ cm, $AE = 3.9$ cm and $DE = 7.4$ cm, find the length of BC.



2. In this figure, A, P, B and Q are points on the circumference of a circle. The chords AB and PQ intersect at right angles at X.

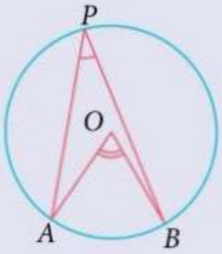
(i) Show that $\triangle AXQ$ and $\triangle PXB$ are similar.

(ii) Given that $AX = 5$ cm, $QX = 10.5$ cm and $PX = 3.4$ cm, find the length of BX.

Reflection

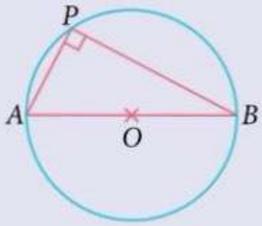
1. Fig. 9.20 summarises the four angle properties of circles. Fill in the blanks.

Property 1:
Angle at Centre
(abbreviation: ∠ at centre = 2 ∠ at Θ^{ce})



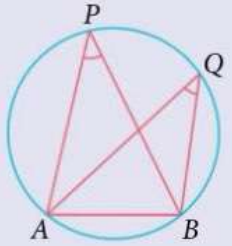
An angle subtended at the centre of a circle is ______ that of any angle subtended at the circumference by the *same arc*, i.e.
$\angle AOB =$ ______ $\times \angle APB$.

Property 2:
Angle in Semicircle
(abbreviation: rt. ∠ in semicircle)



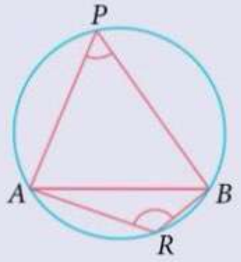
An angle in a *semicircle* is always a ______ angle, i.e.
$\angle APB =$ ______°.

Property 3:
Angles in Same Segment
(abbreviation: ∠s in same segment)



Angles in the *same segment* are ______, i.e.
$\angle APB = \angle$ ______.

Property 4:
Angles in Opposite Segments
(abbreviation: ∠s in opp. segments)



Angles in *opposite segments* are ______, i.e.
$\angle APB + \angle ARB =$ ______°.

Fig. 9.20

2. How are circle angle properties 2, 3 and 4 related to circle angle property 1?
3. Given a semicircle, which of the above properties could I apply?
4. Given two angles subtended at the circumference by the same arc of a circle, which of the above properties could I apply?
5. What have I learnt in this section or chapter that I am still unclear of?

9.3 Alternate Segment Theorem

In Fig. 9.21, *TA* is the tangent to the circle at *A* and *AQ* is a chord at the point of contact. $\angle TAQ$ is the angle between the tangent *TA* and the chord *AQ* at *A*. $\angle QPA$ is the angle subtended by chord *AQ* in the alternate segment. We will now learn a theorem that states how they are related.

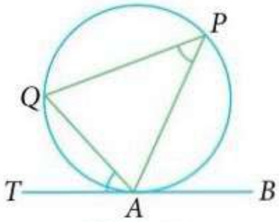
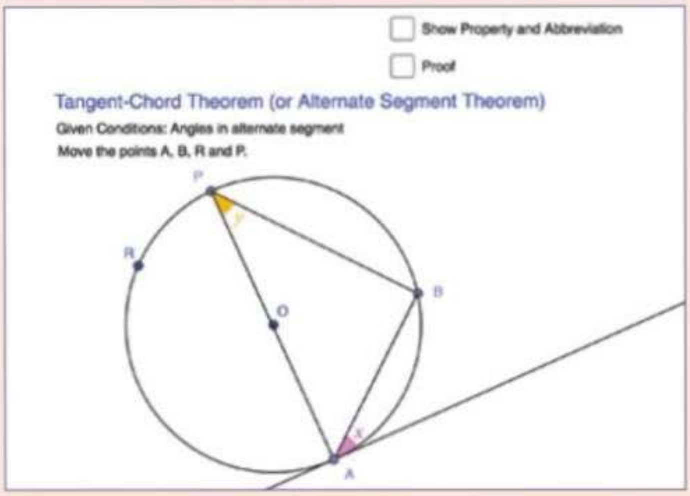


Fig. 9.21

Investigation

Alternate Segment Theorem

Go to www.sl-education.com/tmsoupp4/pg299 and open the geometry software template 'Alternate Segment Theorem' as shown below:



☐ Show Property and Abbreviation

☐ Proof

Tangent-Chord Theorem (or Alternate Segment Theorem)

Given Conditions: Angles in alternate segment

Move the points A, B, R and P.

Fig. 9.22

Given conditions: Angles in alternate segments

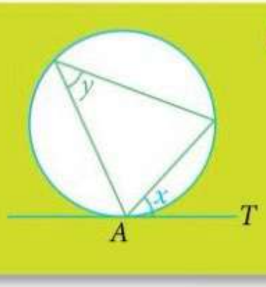
In the above circle, there are two coloured angles. The pink angle, $\angle x$, is the angle between the **tangent** and the **chord** at the point of contact *A*. The yellow angle, $\angle y$, is an angle subtended by the chord in the **alternate segment** with reference to $\angle x$. Notice that the two angles are on different sides of the chord.

1. Click and drag to move the points *A*, *B*, *R* and *P* to change the size of the two coloured angles and the radius of the circle. State what you observe about the relationship between $\angle x$ and $\angle y$.
2. Can you prove that your observation in Question 1 is true for all angles *x* and *y*? You can click the button next to 'Proof' in the template.

This is called the **Tangent-Chord Theorem** (or **Alternate Segment Theorem**).

From the Investigation on page 299, the **Alternate Segment Theorem** states that:

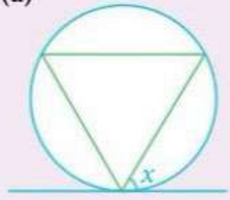
The angle between a tangent and a chord at the point of contact is equal to the angle subtended by the chord in the alternate segment. In the diagram, ∠x = ∠y. (∠s in alt. segments)



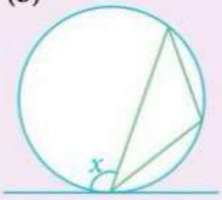
Thinking Time

In each of the circles, identify the angle that is equal to ∠x and label it as y.

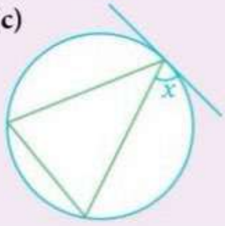
(a)



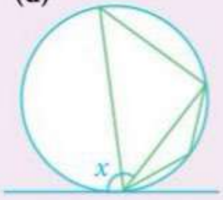
(b)



(c)



(d)

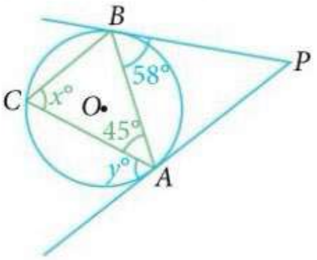


Worked Example 11

Applying Alternate Segment Theorem

Given that PA and PB are tangents to a circle with centre O, ∠ABP = 58° and ∠BAC = 45°, find

(i) the value of x, **(ii)** the value of y.



Just For Fun

Do you see a perfect circle in the figure below?

Solution

(i) $x° = 58°$ (∠s in alt. segments)

$\therefore x = 58$

(ii) ∠PAB = 58° (base ∠s of isosceles △PAB)

$\therefore y° = 180° - 58° - 45°$ (adj. ∠s on a str. line)

$= 77°$

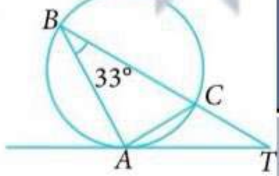
$\therefore y = 77$

Practise Now 11

Similar and Further Questions
Exercise 9B
Question 26

BC is a diameter of the circle and *TA* is the tangent to the circle at *A*. Given that angle $ABC = 33°$, find

(i) angle *CAT*,
(ii) angle *ATC*.



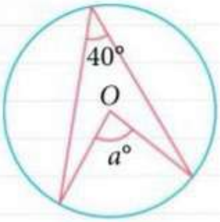
Reflection

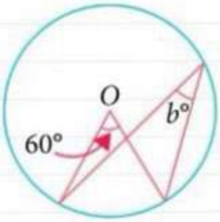
1. When I solve problems related to geometrical proofs, how do I decide whether to apply the Alternate Segment Theorem, or other geometrical properties that I have learnt?
2. What have I learnt in this section or chapter that I am still unclear of?

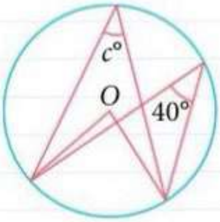
Advanced Intermediate Basic

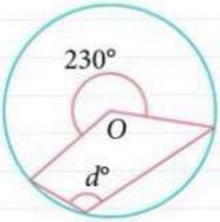
Exercise 9B

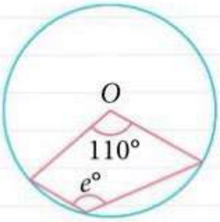
1. Given that *O* is the centre of each of the following circles, find the value of each of the unknowns.

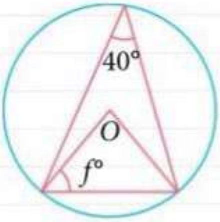
(a)


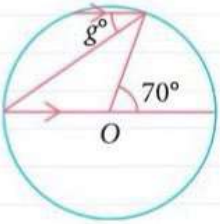
(b)


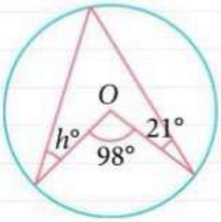
(c)


(d)


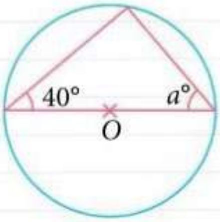
(e)


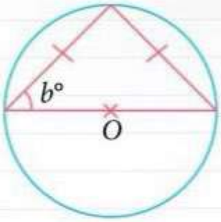
(f)


(g)


(h)


2. Given that *O* is the centre of each of the following circles, find the value of each of the unknowns.

(a)


(b)


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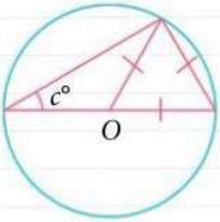
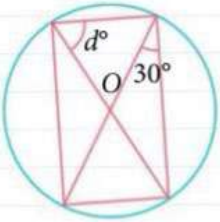
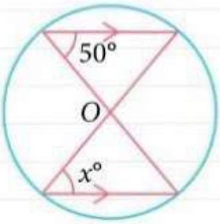
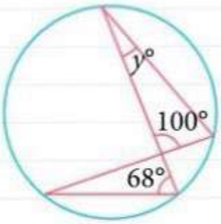


(d)


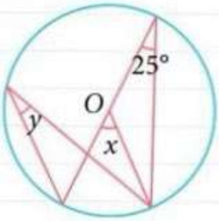
3. Find the value of each of the unknowns.

(a)


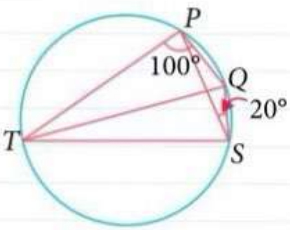
(b)


Exercise 9B

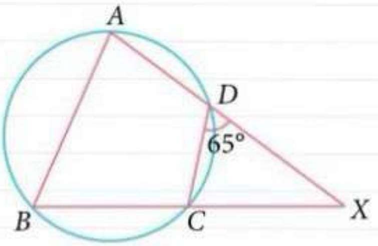
4. Given that O is the centre of the circle, find the angles marked x and y.



5. In the figure below, P, Q, S and T are points on the circle, $\angle TPQ = 100^\circ$ and $\angle PSQ = 20^\circ$.
Find $\angle PQT$.

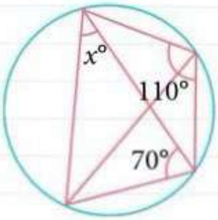


6. In the figure below, A, B, C and D are points on the circle such that AD produced meets BC produced at X. Given that $\angle CDX = 65^\circ$, find $\angle ABC$.

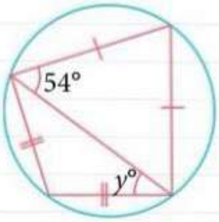


7. Find the values of the unknowns.

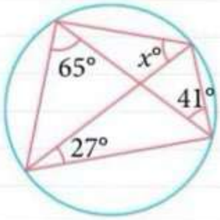
(a)



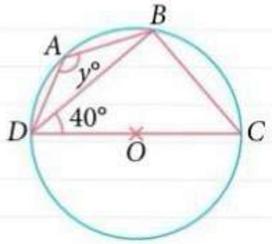
(b)



(c)



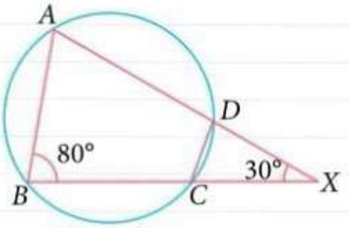
(d) O is the centre of the circle.



8. In the figure below, A, B, C and D are points on the circle such that AD produced meets BC produced at X. Given that $\angle ABC = 80^\circ$ and $\angle AXB = 30^\circ$, find

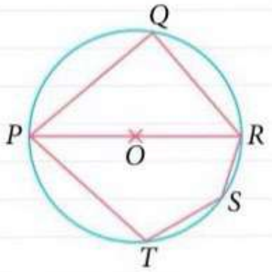
(i) $\angle BAD$,

(ii) $\angle XCD$.

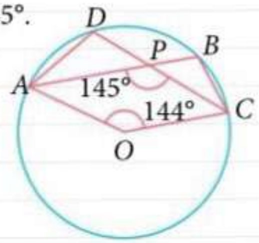


Exercise 9B

9. In the figure below, O is the centre of the circle passing through points P, Q, R, S and T.
Find the sum of $\angle PQR$, $\angle PRS$ and $\angle PTS$.



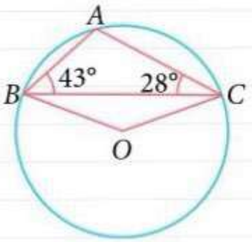
10. In the figure below, O is the centre of the circle and chords AB and CD intersect at P.
$\angle AOC = 144°$ and $\angle APC = 145°$.
Find $\angle BAD$.



11. In the figure below, O is the centre of the circle passing through points A, B and C.
$\angle ABC = 43°$ and $\angle ACB = 28°$.
Find

(i) $\angle OBA$,

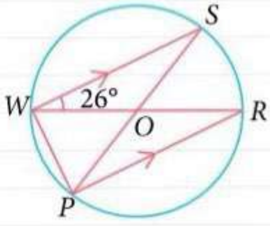
(ii) $\angle OCA$.



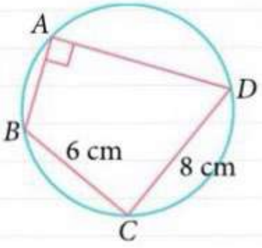
12. In the figure below, O is the centre of the circle with diameters PS and RW.
$\angle SWR = 26°$ and WS is parallel to PR.
Find

(i) $\angle PWR$,

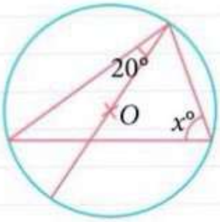
(ii) $\angle SPW$.



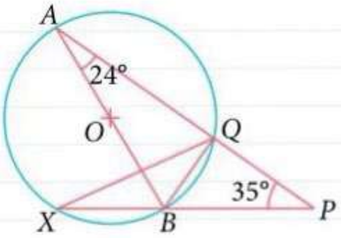
13. In the figure below, A, B, C and D are points on the circle. $\angle BAD = 90°$, $BC = 6$ cm and $CD = 8$ cm.
Find the area of the circle.



14. Given that O is the centre of the circle, find the value of x.

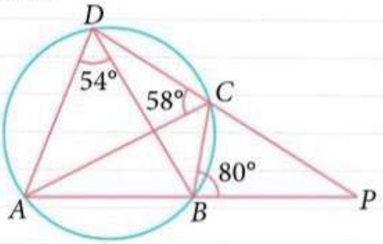


15. In the figure below, A, Q, B and X are points on the circle. AB is a diameter of the circle, and AQ produced meets XB produced at P. Given that $\angle BAP = 24°$ and $\angle BPA = 35°$, find $\angle BQX$.

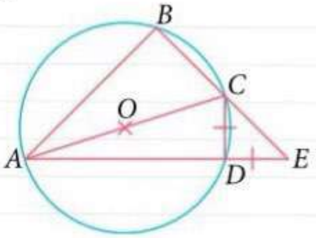


Exercise 9B

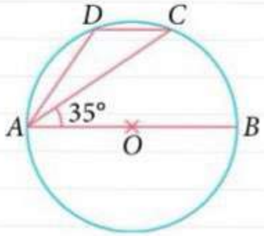
16. In the figure below, chord *DC* produced meets chord *AB* produced at *P*. ∠*ADB* = 54°, ∠*ACD* = 58° and ∠*CBP* = 80°.
Find ∠*APD*.



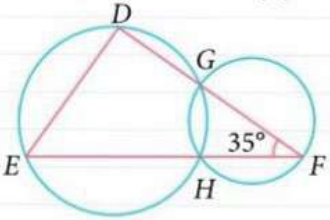
17. In the figure below, *O* is the centre of the circle passing through points *A*, *B*, *C* and *D*. *AD* produced meets *BC* produced at *E*. Given that *CD* = *DE*, find ∠*BAD*.



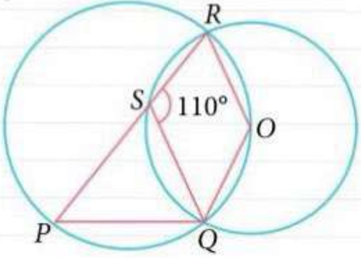
18. In the figure below, *O* is the centre of the circle passing through points *A*, *B*, *C* and *D*. Given that *AB* is a diameter of the circle and ∠*CAB* = 35°, find ∠*ADC*.



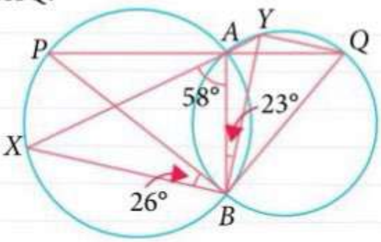
19. In the figure below, two circles intersect at points *G* and *H*. *GF* is a diameter of the circle *GHF* and ∠*GFH* = 35°. *ED* is a chord in the larger circle, and *DGF* and *EHF* are straight lines.
Find
(i) ∠*EDG*, **(ii)** ∠*DEF*.



20. In the figure below, two circles intersect at the points *Q* and *R*. *O* is the centre of the smaller circle, ∠*RSQ* = 110° and *PSR* is a chord in the larger circle. Find ∠*QPS*.



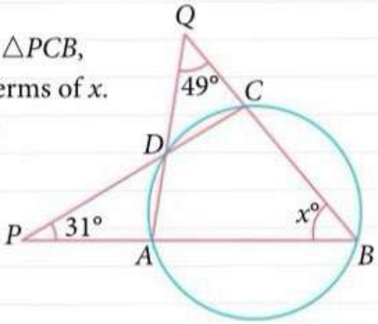
21. In the figure below, points *P*, *A*, *B* and *X* lie on the larger circle and *Q*, *B*, *A* and *Y* lie on the smaller circle. *PAQ* and *XAY* are straight lines, ∠*BAX* = 58°, ∠*PBX* = 26° and ∠*ABY* = 23°.
Find
(i) ∠*AQB*,
(ii) ∠*AYQ*.



Exercise 9B

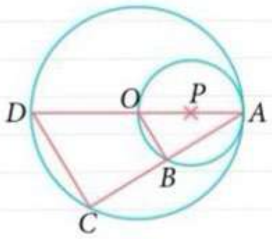
22. In the figure below, A, B, C and D are points on the circumference of a circle. PAB, QCB, PDC and QDA are straight lines. $\angle BPC = 31°$, $\angle AQB = 49°$ and $\angle PBQ = x°$.

- **(i)** By considering $\triangle BAQ$, find $\angle BAQ$ in terms of x.
- **(ii)** By considering $\triangle PCB$, find $\angle PCB$ in terms of x.
- **(iii)** Hence, find the value of x.
- **(iv)** Find $\angle PAD$.



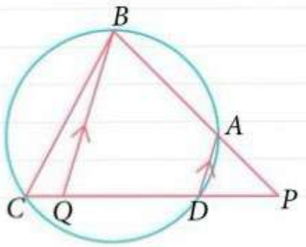
23. In the figure below, O is the centre of the larger circle passing through the points A, C and D with DOA as its diameter. P is the centre of the smaller circle passing through points O, B and A, with OPA as its diameter.

- **(a)** Show that $\triangle ABO$ is similar to $\triangle ACD$.
- **(b)** Given also that $AP = 4$ cm and $OB = 4.5$ cm, find the length of
  - **(i)** OC,
  - **(ii)** CD.



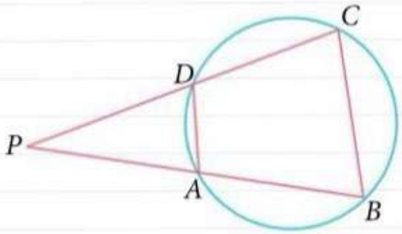
24. In the figure below, A, B, C and D are points on the circle. PAB and $PDQC$ are straight lines. QB is parallel to DA.

- **(i)** Show that $\triangle PAD$ is similar to $\triangle PBQ$.
- **(ii)** Name another triangle that is similar to $\triangle PAD$. Explain your answer.

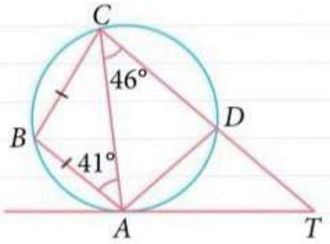


25. In the figure below, A, B, C and D are points on the circle. PAB and PDC are straight lines.

- **(i)** Show that $\triangle PAD$ is similar to $\triangle PCB$.
- **(ii)** Given also that $PA = 12$ cm, $AD = 7$ cm and $PC = 28$ cm, find the length of BC.

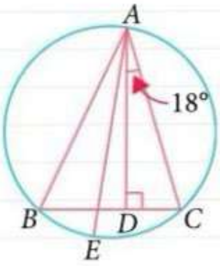


26.



TA is the tangent to the circle at A, $AB = BC$, angle $BAC = 41°$ and angle $ACT = 46°$. Find angle ATC.

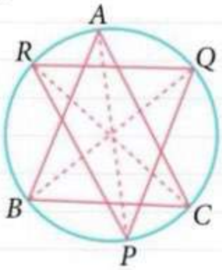
27. In the figure below, A, B, E and C are points on the circle. AE is a diameter of the circle and D lies on BC such that AD is the height of $\triangle ABC$.
Given that $\angle CAD = 18°$, find $\angle BAE$.



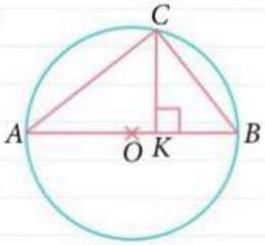
Basic Intermediate Advanced

Exercise 9B

28. In the figure below, A, Q, C, P, B and R are points on the circle. AP, BQ and CR are the angle bisectors of $\angle A$, $\angle B$ and $\angle C$ respectively.
Given that $\angle A = 50°$, $\angle B = 70°$ and $\angle C = 60°$, find $\angle P$, $\angle Q$ and $\angle R$.



29. In the figure below, AOB is a diameter of the circle with centre O. C is a point on the circumference and K lies on AB such that CK is perpendicular to AB.
(i) Show that $\triangle ACK$ is similar to $\triangle CBK$.
(ii) Given also that $AK = 12$ cm and $CK = 10$ cm, find the radius of the circle.

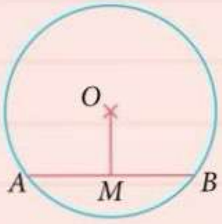
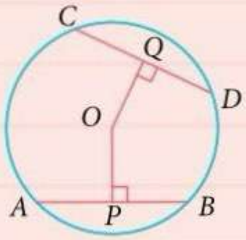
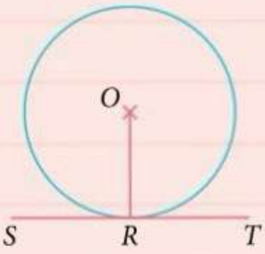
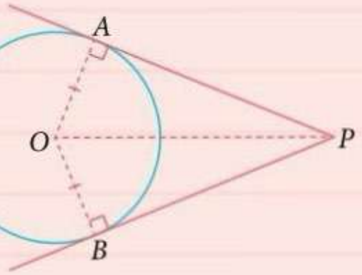


Looking Back

What makes a circle so interesting? In this chapter, we explored some of the symmetric and angle properties that make circles such fascinating shapes. Many of these properties involve the **invariance** of an angle or other property under some transformation. For instance, the angle in any semicircle is a right angle and angles in the same segment are equal. Knowledge about these properties could be useful for modelling buildings and objects that incorporate circles into their design. We could also use these properties to reconstruct a whole plate from a small fragment of a circular plate, or determine which are the best seats in a cinema, given our knowledge of the optimal viewing angle. These are just some examples that show how we could apply our knowledge of circles to solve real-world problems.

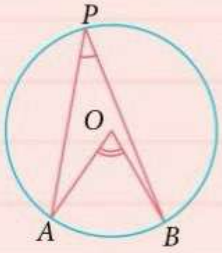
Summary

1. Symmetric properties of circle

	Perpendicular properties	**Equal length properties**
Properties of chords	**Property 1: Perpendicular Bisector of Chord** **(abbreviation: ⊥ bisector of chord)**  Any two of the three conditions imply the third: **(i)** If $OM \perp AB$, then OM bisects the chord AB. **(ii)** If OM bisects the chord AB (which is not the diameter), then $OM \perp AB$. **(iii)** The perpendicular bisector of a chord will pass through the *centre* of the circle.	**Property 2: Equal Chords** **(abbreviation: equal chords)**  **(i)** Equal chords are *equidistant* from the centre of the circle, i.e. if $AB = CD$, then $OP = OQ$. **(ii)** If two chords are equidistant from the centre of the circle, then they are *equal* (in length), i.e. if $OP = OQ$, then $AB = CD$.
Properties of tangents	**Property 3: Tangent Perpendicular to Radius** **(abbreviation: tangent ⊥ radius)**  The tangent to a circle is *perpendicular* to its radius at the point of contact, i.e. $ST \perp OR$.	**Property 4: Tangents from External Point** **(abbreviation: tangents from ext. pt.)**  **(i)** Tangents from an external point are *equal* (in length). **(ii)** OP bisects $\angle APB$. **(iii)** OP bisects $\angle AOB$.

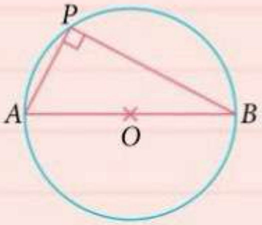
2. Angle properties of circle

Property 1:
Angle at Centre
(abbreviation: $\angle$ at centre = 2 $\angle$ at $\odot^{ce}$)



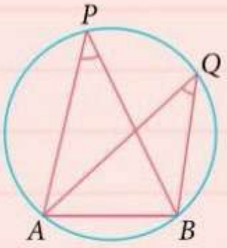
An angle subtended at the centre of a circle is *twice* that of any angle subtended at the circumference by the *same arc*, i.e. $\angle AOB = 2 \times \angle APB$.

Property 2:
Angle in Semicircle
(abbreviation: rt. $\angle$ in semicircle)



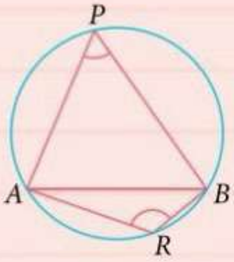
An angle in a semicircle is always a *right angle*, i.e. $\angle APB = 90°$.

Property 3:
Angles in Same Segment
(abbreviation: $\angle$s in same segment)



Angles in the same segment are *equal*, i.e. $\angle APB = \angle AQB$.

Property 4:
Angles in Opposite Segments
(abbreviation: $\angle$s in opp. segments)

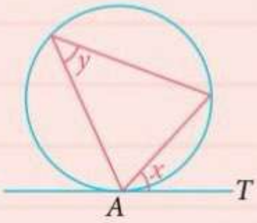


Angles in opposite segments are *supplementary*, i.e. $\angle APB + \angle ARB = 180°$.

3. Alternate Segment Theorem (abbreviation: $\angle$s in alt. segments)

The angle between a tangent and a chord at the point of contact is equal to the angle subtended by the chord in the alternate segment.

In the diagram, $\angle x = \angle y$. ($\angle$s in alt. segments)



CHAPTER 10

Geometrical Transformations

The picture shows escalators in a building for moving people from one floor to another. Translation is illustrated by the motion of escalators in shopping centres, the movement of lifts in high-rise buildings, etc. What other examples can you think of?

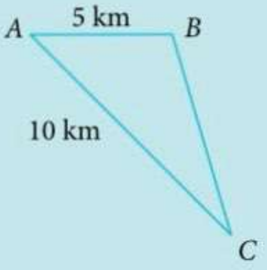
Learning Outcomes

What will we learn in this chapter?

- How to reflect an object, and find the line of reflection
- How to rotate an object, and find the centre and angle of rotation by construction
- How to translate an object
- How to enlarge a figure, and find the centre and scale factor of enlargement
- How to find the image figure of an object under a combination of transformations

Introductory Problem

Yasir has a treasure map which shows the locations of points A, B and C, where C is 10 km to the south-east of A, and B is 5 km to the east of A.



The map also shows the following hints:

(i) The line AC acts as a mirror line, reflecting B to the point D.
(ii) With an imaginary string, locate point X by rotating D 90° anticlockwise about B.
(iii) The treasure is at X.

Can you help Yasir locate the treasure?

In this chapter, we will learn more about reflection, rotation, translation and enlargement, involving the use of coordinates.

10.1 Reflection

A. Reflection in a line

Fig. 10.1 shows the triangle ABC undergoing a reflection in the line $x = 3$ to produce the image $A'B'C'$.

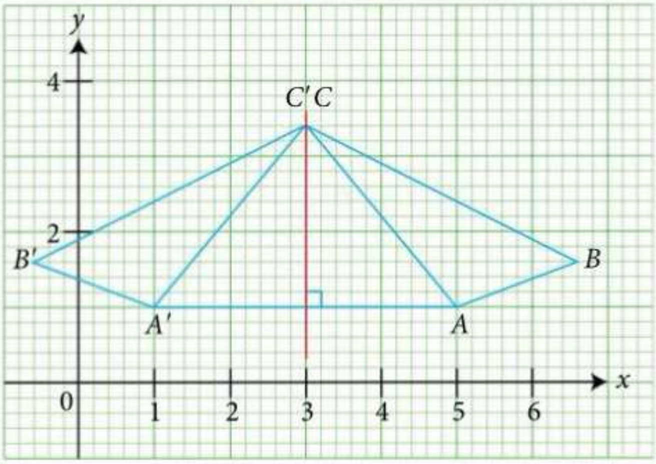


Fig. 10.1

A reflection is defined by its axis of reflection, the line of reflection or simply, the mirror line. In Fig. 10.1, this mirror line is $x = 3$.

(1) In Fig. 10.1, $\triangle ABC$ (read in anticlockwise direction) is reflected to become $\triangle A'B'C'$ (read in clockwise direction). We say that reflection does not preserve *orientation*.

(2) Under reflection, the shape and size of an image are exactly the same as its original figure. We call this gi transformation an **isometric transformation**. In other words, $\triangle ABC$ is *congruent* to $\triangle A'B'C'$.

(3) The line of reflection bisects the line joining any point and its image (e.g. AA' and BB' in Fig. 10.1).

(4) Any point on the mirror line undergoes no change. We say that these points are invariant. In the case of Fig. 10.1, C is the only invariant point, a point that does not undergo any change in a transformation.

From **(3)**, we know that the line of reflection passes through the midpoints formed by the line segments AA' and BB'. We can obtain the coordinates of the midpoints from observation, or by using a formula.

Given two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ where PQ forms a straight line, the coordinates of the midpoint of P and Q are $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

B. Equation of the line of reflection

In Book 3, we learnt how to find the gradient and y-intercept of a straight line and subsequently, obtain the equation of a straight line. In this section, we shall take a look at how we can obtain the equation of the line of reflection.

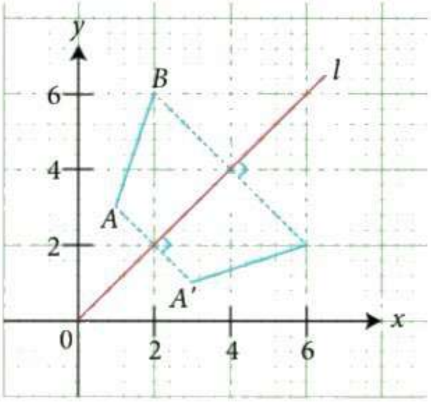


Fig. 10.2

Attention

Given that the line of reflection, l bisects AA' and BB', l will pass through the midpoints of AA' and BB'.

Fig. 10.2 shows the line segment AB and its image $A'B'$, where A, B, A' and B' are the points (1, 3), (2, 6), (3, 1) and (6, 2) respectively.

We can obtain the line of reflection using observation or by first determining the midpoints of AA' and BB', and then finding the equation of the line that passes through the midpoints.

By determining the midpoints of AA' and BB', we get

Midpoint of AA', i.e. $\left(\frac{1+3}{2}, \frac{3+1}{2}\right) = (2, 2)$

Midpoint of BB', i.e. $\left(\frac{2+6}{2}, \frac{6+2}{2}\right) = (4, 4)$

We can obtain the line of reflection by joining the two coordinates i.e. (2, 2) and (4, 4) using a ruler.

Recall that the equation of a straight line is in the form $y = mx + c$, where the constant m is the gradient of the line and the constant c is the y-intercept.

From Fig. 10.2,
Vertical change (or rise) $= 4 - 2$
$= 2$
Horizontal change (or run) $= 4 - 2$
$= 2$

$$\text{Gradient of } l,\ m = \frac{\text{rise}}{\text{run}} = \frac{1}{2} = 1$$

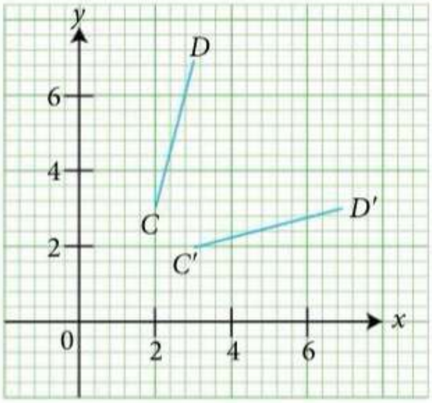
y-intercept, $c = 0$

$\therefore$ the equation of the line of reflection is $y = x$.

Practise Now 1A

Similar and Further Questions
Exercise 10A
Questions 6, 7(a)-(f), 14

The figure shows the line segment *CD* and its image *C′D′*, where *C*, *D*, *C′* and *D′* are the points (2, 3), (3, 7), (3, 2) and (7, 3) respectively.
Find the equation of the line of reflection.



Thinking Time

Fig. 10.3 shows a point *A* under reflection in two lines *l* and *m*. We represent the reflection in line *l* by M_l and that in line *m* by M_m. Hence $M_mM_l(A)$ represents a reflection of point *A* in line *l* followed by line *m*, whereas $M_lM_m(A)$ represents a reflection of point *A* in line *m* followed by line *l*.

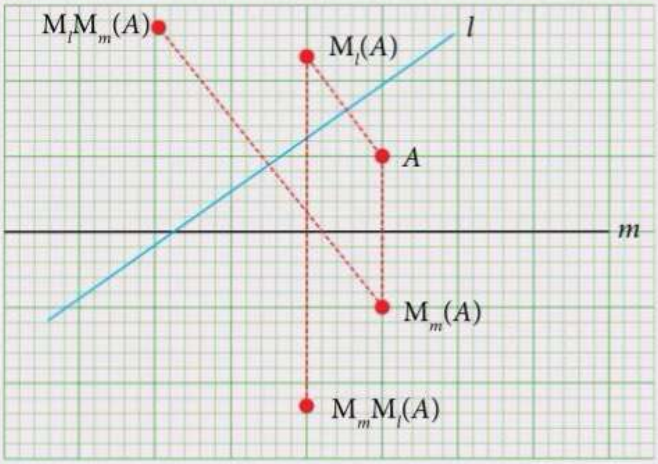


Fig. 10.3

What do you notice about the two points represented by $M_mM_l(A)$ and $M_lM_m(A)$?

Information

In most instances, we use symbols to represent transformations in order to simplify statements. For example, if we represent the enlargement with centre at origin and scale factor 2 as E and a reflection about the *x*-axis as M, then ME represents the transformation of an enlargement followed by a reflection and EM represents a reflection followed by an enlargement, MM (typically written as M^2) represents a reflection followed by another reflection, while EE (E^2) is an enlargement followed by another enlargement.

From Fig. 10.3, we observe that the images of $M_m M_l(A)$ and $M_l M_m(A)$ are not the same. Therefore, we can conclude that the combination of reflections is not commutative.

Worked Example 1

Reflection of points and line segments

The coordinates of A and B are $(-2, 2)$ and $(1, 4)$ respectively. The line joining A and B is reflected in the x-axis to $A'B'$.

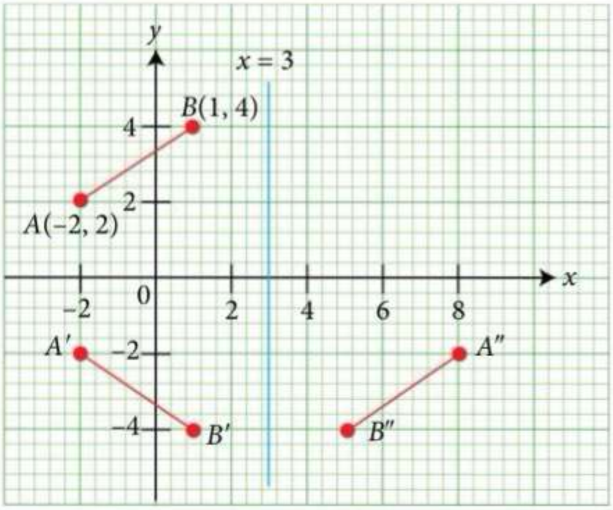
(i) Find the coordinates of A' and B'.

$A'B'$ is then reflected in the line $x = 3$ to give $A''B''$.

(ii) Find the coordinates of A'' and B''.

Show your working on a sheet of graph paper.

Solution



From the graph, the coordinates of A' and B' are $(-2, -2)$ and $(1, -4)$ respectively. A'' is the point $(8, -2)$ and B'' is the point $(5, -4)$.

Practise Now 1B

Similar and Further Questions
Exercise 10A
Questions 1(a)–(c), 2(a), (b), 3–5, 8–12

The coordinates of A and B are $(-3, 1)$ and $(-1, 5)$ respectively. The line joining A and B is reflected in the x-axis to $A'B'$.

(i) Find the coordinates of A' and B'.

$A'B'$ is then reflected in the line $x = 2$ to give $A''B''$.

(ii) Find the coordinates of A'' and B''.

Show your working on a sheet of graph paper.

Reflection

1. How do I construct the image of a two-dimensional figure under a reflection?
2. How do I identify from an object and its image that a transformation is a reflection?

Exercise 10A

1. Write down the coordinates of each of the following points after undergoing a reflection in

(a) the x-axis, **(b)** the y-axis, and

(c) the line $y = x$.

(i) $(3, 4)$ **(ii)** $(-1, 3)$

(iii) $(3, 3)$ **(iv)** $(-3, -4)$

(v) $(3, -2)$ **(vi)** (p, q)

2. **(a)** State the coordinates of the final image when the point $A(-1, 3)$ is reflected in

(i) the x-axis and then in the line $y = 4$,

(ii) the line $y = 4$ and then in the x-axis.

(b) Is the answer in **(i)** the same as that in **(ii)**?

3. State the coordinates of the reflection of the point $(3, 2)$ in the line $x = 2$.

4. The reflection of the origin in the line $y = x - 2$ is the point O'. On a sheet of graph paper,

(i) draw the line $y = x - 2$,

(ii) find the coordinates of O'.

5. The point $P(2, 1)$ is transformed by M_1, a reflection in the y-axis and then M_2, a reflection in the line $x = 4$. Give the coordinates of

(i) $M_1(P)$, **(ii)** $M_2(P)$,

(iii) $M_1M_2(P)$, **(iv)** $M_2M_1(P)$.

6. Under a reflection in the line l_1, the point $(3, 5)$ is mapped onto $(5, 3)$.

(i) Find the equation of l_1.

The point $(5, 3)$ is then reflected in the line l_2 and the coordinates of the final image is $(-5, 3)$.

(ii) Find the equation of l_2.

7. The coordinates of the point A and its image A' under a reflection are given below. Plot the points A and A' on a sheet of graph paper, construct the line of reflection and find its equation in each case.

(a) $A(1, 1), A'(3, 1)$ **(b)** $A(1, -1), A'(1, 9)$

(c) $A(2, 1), A'(0, 3)$ **(d)** $A(0, 1), A'(1, 2)$

(e) $A(0, -1), A'(2, 1)$ **(f)** $A(-1, 1), A'(3, -1)$

8. The point $A(3, 4)$ is reflected in the line $x = 2$ and then reflected in the line $y = 1$.

(i) Find the coordinates of the image of A under these two reflections.

(ii) State the coordinates of the point which remains invariant under these two reflections.

9. Find the coordinates of the image of the point $A(2, 3)$ under a reflection in the line $x = 6$ followed by a reflection in the line $y = x$. Show your working on a sheet of graph paper.

10. The image of the origin under a reflection in the line $y = x + 2$ is point A. On a sheet of graph paper,

(i) draw the line $y = x + 2$,

(ii) find the coordinates of A.

11. **(i)** State the coordinates of the final image when

(a) the point $A(1, 4)$ is reflected in the line $y = x$ followed by another reflection in the line $x + y = 6$,

(b) the point $A(1, 4)$ is reflected in the line $x + y = 6$ and then in the line $y = x$.

(ii) Is the answer in **(a)** the same as that in **(b)**?

(iii) State the coordinates of the invariant point under these two reflections.

12. **(i)** State the coordinates of the final image when

(a) the point $A(1, 2)$ is reflected in the line $x + y = 6$ followed by another reflection in the line $x = 4$,

(b) the point $A(1, 2)$ is reflected in the line $x = 4$ and then in the line $x + y = 6$.

(ii) Is the reflection commutative in this case?

(iii) State the coordinates of the invariant point under these two reflections.

10.2 Rotation

A. Rotation about a point

Fig. 10.4 shows the flag $ABCD$ rotated through 90° anticlockwise about the origin. The image of $ABCD$ is $A'B'C'D'$.

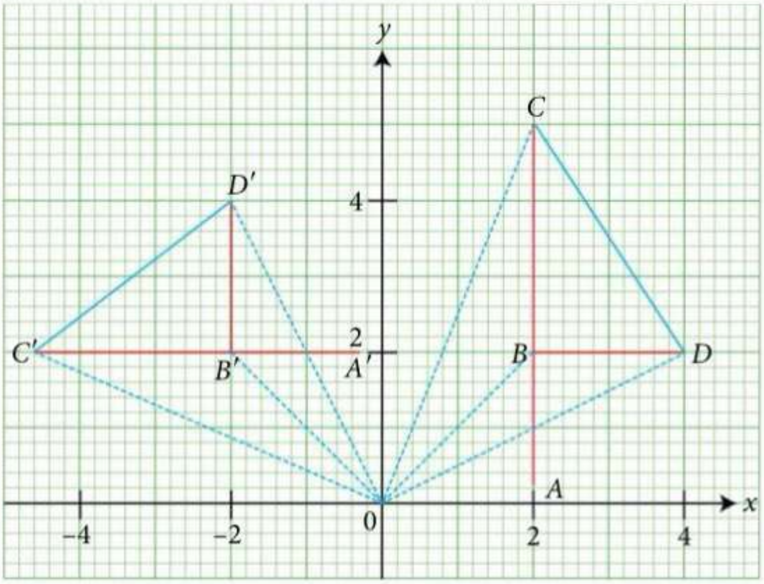


Fig. 10.4

The kite $PQRS$ in Fig. 10.5 is rotated through 180° about $P(0, 2)$. The image is $P'Q'R'S'$. Notice that P is invariant under the rotation.

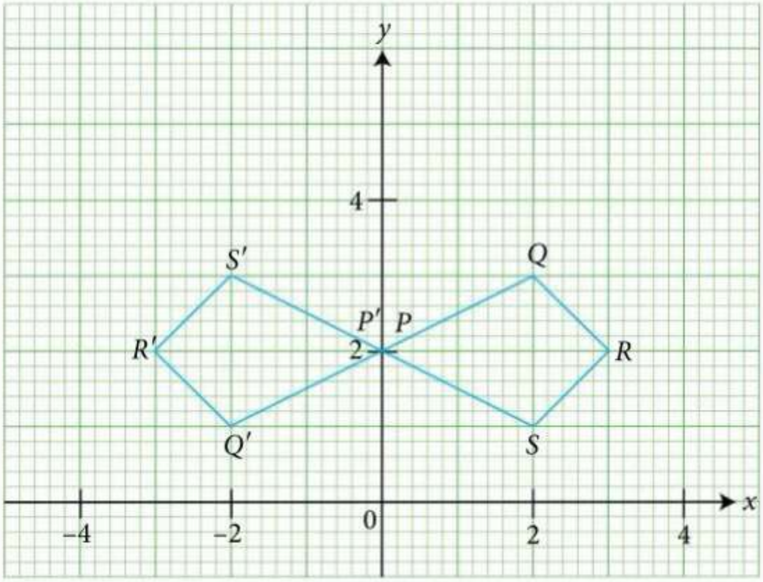


Fig. 10.5

Thinking Time

Can the transformation in Fig. 10.5 also be a reflection? Explain your answer.

From the Fig. 10.4 and Fig. 10.5, we observe the following:

1. A figure can be rotated *clockwise* or *anticlockwise*.
2. A 180° rotation is sometimes referred to as a half turn.
3. In a rotation, every point on the original figure is rotated through the *same angle* about the **centre of rotation**. If the centre of rotation lies on the figure, then it is the *invariant* point.
4. Rotation preserves *orientation* and it is an **isometric transformation**, i.e. the image is of the same *size* and *shape*.
5. We represent rotation with R. If R represents a rotation of 90°, then R^2 represents a rotation in the same direction with twice the angle of rotation i.e. rotation of 180° and R^{-1} represents a rotation in the opposite direction, i.e. rotation of –90°.

Reflection

Does it matter if a rotation is clockwise or anticlockwise? What if the rotation is 180°?

B. Locating the centre and angle of rotation

How do we find the centre and angle of a rotation? Consider Fig. 10.6, which shows $\triangle ABC$ mapped onto $\triangle A'B'C'$ by rotation.

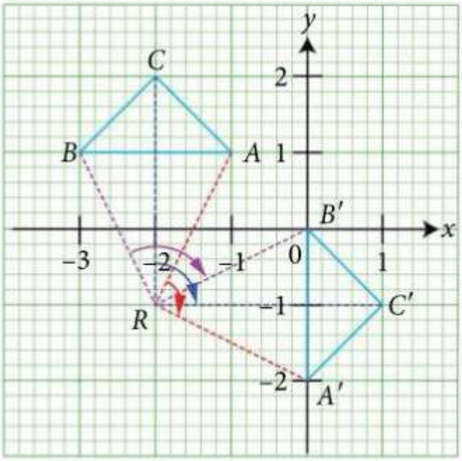
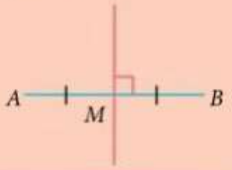


Fig. 10.6

Recall

The perpendicular bisector of a line segment AB is a line that is perpendicular to AB and bisects AB ($AM = MB$).



Before we can find the angle of rotation, we must first find the centre of rotation.
The following steps describe how we can do this.

Step 1: Draw a line connecting a selected point and its corresponding image (e.g. A and A'), and find the midpoint M_1 of AA'.

Step 2: With the aid of the grids, construct the line m_1 passing through M_1 and perpendicular to AA'.

Step 3: Repeat steps 1 and 2 with another point and its image (e.g. B and B'). Let this perpendicular line be m_2.

Step 4: The point of intersection of m_1 and m_2 is the centre of rotation.

Following the steps above, we see that the centre of rotation is at $R(-2, -1)$.

Then, to find the angle of rotation, as indicated by the arrows, we observe each pair of vertices to determine that the angle of rotation is 90° clockwise.

Information

Another way of constructing perpendicular bisectors is with the use of a pair of compasses, which is not in the scope of the syllabus. You can search the internet for more information.

Worked Example 2

Problem involving rotation

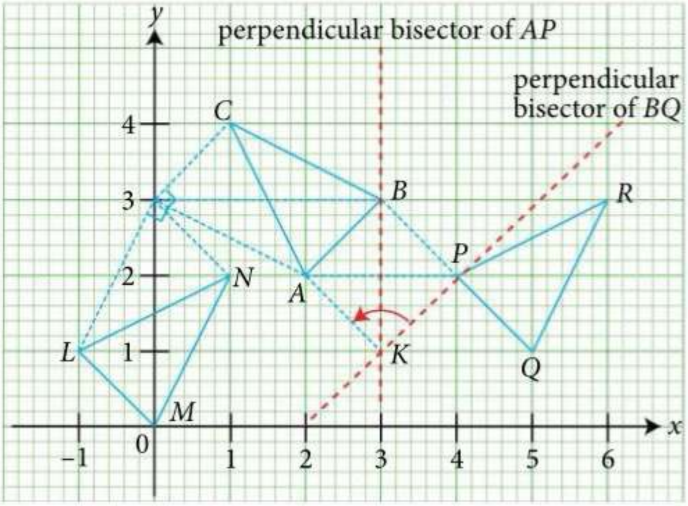
Using a scale of 1 cm to represent 1 unit on both axes, draw △*ABC* and △*PQR* with vertices *A*(2, 2), *B*(3, 3), *C*(1, 4), *P*(4, 2), *Q*(5, 1) and *R*(6, 3).

(a) △*ABC* is mapped onto △*LMN* by a 90° clockwise rotation with centre of rotation at (0, 3). Draw △*LMN* and label the vertices clearly.

(b) △*ABC* is the image of △*PQR* under a rotation. Find the centre of rotation and state the angle of rotation.

*Solution

(a) The vertices of △*LMN* are *L*(–1, 1), *M*(0, 0) and *N*(1, 2).



Problem-solving Tip

From the plot, the line connecting point *P* and its image *A* is parallel to the *x*-axis and has a length of 2 units. Thus, the perpendicular bisector of *AP* is the line *x* = 3. Since the midpoint of *BQ* is *P*, the perpendicular bisector of *BQ* passes through *P*.

(b) From the intersection of the two perpendicular bisectors, the centre of rotation is *K*(3, 1) and the angle of rotation is 90° anticlockwise.

Attention

When stating the angle of rotation, remember to also state if the rotation is anticlockwise or clockwise.

Practise Now 2

Similar and Further Questions

Exercise 10B

Questions 1, 2(a)–(c), 3(a), (b), 4, 5, 6(a)–(c), 7, 8

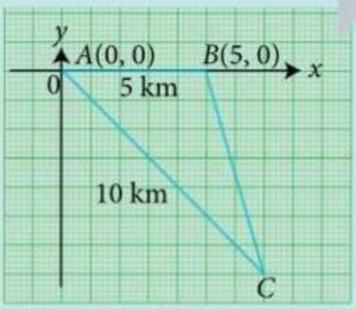
Using a scale of 1 cm to represent 1 unit on both axes, draw △*ABC* and △*PQR* with vertices *A*(1, 6), *B*(4, 5), *C*(1, 4), *P*(7, 6), *Q*(6, 3) and *R*(5, 6).

(a) △*ABC* is mapped onto △*LMN* by a 90° anticlockwise rotation with centre of rotation at (0, 5). Draw △*LMN* and label the vertices clearly.

(b) △*PQR* is the image of △*ABC* under a rotation. Find the centre of rotation and state the angle of rotation.

Introductory Problem Revisited

Did you manage to solve the Introductory Problem? Let us first write the coordinates of A and B as $A(0, 0)$ and $B(5, 0)$ respectively. Can you apply what you have learnt in Sections 10.1 and 10.2 to solve this problem?



Reflection

1. How do I construct the image of a two-dimensional figure under a rotation?
2. Given an object and its image, how do I identify that a transformation is a rotation?

Basic Intermediate Advanced

Exercise 10B

1. The vertices of $\triangle ABC$ are $A(3, 1)$, $B(4, 1)$ and $C(4, 5)$. $\triangle ABC$ is mapped onto $\triangle PQR$ by a 90° clockwise rotation about the point $(2, 1)$.
On a sheet of graph paper, draw and label $\triangle ABC$ and $\triangle PQR$.

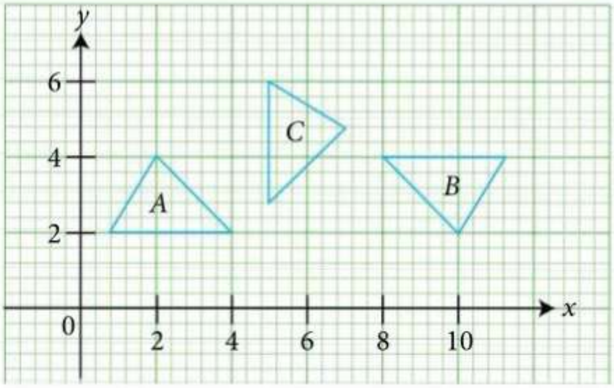
2. Given that P is the point $(2, 4)$, Q is the point $(4, -1)$ and R is the point $(-1, 0)$, find
- **(a)** the image of P under a clockwise rotation of 90° about R,
- **(b)** the image of Q under an anticlockwise rotation of 90° about P,
- **(c)** the image of R under a 180° rotation about Q.

3. Find the coordinates of the image of the point $(1, 4)$ under a clockwise rotation of
- **(a)** 90° about the centre $(4, 2)$,
- **(b)** 180° about the centre $(4, 2)$.

4. If R represents an anticlockwise rotation of 240° about the origin, describe R^2 and R^4.

5. The coordinates of $\triangle ABC$ are $A(4, 1)$, $B(6, 1)$ and $C(4, 6)$ while the coordinates of its image $\triangle A'B'C'$ under a rotation are $A'(0, -1)$, $B'(-2, -1)$ and $C'(0, -6)$. On a sheet of graph paper,
- **(i)** draw $\triangle ABC$ and $\triangle A'B'C'$,
- **(ii)** find the centre of rotation and state the angle of rotation.

6.



- **(a)** Triangle A can be mapped onto triangle B by a rotation. Find
  - **(i)** the coordinates of the centre of rotation,
  - **(ii)** the angle of rotation.

Exercise 10B

(b) Triangle C can be mapped onto triangle A by a rotation. Find

(i) the coordinates of the centre of rotation,

(ii) the angle of rotation.

(c) Triangle B is rotated through 90° clockwise about the point (4, 6). Find the coordinates of the vertices of the image of triangle B.

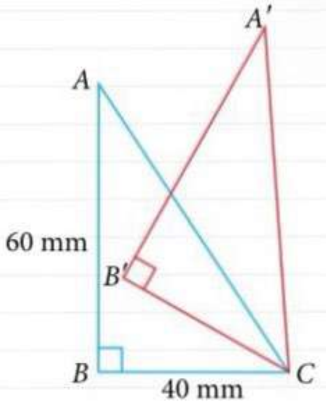
7. Under a rotation, the line $P'Q'$ is the image of the line PQ. Given that their coordinates are $P(1, 1)$, $Q(1, 4)$, $P'(3, 1)$ and $Q'(k, 1)$, where $k > 0$, find

(i) the value of k,

(ii) the image of the point $\left(1, 2\frac{1}{2}\right)$,

(iii) the coordinates of the point whose image is $\left(5\frac{1}{2}, 1\right)$.

8. The triangle $A'B'C$ is the image of the triangle ABC under a clockwise rotation of 25° about C.



Calculate, giving your answer correct to the nearest degree,

(i) $C\hat{A}A'$,

(ii) $A\hat{C}B'$.

10.3 Translation

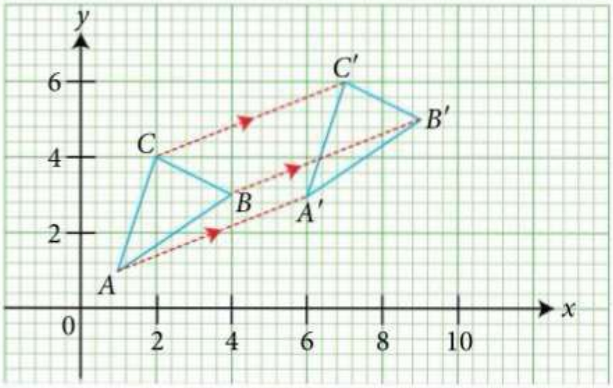


Fig. 10.7

Fig. 10.7 shows $\triangle ABC$ being translated to $\triangle A'B'C'$. A **translation** is an **isometric transformation** and it preserves *orientation*. We represent the translation, T, of point A to A' by writing T(A).

Thinking Time

Can there be invariant points under translation?

When an object moves a units along the x-axis and b units along the y-axis, the translation can be represented b a *column vector* $\begin{pmatrix} a \\ b \end{pmatrix}$. In Fig. 10.7, the *vector equation* representing the translation is $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}$, where $\begin{pmatrix} a \\ b \end{pmatrix}$ is the **translation vector** and $\begin{pmatrix} x' \\ y' \end{pmatrix}$ is the **image** of $\begin{pmatrix} x \\ y \end{pmatrix}$.

For column vectors,

$$\begin{pmatrix} p \\ q \end{pmatrix} + \begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} p+r \\ q+s \end{pmatrix} \text{ and } \begin{pmatrix} p \\ q \end{pmatrix} - \begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} p-r \\ q-s \end{pmatrix}$$

In this section, we will apply what we have learnt about vectors in Chapter 4.

Worked Example 3

Problem involving translation

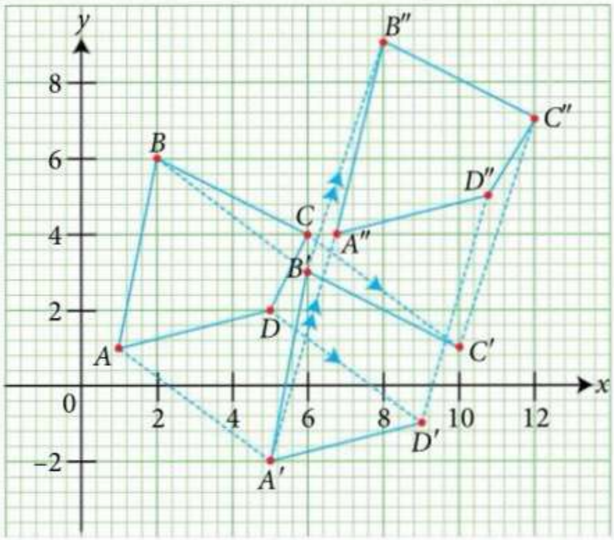
The vertices of a quadrilateral are $A(1, 1)$, $B(2, 6)$, $C(6, 4)$ and $D(5, 2)$.

(a) Find the coordinates of the vertices of the image of the quadrilateral $ABCD$ under a translation T_1 represented by $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$.

(b) Draw the image of the new quadrilateral if it undergoes another translation T_2 represented by $\begin{pmatrix} 2 \\ 6 \end{pmatrix}$. Show your working on a sheet of graph paper.

Solution

The image of the quadrilateral $ABCD$ is obtained as shown below.



(a) $T_1(A) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 4 \\ -3 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$

$T_1(B) = \begin{pmatrix} 2 \\ 6 \end{pmatrix} + \begin{pmatrix} 4 \\ -3 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \end{pmatrix}$

$T_1(C) = \begin{pmatrix} 6 \\ 4 \end{pmatrix} + \begin{pmatrix} 4 \\ -3 \end{pmatrix} = \begin{pmatrix} 10 \\ 1 \end{pmatrix}$

$T_1(D) = \begin{pmatrix} 5 \\ 2 \end{pmatrix} + \begin{pmatrix} 4 \\ -3 \end{pmatrix} = \begin{pmatrix} 9 \\ -1 \end{pmatrix}$

Therefore, the coordinates of the vertices of the image are $A'(5, -2)$, $B'(6, 3)$, $C'(1$ $)$ and $D'(9, -1)$. $A'B'C'D'$ is shown on the figure.

(b) The image of the new quadrilateral $A''B''C''D''$ under T_2 is obtained similarly.

$$T_2(A) = \begin{pmatrix} 5 \\ -2 \end{pmatrix} + \begin{pmatrix} 2 \\ 6 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \end{pmatrix}$$

$$T_2(B) = \begin{pmatrix} 6 \\ 3 \end{pmatrix} + \begin{pmatrix} 2 \\ 6 \end{pmatrix} = \begin{pmatrix} 8 \\ 9 \end{pmatrix}$$

$$T_2(C) = \begin{pmatrix} 10 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 6 \end{pmatrix} = \begin{pmatrix} 12 \\ 7 \end{pmatrix}$$

$$T_2(D) = \begin{pmatrix} 9 \\ -1 \end{pmatrix} + \begin{pmatrix} 2 \\ 6 \end{pmatrix} = \begin{pmatrix} 11 \\ 5 \end{pmatrix}$$

The quadrilateral, with vertices $A''(7, 4)$, $B''(8, 9)$, $C''(12, 7)$ and $D''11, 5)$, are also shown on the figure above.

Practise Now 3

Similar and Further Questions
Exercise 10C
Questions 1, 2

The vertices of a quadrilateral are $A(1, 1)$, $B(2, 4)$, $C(4, 4)$ and $D(5, 1)$.

(a) Find the coordinates of the vertices of the image of the quadrilateral $ABCD$ under a translation T_1 represented by $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$.

(b) Draw the image of the new quadrilateral if it undergoes another translation T_2 represented by $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$. Show your working on a sheet of graph paper.

Thinking Time

(a) Would the result be the same if T_2 is performed first in Worked Example 3?

(b) Can you give a single vector that would produce the same result as the two successive transformations?

Journal Writing

Find out about Escher's tessellations and the transformations he used in his tessellations.

Worked Example 4

Problem involving translation

T_1 is the translation $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and T_2 is a translation that will move the point (1, 2) to (−2, 5).

(i) Find the image of the point $A(7, 4)$ under T_1.

(ii) Find the translation vector represented by T_2.

(iii) What is the image of the point $B(8, -4)$ under T_2?

Solution

(i)
$$\begin{aligned} A' &= T_1(A) \\ &= \begin{pmatrix} 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 7 \\ 4 \end{pmatrix} \\ &= \begin{pmatrix} 9 \\ 7 \end{pmatrix} \end{aligned}$$

$\therefore$ image of A under T_1 is (9, 7).

(ii) Let the translation vector of T_2 be $\begin{pmatrix} a \\ b \end{pmatrix}$.

$$\begin{pmatrix} -2 \\ 5 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\begin{aligned} \begin{pmatrix} a \\ b \end{pmatrix} &= \begin{pmatrix} -2 \\ 5 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} -3 \\ 3 \end{pmatrix} \end{aligned}$$

$\therefore$ the translation vector of T_2 is $\begin{pmatrix} -3 \\ 3 \end{pmatrix}$.

(iii)
$$\begin{aligned} B' &= T_2(B) \\ &= \begin{pmatrix} -3 \\ 3 \end{pmatrix} + \begin{pmatrix} 8 \\ -4 \end{pmatrix} \\ &= \begin{pmatrix} 5 \\ -1 \end{pmatrix} \end{aligned}$$

$\therefore$ the image of B under T_2 is (5, −1).

Practise Now 4

Similar and Further Questions
Exercise 10C
Questions 3, 4(a)–(e), 5

T_1 is the translation $\begin{pmatrix} 4 \\ 5 \end{pmatrix}$ and T_2 is a translation that will move the point (3, 2) to (5, 7).

(i) Find the image of the point $A(2, 4)$ under T_1.

(ii) Find the translation vector represented by T_2.

(iii) What is the image of the point $B(6, -3)$ under T_2?

Reflection

1. How do I construct the image of a two-dimensional figure under a translation?
2. Given an object and its image, how do I identify that a transformation is a translation?

Exercise 10C

1. The vertices of a quadrilateral are $A(2, 1)$, $B(3, 3)$, $C(5, 3)$ and $D(5, 2)$.
   (a) Find the coordinates of the vertices of the image of the quadrilateral $ABCD$ under a translation T_1 represented by $\begin{pmatrix} 5 \\ -4 \end{pmatrix}$.
   (b) Draw the image of the new quadrilateral if it undergoes another translation T_2 represented by $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$.
   Show your working on a sheet of graph paper.

2. The vertices of $\triangle PQR$ are $P(1, 3)$, $Q(7, 5)$ and $R(2, 0)$. Find the coordinates of the vertices of the image of $\triangle PQR$ under a translation T represented by $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$.

3. A translation T maps the point (6, 2) onto the point (2, 7) and the point (–1, –5) onto the point P. Find the column vector representing the translation T and the coordinates of the point P.

4. Under a translation T_1, the image of the point (5, –1) is (2, 3). Under a translation T_2, the image of the point (–2, 5) is (4, –5). Find the image of the point (7, 6) under the following transformations.
   (a) T_1 **(b)** T_2
   (c) T_1T_2 **(d)** T_2T_1
   (e) T_1^2

5. T is the translation $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$, A is the point (2, 4), B is (p, q) and C is (h, k).
   (i) Find the coordinates of the image of the point A under T.
   (ii) Given that $T(B) = A$, find the value of p and of q.
   (iii) Given that $T^2(A) = C$, find the value of h and of k.
   (iv) Find the coordinates of the point D such that $T^2(D) = A$.

10.4 Enlargement

A. Enlargement (Recap)

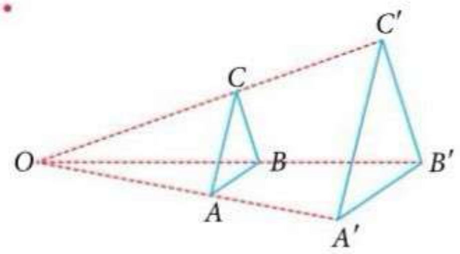


Fig. 10.8

Fig. 10.8 shows two similar triangles ABC and $A'B'C'$. $\triangle A'B'C'$ is an enlargement to $\triangle ABC$. We say that $\triangle ABC$ is transformed onto $\triangle A'B'C'$ by an enlargement with centre O and scale factor $\frac{OA'}{OA}$.

B. Enlargement of a Figure

Fig. 10.9 shows a triangle ABC being enlarged 3 times (scale factor 3) to triangle $A_1B_1C_1$ with origin, O, as the *centre of enlargement* or *point of enlargement*.

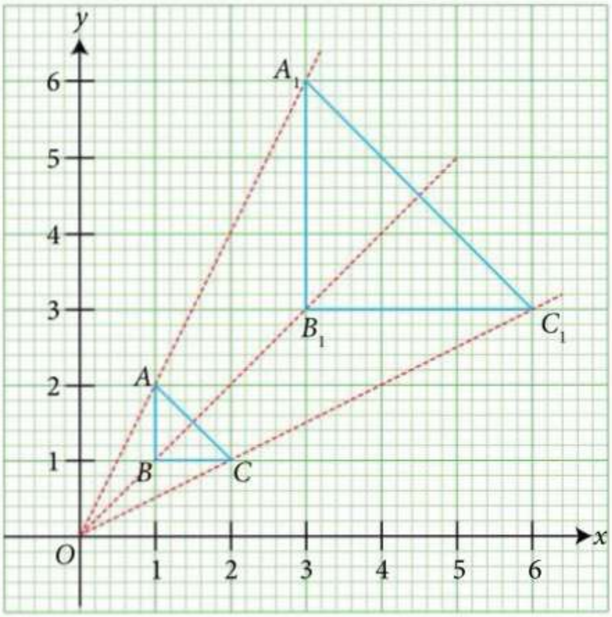


Fig. 10.9

Step 1: Join the point of enlargement O to A and produce OA.
Step 2: From O, mark off a distance equal to 3 times the length of OA on OA produced to get the point A_1.
Step 3: Repeat the above procedure for points B and C to get B_1 and C_1.
Step 4: Join A_1B_1, B_1C_1 and A_1C_1 to get the enlarged figure $A_1B_1C_1$.

How many times larger is the image as compared to the original triangle in Fig. 10.9?

In reflection, rotation and translation, there is no change in the shape and size of the image. They are called **isometric transformations**. In enlargement, there is no change in the shape, angle and orientation of the figure but the size of the image changes. Enlargement is a non-isometric transformation. We learnt in Book 2 that the images formed under enlargement yield similar figures.

Fig. 10.10 shows a quadrilateral $ABCD$ being enlarged to quadrilateral $A'B'C'D'$ with E as the centre of enlargement and scale factor 2.

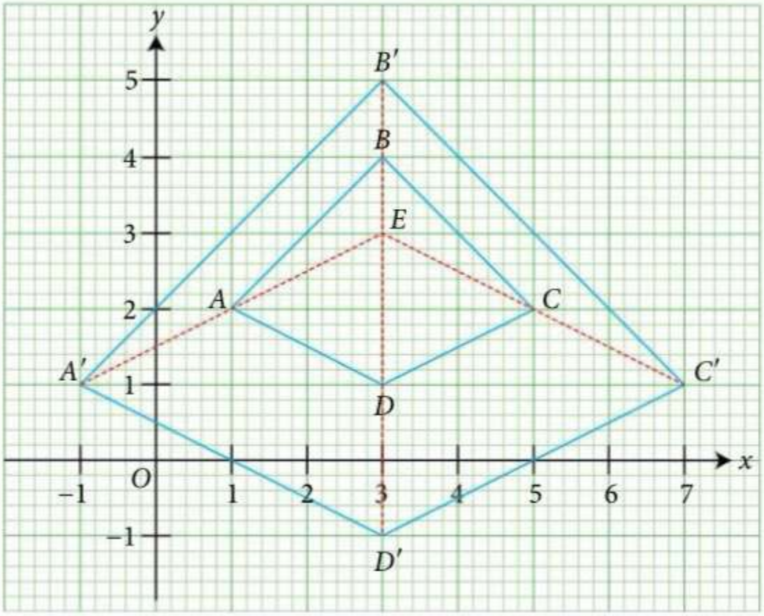


Fig. 10.10

We can also say that quadrilateral $A'B'C'D'$ is being "enlarged" to quadrilateral $ABCD$ with E as the centre of enlargement and scale factor $\frac{1}{2}$, even though the image $ABCD$ is smaller than the original figure $A'B'C'D'$.
The term enlargement in mathematics may thus refer to the enlarging or diminishing of a figure depending on the scale factor involved.

What if the scale factor is negative? Fig. 10.11 shows $\triangle ABC$ being enlarged with a scale factor of −2 and E as the centre of enlargement.

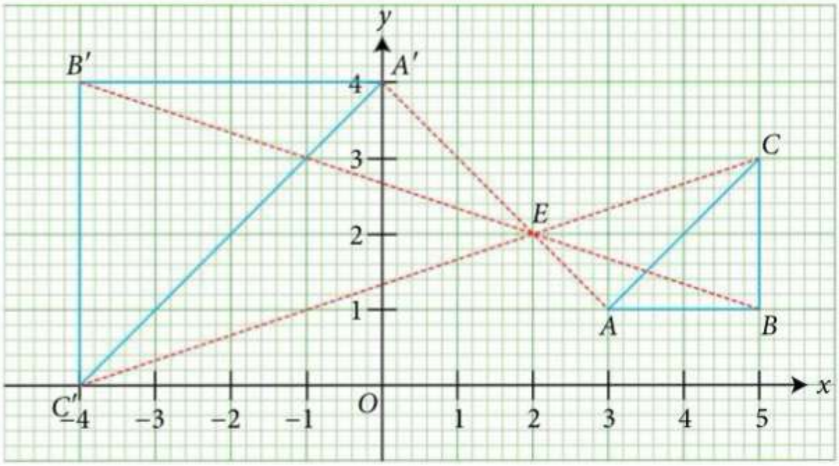


Fig. 10.11

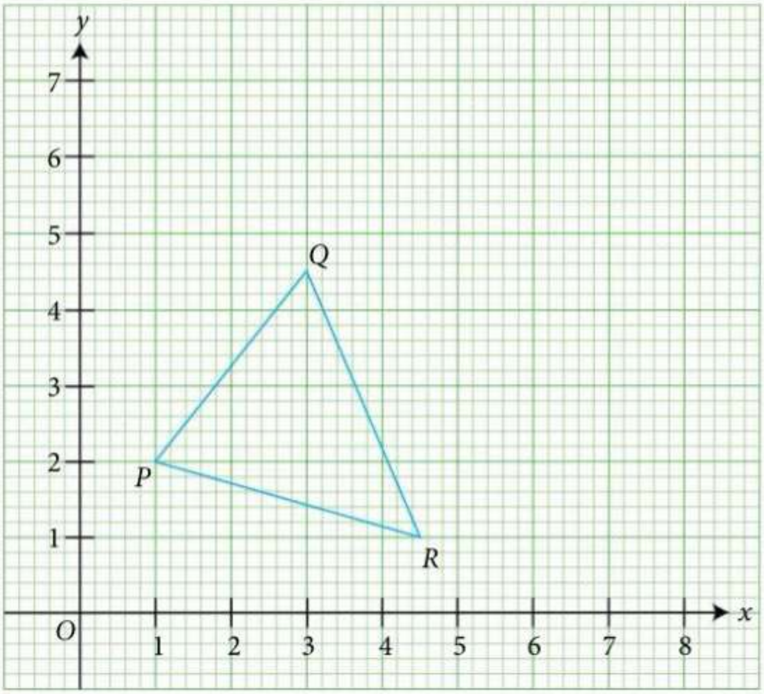
Step 1: Join A to E and produce in the direction of AE.
Step 2: With E as the centre and radius equal to $2AE$, mark off the image A' on AE produced.
Step 3: Repeat **Step 1** and **Step 2** for points B and C to get the images B' and C'.

We have learnt that if the scale factor is greater than 1, the image is enlarged; if it is between 0 and 1, the image diminished. Compare the points of A and A' For a negative scale factor, what can you say about the corresponding points of the image and the original figure?

Worked Example 5

Enlargement of figures with scale factor greater than 1

Enlarge $\triangle PQR$ with P as the point of enlargement and scale factor 2.



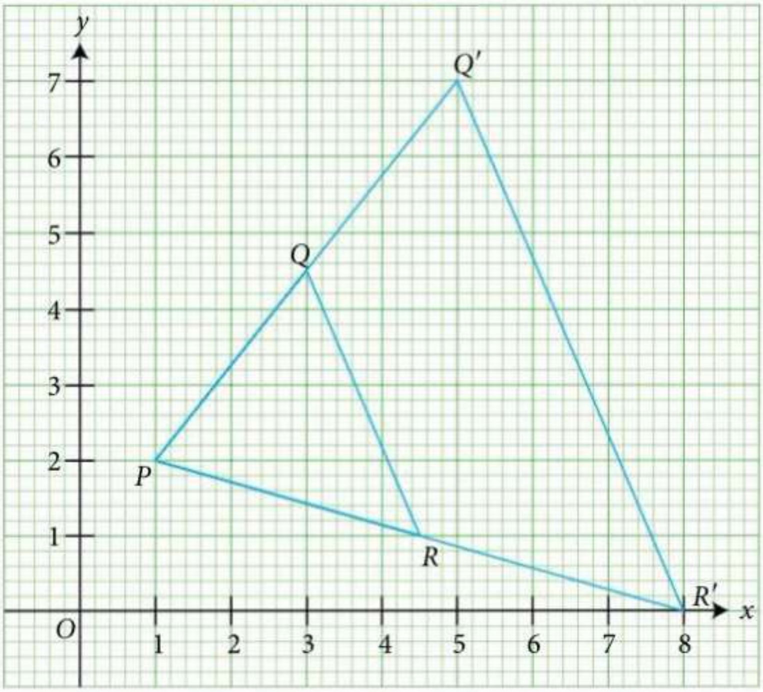
Solution

Step 1: Produce PQ.

Step 2: With P as centre, mark the point Q' on PQ produced such that $PQ' = 2PQ$.

Step 3: Repeat the procedure for the point R to get R'.

The diagram below shows the enlarged triangle $PQ'R'$.



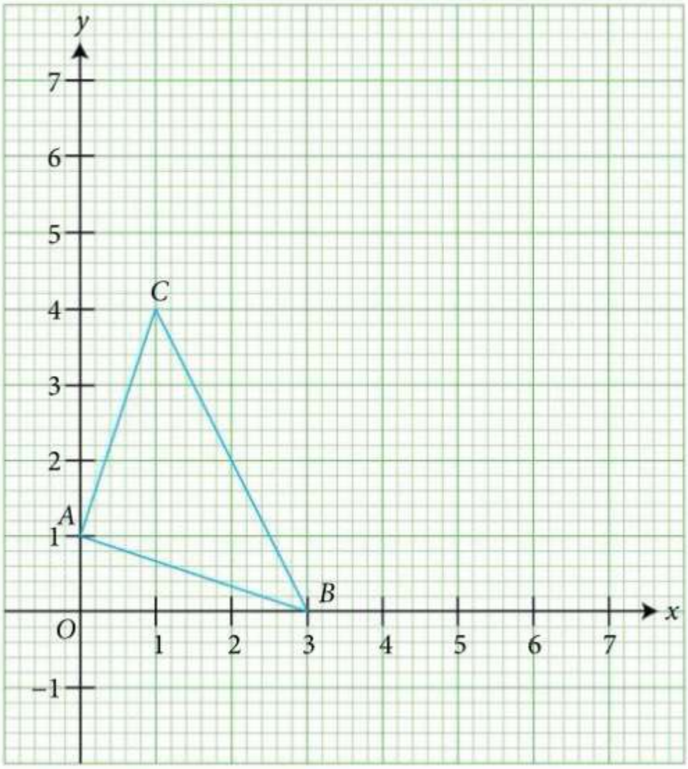
Attention

Notice that all points in the triangle have been transformed except point P, which is the only **invariant** point.

Practise Now 5

Similar and Further Questions
Exercise 10D
Questions 1–3, 4(a)–(c), 5, 6(a)–(d), 10(a), 13

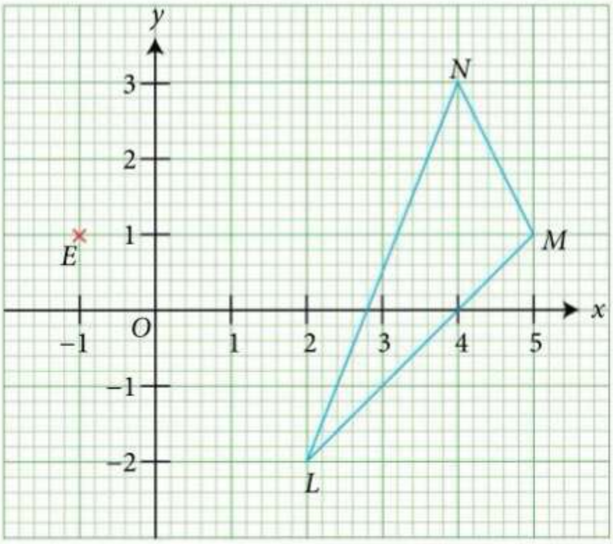
Enlarge $\triangle ABC$ with A as the point of enlargement and scale factor 2.



Worked Example 6

Enlargement of figures with scale factor between 0 and 1

Enlarge $\triangle LMN$ with E as the point of enlargement and scale factor $\frac{1}{2}$.



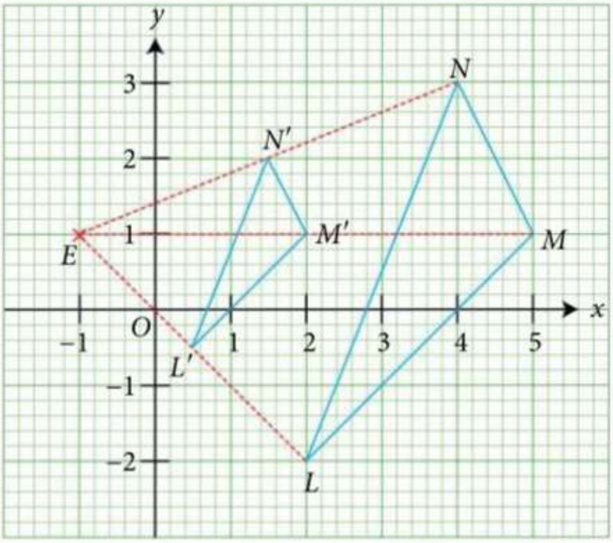
Solution

Step 1: Join E to L.

Step 2: With E as the centre, mark off the point L' on EL produced such that $EL' = \frac{1}{2}EL$.

Step 3: Repeat the above procedure for points M and N to get points M' and N'.

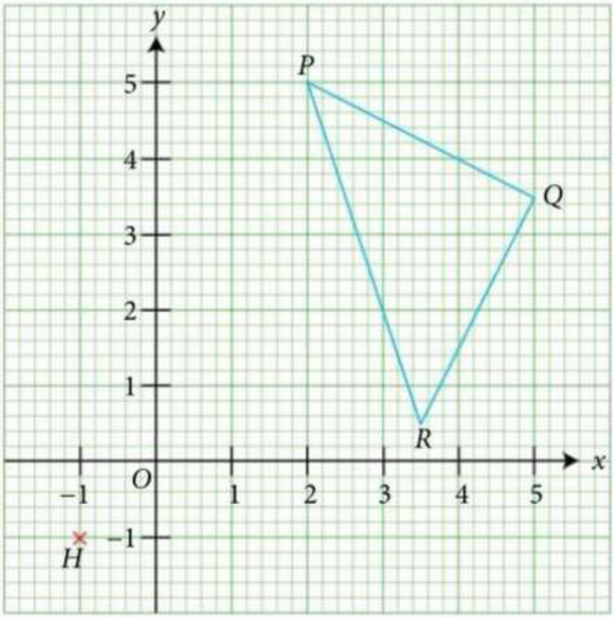
Step 4: Join $L'M'$, $M'N'$ and $L'N'$ to obtain the image triangle $L'M'N'$.



Practise Now 6

Similar and Further Questions
Exercise 10D
Questions 4(d), 6(e), 7, 8, 10(b), 14, 15

Enlarge $\triangle PQR$ with H as the point of enlargement and scale factor $\frac{1}{3}$.



Class Discussion

Enlargement in our surroundings

Explore your surroundings to find examples of enlargement being put into use. Discuss your observations in class.

C. Centre of enlargement and scale factor

Fig. 10.12 shows how to find the centre of enlargement and the scale factor, with *ABCD* as the original figure and $A_1B_1C_1D_1$ as the image.

Step 1: Join any two corresponding points from the original figure and the image (e.g. A_1A and B_1B).

Step 2: Extend A_1A and B_1B.

Step 3: The point of intersection of these lines yields the centre of enlargement *E*.

Step 4: The scale factor, $k = \frac{A_1B_1}{AB} = \frac{B_1C_1}{BC} = \frac{C_1D_1}{CD} = \frac{D_1A_1}{DA}$.

Alternatively, $k = \frac{A_1E}{AE} = \frac{B_1E}{BE} = \frac{C_1E}{CE} = \frac{D_1E}{DE}$.

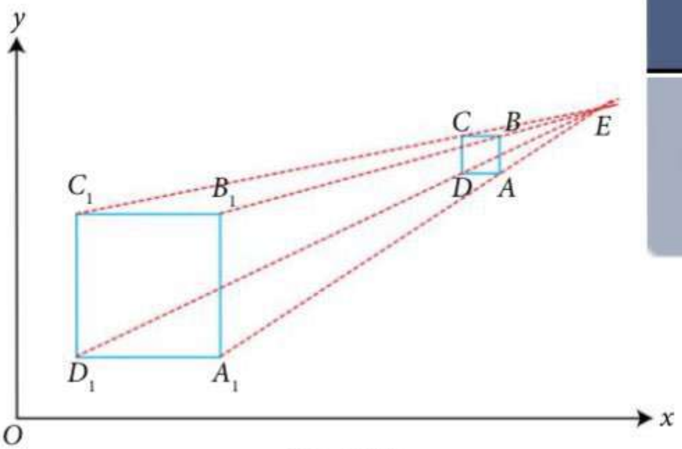


Fig. 10.12

Worked Example 7

Finding centre and scale factor of enlargement

△*PQR* has vertices *P*(1, 2), *Q*(5, 1) and *R*(4, 4). An enlargement maps △*PQR* onto △*PAB*. Given that the coordinates of *A* are (*m*, 0), draw △*PQR* on a sheet of graph paper and find

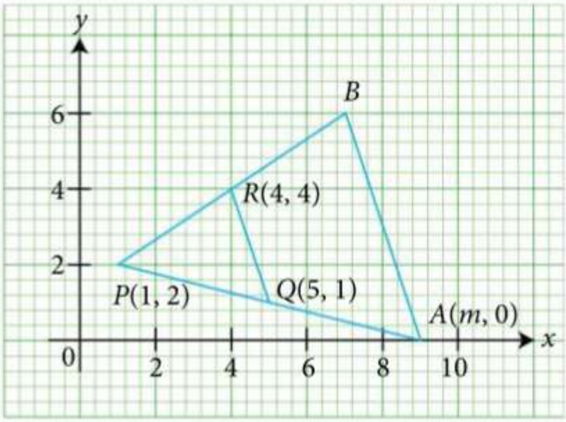
(a) the centre of enlargement,

(b) the value of *m*,

(c) the scale factor of the enlargement,

(d) the coordinates of the point *B*.

Solution



(a) From the graph, we see that *P* is the invariant point and hence, the centre of enlargement is (1, 2).

(b) *PQ* is produced to meet *A* on the *x*-axis. *A* is the point (9, 0).

$\therefore m = 9$

(c) *PQ* = *QA*.

$\therefore$ scale factor $= \frac{PA}{PQ} = \frac{2}{1} = 2$

(d) Produce *PR*. *B* is the point on *PR* produced such that *PB* = 2*PR*.

The coordinates of *B* are (7, 6).

Practise Now 7

Similar and Further Questions
Exercise 10D
Questions 9, 11(a)–(d), 12(a)–(f), 16

The vertices of $\triangle ABC$ are $A(1, 1)$, $B(1, -1)$ and $C(2, 2)$. Under an enlargement, $\triangle ABC$ is mapped onto $\triangle PQR$ whose vertices have coordinates $P(3, 2)$, $Q(3, -2)$ and $R(5, 4)$. Draw these two triangles on a sheet of graph paper and find

(a) the coordinates of the centre of enlargement,
(b) the scale factor.

Reflection

1. How do I identify from the scale factor of an enlargement, if the image is enlarged or diminished?
2. When the scale factor of an enlargement is negative, where does the image lie with respect to the object?

Basic **Intermediate** **Advanced**

Exercise 10D

1. On a sheet of graph paper, draw $\triangle ABC$ with vertices $A(2, 1)$, $B(2, 5)$ and $C(4, 2)$. Enlarge $\triangle ABC$ with the origin as the point of enlargement and scale factor 2.

2. On a sheet of graph paper, draw the $\triangle PQR$ with vertices $P(2, 2)$, $Q(5, 3)$ and $R(3, 5)$. Enlarge $\triangle PQR$ with Q as the point of enlargement and scale factor 3.

3. On a sheet of graph paper, draw the quadrilateral $PQRS$ with vertices $P(2, 2)$, $Q(7, 2)$, $R(6, 6)$ and $S(4, 6)$. Enlarge $PQRS$ with $E(4, 4)$ as the centre of enlargement and scale factor $1\frac{1}{2}$.

4. The coordinates of the quadrilateral $ABCD$ are $A(2, 3)$, $B(6, 2)$, $C(10, 5)$ and $D(8, 8)$. Find the image of the point

(a) A under an enlargement with centre at $(0, 2)$ and scale factor 2,
(b) B under an enlargement with centre at $(4, 0)$ and scale factor 3,
(c) C under an enlargement with centre at $(8, 4)$ and scale factor -2,
(d) D under an enlargement with centre at $(1, 2)$ and scale factor $\frac{1}{2}$.

5. $\triangle ABC$ with vertices $A(2, 1)$, $B(4, 3)$ and $C(3, 6)$ is transformed into $\triangle A'B'C'$ under an enlargement, with centre $(1, 1)$ and scale factor 3. Illustrate these points on a clearly-labelled diagram, marking the positions of $\triangle ABC$ and $\triangle A'B'C'$.

6. Enlarge the following triangles with the centre of enlargement E and scale factor k as given.

(a) $A(1, 3)$, $B(2, 5)$ and $C(6, 1)$; $E(0, 0)$; $k = 2$
(b) $P(1, 4)$, $Q(4, 1)$ and $R(5, 6)$; $E(1, 2)$; $k = -2$
(c) $X(1, 1)$, $Y(2, 3)$ and $Z(4, 2)$; $E(1, 1)$; $k = 3$
(d) $L(4, 1)$, $M(4, 3)$ and $N(1, 3)$; $E(1, 0)$; $k = -3$
(e) $J(4, 4)$, $H(6, 7)$ and $K(3, 9)$; $E(8, 4)$; $k = \frac{1}{2}$

7. The vertices of $\triangle ABC$ are $A(1, 1)$, $B(3, -1)$ and $C(0, 0)$. $\triangle ABC$ is enlarged to $\triangle PQR$ with $E(4, 4)$ as the centre of enlargement and scale factor $\frac{1}{2}$. Find the coordinates of P, Q and R.

Exercise 10D

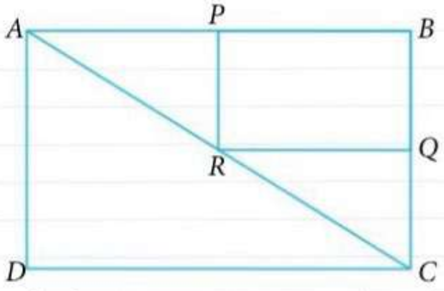
8. The vertices of $\triangle PQR$ are $P(1, 2)$, $Q(2, 6)$ and $R(8, 1)$. $\triangle PQR$ is mapped onto $\triangle LMN$ by an enlargement with centre at $E(2, 4)$ and scale factor $\frac{1}{2}$. Find the coordinates of L, M and N.

9. The coordinates of $\triangle ABC$ are $A(2, 1)$, $B(7, 1)$ and $C(4, 4)$. $\triangle ABC$ is mapped onto $\triangle APQ$ by an enlargement scale factor 2.

(i) State the centre of enlargement.

(ii) Find the coordinates of P and Q.

10.



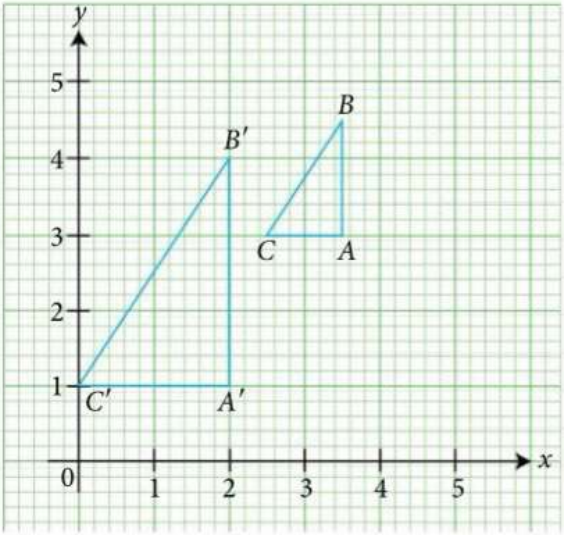
In the figure, $ABCD$ is a rectangle and P and Q are the midpoints of AB and BC respectively.

(a) $\triangle APR$ is mapped by an enlargement with centre at A and scale factor 2. Name the image figure.

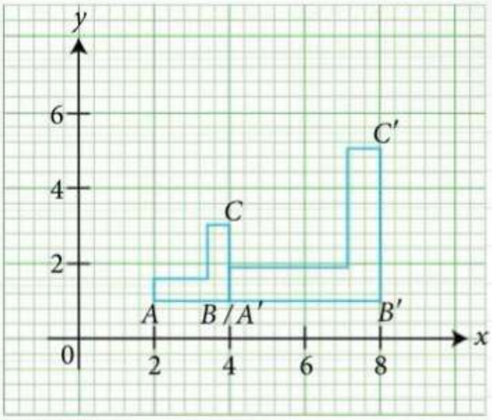
(b) $ABCD$ is mapped by an enlargement with centre at B and scale factor $\frac{1}{2}$. Name the image figure.

11. Find the centre of enlargement and scale factor for each of the following enlargements which map ABC onto $A'B'C'$.

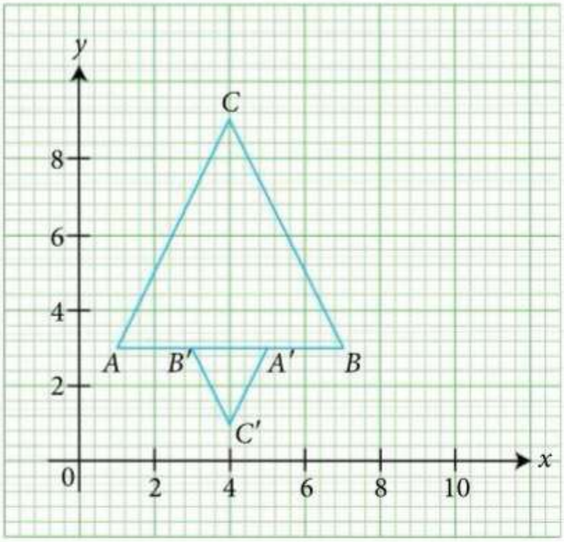
(a)



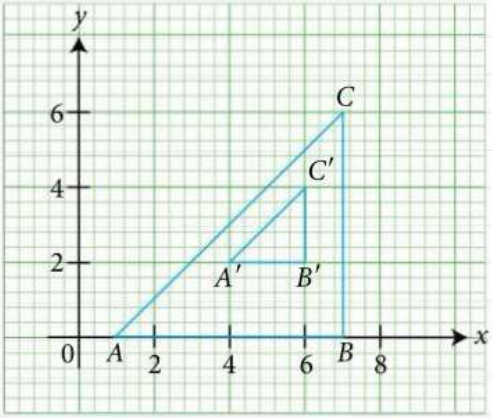
(b)



(c)



(d)



Exercise 10D

12. In each of the following, $\triangle ABC$ is mapped onto $\triangle PQR$ by an enlargement. Draw the object and image, and find the centre of enlargement and the scale factor in each case.

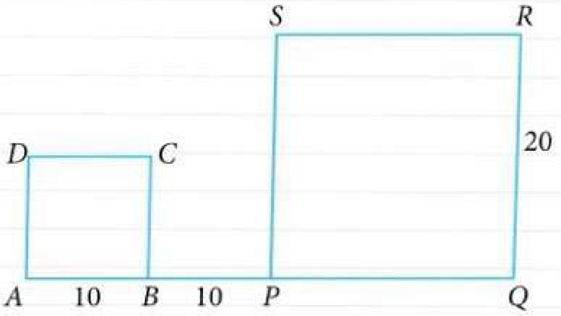
	Object	Image
(a)	$A(4, 4)$, $B(7, 4)$, $C(4, 6)$	$P(1, 2)$, $Q(7, 2)$, $R(1, 6)$
(b)	$A(3, 3)$, $B(5, 1)$, $C(2, 1)$	$P(5, 7)$, $Q(11, 1)$, $R(2, 1)$
(c)	$A(10, 4)$, $B(10, 8)$, $C(6, 8)$	$P(13, 3)$, $Q(13, 9)$, $R(7, 9)$
(d)	$A(7, 3)$, $B(13, 3)$, $C(13, 0)$	$P(3, 7)$, $Q(1, 7)$, $R(1, 8)$
(e)	$A(2, 4)$, $B(5, 3)$, $C(4, 6)$	$P(1, 6)$, $Q(7, 4)$, $R(5, 10)$
(f)	$A(3, 1)$, $B(6, 3)$, $C(1, 4)$	$P(6, 13)$, $Q(0, 9)$, $R(10, 7)$

13. Under an enlargement with centre (4, 3) and scale factor –2, the line AB with coordinates $A(9, 2)$ and $B(6, 6)$ is mapped onto the line PQ. Find
(i) the coordinates of P and Q,
(ii) the length of PQ.

14. $\triangle ABC$ is enlarged onto $\triangle A'B'C'$ with the origin as the centre of enlargement and scale factor –2. If the coordinates of $\triangle A'B'C'$ are $A'(-2, -2)$, $B'(-10, -4)$ and $C'(-4, -6)$, find the coordinates of $\triangle ABC$.

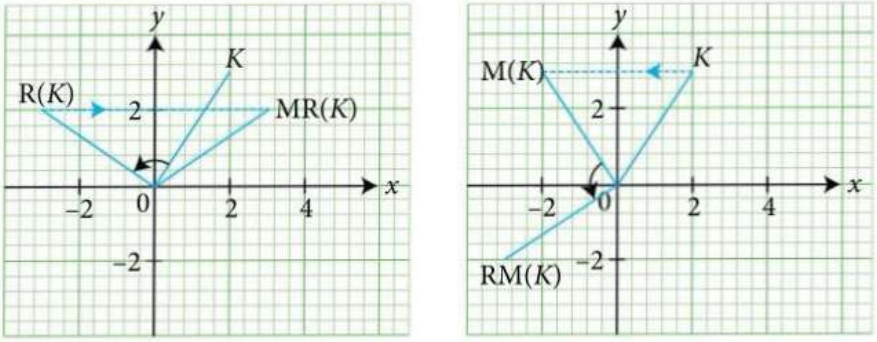
15. $\triangle ABC$ is enlarged onto $\triangle A'B'C'$ with B as the centre of enlargement and scale factor –2. If the coordinates of $\triangle A'B'C'$ are $A'(7, 7)$, $B'(3, 3)$ and $C'(9, 3)$, and B is the point (3, 3), find the coordinates of A and C.

16. In the figure, $ABPQ$ is a straight line. The square $PQRS$ is the image of $ABCD$ under an enlargement. Given that $AB = 10$ cm, $BP = 10$ cm and $RQ = 20$ cm, find the distance of the centre of enlargement from the point A.



10.5 Combined transformations

Now, let us look at how transformations can be combined.
If M represents a reflection in the y-axis and R represents a 90° anticlockwise rotation about the origin, then MR represents a 90° anticlockwise rotation about the origin followed by a reflection in the y-axis, while RM denotes a reflection in the y-axis followed by a 90° anticlockwise rotation about the origin.
Consider the point $K(2, 3)$. Under MR, K will be mapped onto (–3, 2) under R and then onto (3, 2) under M, i.e. MR(2, 3) = (3, 2) [see Fig. 10.13(**a**)].
Under RM, $K(2, 3)$ will be mapped onto (–2, 3) under M and then onto (–3, –2) under R, i.e. RM(2, 3) = (–3, –2) [see Fig. 10.13(**b**)].



(a) (b)

Fig. 10.13

From Fig. 10.13, we observe that MR ≠ RM.
In general, the combination of two transformations is *non-commutative*.

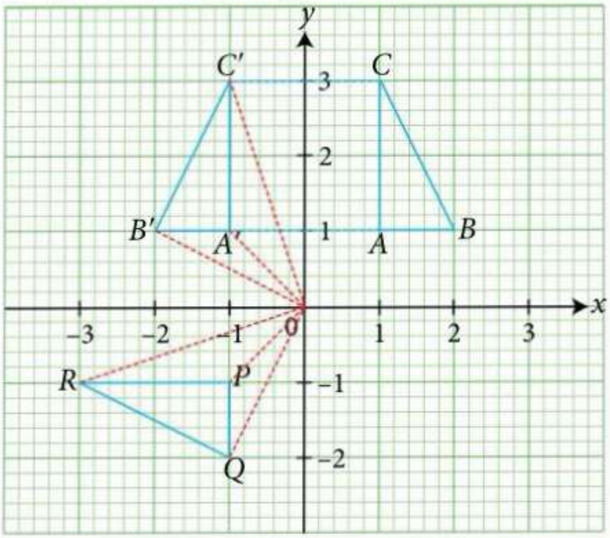
Worked Example 8

Finding image points under combined transformations

On a sheet of graph paper, draw △*ABC* with *A*(1, 1), *B*(2, 1) and *C*(1, 3). △*ABC* is reflected in the *y*-axis followed by a rotation of 90° anticlockwise about the origin to obtain △*PQR*. Draw △*PQR* on the same axes.

***Solution**

We observe that △*ABC* is reflected in the *y*-axis to △*A′B′C′*. △*A′B′C′* is then rotated through 90° anticlockwise about the origin to obtain △*PQR* whose coordinates are *P*(−1, −1), *Q*(−1, −2) and *R*(−3, −1).



Thinking Time

Would you obtain the same result if △*ABC* is first rotated through 90° anticlockwise about *O* and then reflected in the *y*-axis? Show your working on a sheet of graph paper.

Practise Now 8

Similar and Further Questions
Exercise 10E
Questions 1(a), (b), 2, 3(a), (b), 4, 5(a), (b)

Using a scale of 1 cm to represent 1 unit on each axis, draw *x*- and *y*-axes for $-5 \leqslant x \leqslant 5$ and $-5 \leqslant y \leqslant 5$.

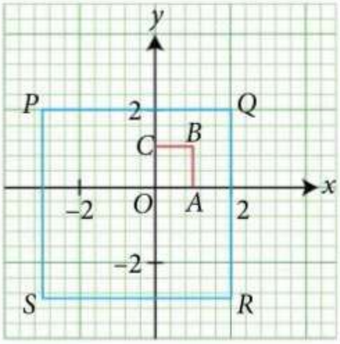
(a) The vertices of △*ABC* are *A*(1, 2), *B*(5, 1) and *C*(3, 4). Draw and label △*ABC*.

(b) △*ABC* undergoes a double transformation: a reflection in the *x*-axis (M), followed by a translation (T) of 5 units in the negative *x*-direction and 5 units in the positive *y*-direction. Plot the image of △*ABC* under TM.

Worked Example 9

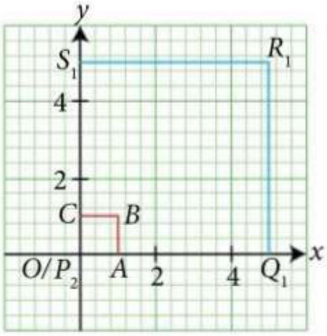
Finding image points under successive transformations

Sketch and describe three successive transformations under which *OABC* will be mapped onto *PQRS*.

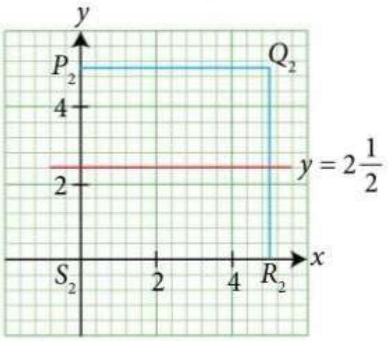


Solution

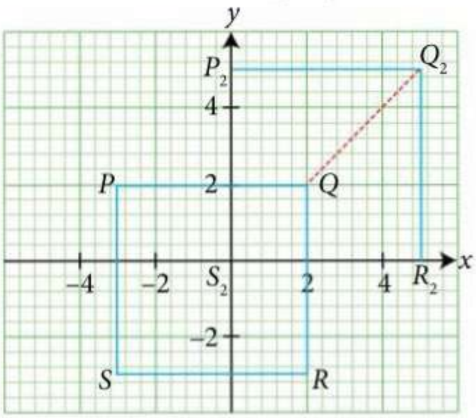
Step 1: *OABC* is mapped onto $P_1Q_1R_1S_1$ under an enlargement with centre *O*, scale factor 5.



Step 2: $P_1Q_1R_1S_1$ is reflected in the line $y = 2\frac{1}{2}$ to $P_2Q_2R_2S_2$.



Step 3: $P_2Q_2R_2S_2$ is then translated to *PQRS* by $\begin{pmatrix} -3 \\ -3 \end{pmatrix}$.



Practise Now 9

Similar and Further Questions
Exercise 10E
Questions 6, 7

The coordinates of $\triangle ABC$ are $A(7, 4)$, $B(7, -0)$ and $C(5, 4)$. The coordinates of $\triangle PQR$ are $P(10, 4)$, $Q(10, 10)$ and $R(7, 4)$. Draw $\triangle ABC$ and $\triangle PQR$ on a sheet of graph paper and describe two successive transformations that will map $\triangle ABC$ onto $\triangle PQR$.

Worked Example 10

Finding image points under successive transformations

Using a scale of 1 cm to 1 unit on each axis, draw the x- and y-axes for $-3 \leqslant x \leqslant 6$ and $-5 \leqslant y \leqslant 5$.

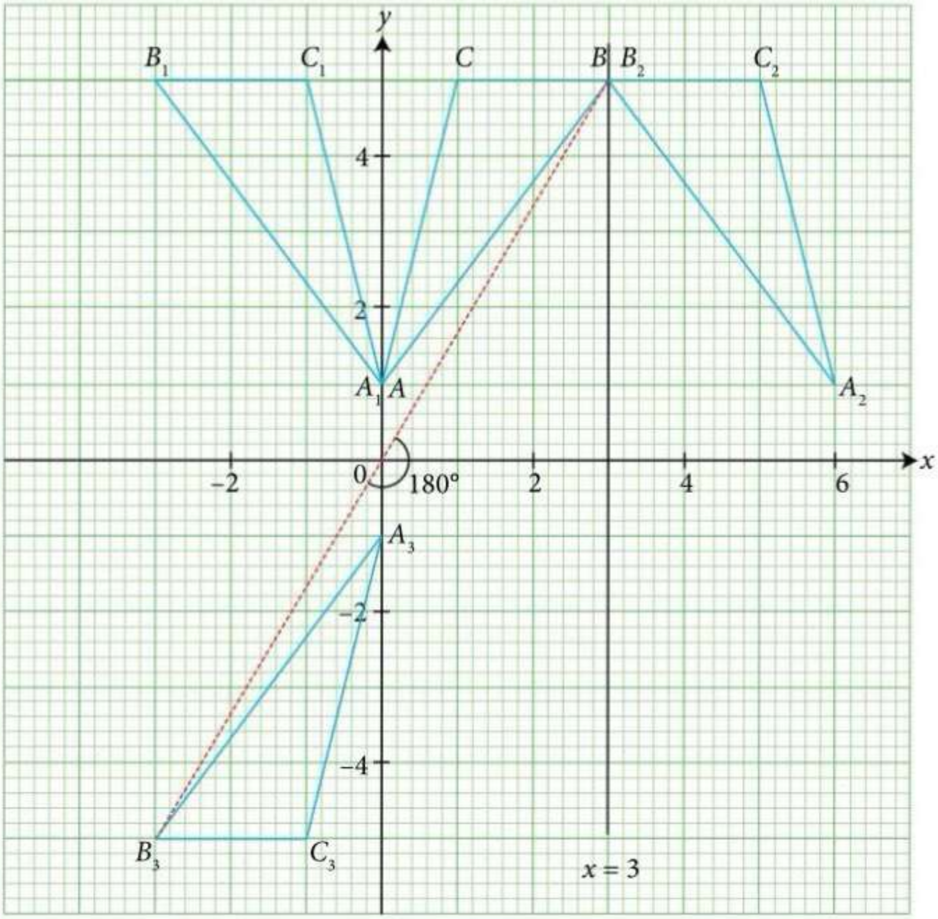
(a) The vertices of $\triangle ABC$ are $A(0, 1)$, $B(3, 5)$ and $C(1, 5)$. Draw and label $\triangle ABC$.

(b) $\triangle ABC$ is mapped onto $\triangle A_1B_1C_1$ by a reflection in the y-axis. $\triangle A_1B_1C_1$ is then mapped to $\triangle A_2B_2C_2$ by a translation T represented by $\begin{pmatrix} 6 \\ 0 \end{pmatrix}$. Draw and label $\triangle A_1B_1C_1$ and $\triangle A_2B_2C_2$. State the single transformation that will map $\triangle ABC$ directly onto $\triangle A_2B_2C_2$.

(c) $\triangle ABC$ is mapped onto $\triangle A_3B_3C_3$ by an enlargement with scale factor -1 and centre at origin. Draw and label $\triangle A_3B_3C_3$. Describe another single transformation that maps $\triangle ABC$ onto $\triangle A_3B_3C_3$.

***Solution**

(a)



(b) The transformation that will map $\triangle ABC$ onto $\triangle A_2B_2C_2$ is a reflection in the line $x = 3$.

(c) The transformation represents a rotation of 180° about the origin.

Practise Now 10

Similar and Further Questions

Exercise 10E

Questions 8(a), (b), 9(a)–(c), 10, 11, 12(a), (b), 13-16

Using a scale of 1 cm to represent 1 unit on each axis, draw x- and y-axes for $-5 \leqslant x \leqslant 5$ and $-5 \leqslant y \leqslant 5$.

(a) The vertices of $\triangle ABC$ are $A(1, 2)$, $B(4, 1)$ and $C(3, 4)$. Draw and label $\triangle ABC$.

(b) $\triangle ABC$ undergoes a double transformation: a reflection in the y-axis (M), followed by an anticlockwise rotation of 90° about the origin (R). Draw the image of $\triangle ABC$ under

(i) M, **(ii)** RM.

(c) Describe a single transformation that will map $\triangle ABC$ onto $\triangle A''B''C''$, where $\triangle A''B''C''$ is the image of $\triangle ABC$ under RM.

Reflection

1. If an object A undergoes two combined transformations involving translation, reflection and/or rotation to image B, is it always possible to find a single transformation that maps the object A directly onto the image B?
2. What have I learnt in this section or chapter that I am unclear of?

Basic **Intermediate** **Advanced**

Exercise 10E

1. If R represents a reflection in the y-axis and T a translation of 2 units in the positive x-direction and 4 units in the positive y-direction, find the image of the point (1, 3) under the combined transformation represented by

(a) RT, **(b)** TR.

2. Find the coordinates of the image of the point (5, 2) under a reflection in the x-axis followed by a clockwise rotation of 90° about the origin.

3. Points are reflected in the y-axis and their images are rotated through 90° anticlockwise about O. Find the coordinates of the final image of the point

(a) (2, −3), **(b)** (−4, −1).

4. Find the coordinates of the image of (7, −2) under a translation of 4 units in the negative x-direction and 3 units in the positive y-direction followed by a 180° rotation about the origin.

5. E is an enlargement with centre (0, 0) and scale factor 2. T is a translation represented by 2 units in the positive x-direction and 1 unit in the positive y-direction. Find the final image of the point (2, 1) under

(a) ET, **(b)** T^2.

6. **(i)** Under a reflection in the line $y = 3$, the point $A(5, 1)$ is mapped onto A_1. Find the coordinates of A_1.

(ii) A reflection in the line $y = 8$ will map the point A_1 onto the point A_2. Find the coordinates of A_2.

(iii) Given that A_2 is the reflection of A in the line $y = k$, find the value of k.

7. A reflection in the line $y = 0$ will map the point P onto P_1 and a reflection in the line $x = 0$ will map the point P_1 onto P_2. Describe a single transformation that will map P onto P_2.

Exercise 10E

8. The coordinates of $\triangle ABC$ are $A(2, 2)$, $B(5, 2)$ and $C(3, 4)$. $\triangle ABC$ is mapped onto $\triangle PQR$ by means of the following successive transformations.

(a) A clockwise rotation of 90° about (0, 0).

(b) A reflection in the line $x = 0$.

Draw $\triangle ABC$ and $\triangle PQR$ on a sheet of graph paper and describe a single transformation that will map $\triangle ABC$ onto $\triangle PQR$.

9. The coordinates of $\triangle ABC$ are $A(4, 0)$, $B(5, 0)$ and $C(5, 2)$. $\triangle ABC$ is mapped onto $\triangle PQR$ by means of the following successive transformations.

(a) An enlargement scale factor 2, centre at (4, 0).

(b) A reflection in the line $x = 4$.

(c) A 90° clockwise rotation about the point (1, 1).

Draw $\triangle ABC$ and $\triangle PQR$ on a sheet of graph paper and label the vertices clearly.

10. The transformation R is a 90° clockwise rotation about (0, 2) and the translation T is given by the column vector $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$. If A is the point (4, 5), find the coordinates of R(A) and TR(A).

11. A is the point (5, 1) and R is a transformation which gives a 90° anticlockwise rotation about the origin. T is a translation represented by the column vector $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$. If RT($A$) = B, TR(A) = C and R^2(A) = D, find the coordinates of B, C and D.

12. The coordinates of $\triangle ABC$ are $A(1, -1)$, $B(1, 0)$ and $C(3, -1)$. $\triangle ABC$ is transformed into $\triangle A_1B_1C_1$ by the following successive transformations.

(a) A reflection in the x-axis.

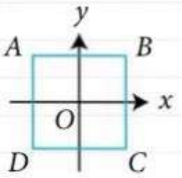
(b) An enlargement, centre origin, scale factor 3.

Draw $\triangle ABC$ on a sheet of graph paper and construct $\triangle A_1B_1C_1$ on the same graph. State the coordinates of A_1.

13. R represents a 90° anticlockwise rotation about O and M represents a reflection in the x-axis.

(i) On a copy of the diagram, sketch and label the image of the square $ABCD$ under the transformation MR.

(ii) State a single transformation which is equivalent to MR.



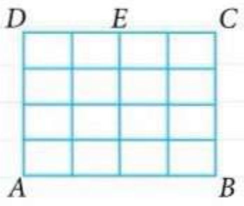
14. (i) $\triangle ABC$ with vertices $A(1, 1)$, $B(2, 3)$ and $C(-1, 4)$ is first reflected in the x-axis and then rotated through 90° anticlockwise about the origin. Find the new coordinates of A, B and C.

(ii) A square $PQRS$ undergoes the same transformations in part (i). Given that the coordinates are $P(0, 5)$, $Q(3, 5)$, $R(3, 8)$ and $S(0, 8)$, find the new coordinates of P, Q, R and S.

15. (i) A square $ABCD$ with vertices $A(2, 4)$, $B(4, 4)$, $C(4, 6)$ and $D(2, 6)$ is enlarged with centre $P(0, 5)$ and scale factor 3, to $A_1B_1C_1D_1$. Construct $A_1B_1C_1D_1$ and state the coordinates of C_1.

(ii) Taking A as the centre of enlargement and scale factor −3, construct $A_2B_2C_2D_2$, the square formed under this transformation and state the coordinates of C_2.

16. The rectangle $ABCD$ is divided into 16 equal rectangles. The point P is such that the area of $\triangle APB$ is equal to one quarter of the area of rectangle $ABCD$, and the point Q, lying on AB, is such that $\triangle AEB$ is an enlargement of $\triangle APQ$.



Mark P and Q clearly on a copy of the diagram.

Looking Back

In this chapter, we extend our study of congruent figures to three isometric transformations: reflection, rotation and translation. Furthermore, we have learnt to construct images under an enlargement, which is a non-isometric transformation. These transformations may involve points which are **invariant**.

Knowledge of such transformations has led to several real-life applications in areas such as computer animation and photocopying. While we have learnt to use **diagrams** to perform these transformations, you might learn to do so with the use of **functions** or matrices in future.

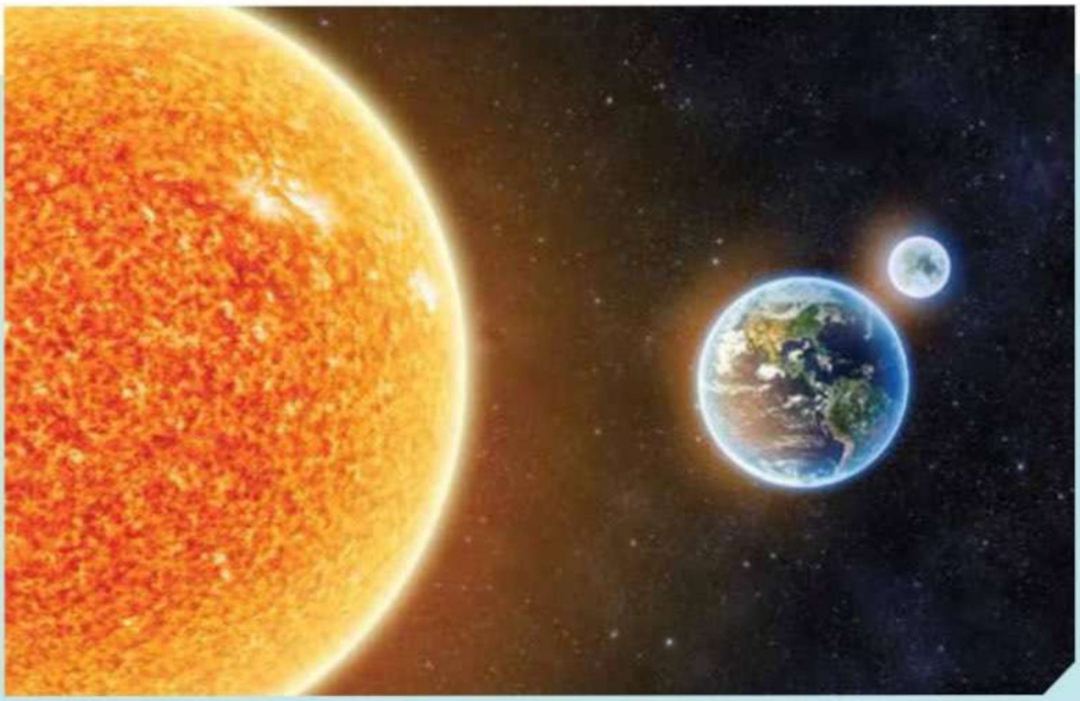
Summary

1. An object can undergo any of the following transformations.
   - **(a)** Under a **reflection**, the object and its image are symmetrical about the mirror line. A reflection does not preserve orientation. Points on the mirror line are invariant.
   - **(b)** A **rotation** is defined by its centre, angle and direction (clockwise or anticlockwise) of rotation. The centre of rotation is the only invariant point.
   - **(c)** A **translation** moves all the points of an object on a plane over the same distance and in the same direction. It preserves orientation and has no invariant points.
   - **(d)** An **enlargement** is defined by its *centre* and *scale factor*. The scale factor affects the size and position of the image. The centre of enlargement is the only possible invariant point.

2. Reflection, rotation and translation are **isometric transformations** while enlargement is a **non-isometric transformation**. Isometric transformations preserve the *shape* and *size* of an object.

CHAPTER 11

Area and Volume of Similar Figures and Solids



Is there life on other planets? To answer this question, astro-physicists have been searching for planets similar to Earth in other parts of the universe for quite some time now. These planets have to be similar in size in relation to the star they are orbiting as Earth is to our Sun. Since then, several "candidate" planets have been found. But how can we measure the mass, volume or surface area of these planets since direct measurement is impossible?

Given that all spheres are similar, we can estimate these measurements using ideas of similarity. The use of similarity here is the same as how we can use scale drawings of a building to estimate the amount of material needed for the actual structure. In this chapter, we will apply the idea of **proportionality** to investigate the area and volume of similar figures and solids.

Learning Outcomes

What will we learn in this chapter?

- How to compare ratios between the lengths, areas and volumes of similar figures and solids
- How the change in scale factor affects the area and volume of similar figures and solids
- How to solve real-world problems using the relationship between similar figures and solids

Introductory Problem

Two similar cuboids are made of the same material. The smaller cuboid has a length of 15 cm. The volume of the larger cuboid is 60 times that of the smaller cuboid. Find the length of the larger cuboid.

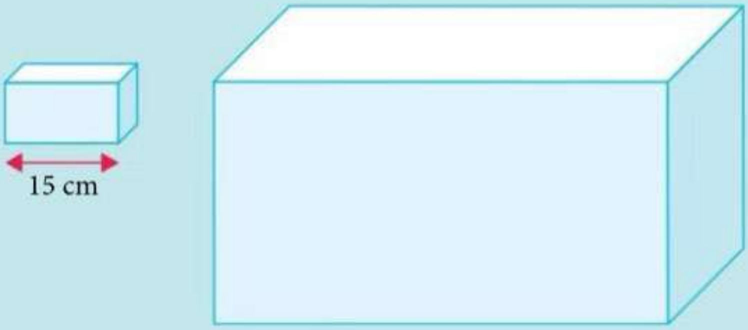


Fig. 11.1

In Book 2 and the previous chapter, we learnt about similar figures. In this chapter, we will learn how to find the area of similar figures and the volume of similar solids.

11.1 Area of similar figures

Investigation Areas of similar figures

1. Table 11.1 shows three squares. Are they similar? Explain your answer.

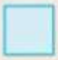
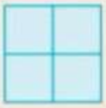
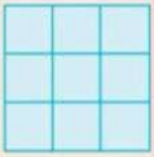
Square			
Length of square	1 unit	2 units	3 units
Area of square			

Table 11.1

2. Complete Table 11.1 to find the area of each square.
3. (a) The length of the second square is double that of the first square.
What is the relationship between their areas?
(b) The length of the third square is three times that of the first square.
What is the relationship between their areas?

4. Let the length and the area of a square be l_1 and A_1 respectively.
Let the length and the area of a second square be l_2 and A_2 respectively.
Note that the two squares are **similar**.
Express the following ratio of areas in terms of l_1 and l_2.

$$\frac{A_2}{A_1} = \frac{\square}{\square}$$

5. Is the formula in Question 4 always true?

Let us investigate what happens if we have similar triangles instead.

Table 11.2 shows three **similar** triangles.

Triangle			
Length of corresponding side of triangle	1 unit	2 units	3 units
Area of triangle	1 square unit		

Table 11.2

6. (a) The length of a side of the second triangle is double that of the corresponding side of the first triangle. What is the relationship between their areas?
   (b) The length of a side of the third triangle is three times that of the corresponding side of the first triangle. What is the relationship between their areas?

7. Let the length and the area of a triangle be l_1 and A_1 respectively.
Let the length and the area of a second *similar* triangle be l_2 and A_2 respectively.
Express the following ratio of areas in terms of l_1 and l_2.

$$\frac{A_2}{A_1} = \frac{\square}{\square}$$

In general, the ratio of the areas of two **similar** figures is the square of the ratio of their corresponding lengths, i.e.

$$\frac{A_2}{A_1} = \left(\frac{l_2}{l_1}\right)^2,$$

where A_1 and l_1 are the area and the length of the first figure respectively, and A_2 and l_2 are the area and the length of the second similar figure respectively.

Big Idea

Proportionality

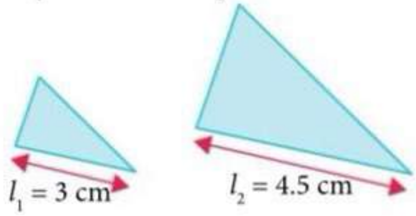
Recall that we learnt in Book 2 that if two figures are similar, then all the ratios of the lengths of the corresponding sides are equal, i.e. all the lengths of the corresponding sides of the two figures are proportional. However, the ratio of the areas of two similar figures is equal to the square of the ratio of the lengths of the corresponding sides, i.e. the areas of the two similar figures are proportional to the squares of the lengths of the corresponding sides. This means that if a figure is enlarged by a scale factor of 3, then the area of the similar figure is $3^2 = 9$ times the area of the original figure.

Worked Example 1

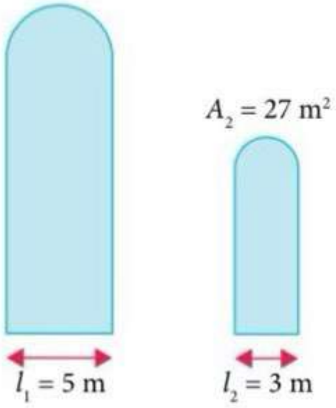
Finding the area of similar figures

Find the unknown area of each of the following pairs of similar figures.

(a) $A_1 = 4 \text{ cm}^2$ $A_2 = ?$



(b) $A_1 = ?$



Solution

(a)

$$\frac{A_2}{A_1} = \left(\frac{l_2}{l_1}\right)^2$$

$$\frac{A_2}{4} = \left(\frac{4.5}{3}\right)^2$$

$$= \frac{9}{4}$$

$$A_2 = \frac{9}{4} \times 4$$

$$= 9 \text{ cm}^2$$

(b)

$$\frac{A_1}{A_2} = \left(\frac{l_1}{l_2}\right)^2$$

$$\frac{A_1}{27} = \left(\frac{5}{3}\right)^2$$

$$= \frac{25}{9}$$

$$A_1 = \frac{25}{9} \times 27$$

$$= 75 \text{ m}^2$$

Attention

To simplify $\left(\frac{4.5}{3}\right)^2$ without using a calculator, we have

$$\left(\frac{4.5}{3}\right)^2 = \left(\frac{4.5 \times 2}{3 \times 2}\right)^2$$

$$= \left(\frac{9}{6}\right)^2$$

$$= \left(\frac{3}{2}\right)^2$$

$$= \frac{9}{4}$$

Problem-solving Tip

For **(b)**, write the unknown A_1 first. It will make subsequent algebraic manipulations easier.

Practise Now 1

Similar and Further Questions
Exercise 11A
Questions 1(a)–(f), 2, 3, 5–9

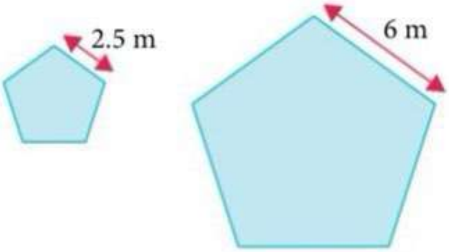
Find the unknown area of each of the following pairs of similar figures.

(a)

$A_2 = ?$

$A_1 = 32 \text{ cm}^2$

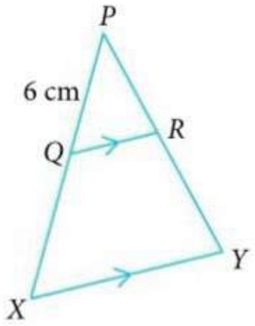
(b) $A_1 = ?$ $A_2 = 61.9 \text{ m}^2$



Worked Example 2

Finding the length of similar figures

In the figure, *PQX* and *PRY* are straight lines. *QR* is parallel to *XY*, $PQ = 6$ cm and the areas o $\triangle PQR$ and $\triangle PXY$ are 9 cm^2 and 64 cm^2 respectively. Find the length of *QX*.



Solution

$\angle QPR = \angle XPY$ (common $\angle$)
$\angle PQR = \angle PXY$ (corr. $\angle$s, $QR \,//\, XY$)
$\therefore \triangle PQR$ is similar to $\triangle PXY$ (AA Similarity Test).

$$\left(\frac{PX}{PQ}\right)^2 = \frac{\text{Area of } \triangle PXY}{\text{Area of } \triangle PQR}$$

$$\left(\frac{PX}{6}\right)^2 = \frac{64}{9}$$

$$\frac{PX}{6} = \sqrt{\frac{64}{9}} \text{ (since } \frac{PX}{6} > 0)$$

$$= \frac{8}{3}$$

$$PX = \frac{8}{3} \times 6$$

$$= 16 \text{ cm}$$

$$QX = PX - PQ$$

$$= 16 - 6$$

$$= 10 \text{ cm}$$

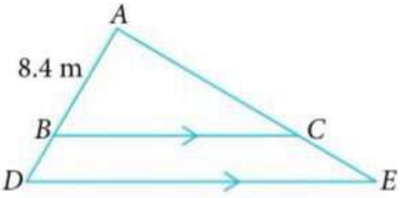
Attention

We need to prove that $\triangle PQR$ is similar to $\triangle PXY$ before using the properties of similar triangles.

Practise Now 2

Similar and Further Questions
Exercise 11A
Questions 4(a)–(d), 10–12

In the figure, *ABD* and *ACE* are straight lines. *BC* is parallel to *DE*, $AB = 8.4$ m and the areas of $\triangle ABC$ and $\triangle ADE$ are 49 m^2 and 100 m^2 respectively. Find the length of *BD*.



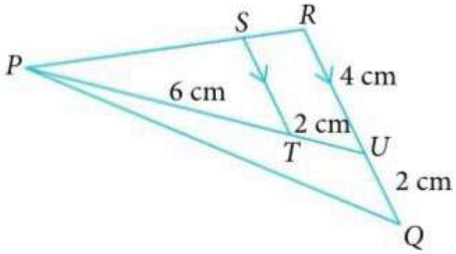
Worked Example 3

Solving problem involving similar figures

In the figure, PSR, PTU and RUQ are straight lines. ST is parallel to RU, $RU = 4$ cm, $UQ = 2$ cm, $PT = 6$ cm and $TU = 2$ cm. Given that the area of $\triangle PUR$ is 12 cm^2, find the area of

(i) $\triangle PTS$,

(ii) $\triangle PQU$.



Solution

(i) $T\hat{P}S = U\hat{P}R$ (common ∠)

$P\hat{T}S = P\hat{U}R$ (corr. ∠s, $ST \parallel RU$)

$\therefore \triangle PST$ is similar to $\triangle PRU$ (AA Similarity Test).

$$\frac{\text{Area of } \triangle PTS}{\text{Area of } \triangle PUR} = \left(\frac{PT}{PU}\right)^2$$

$$\frac{\text{Area of } \triangle PTS}{12} = \left(\frac{6}{8}\right)^2$$

$$= \frac{9}{16}$$

$$\text{Area of } \triangle PTS = \frac{9}{16} \times 12$$

$$= 6.75 \text{ cm}^2$$

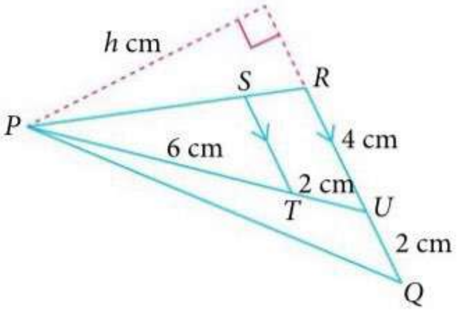
Problem-solving Tip

$P \leftrightarrow P$, $T \leftrightarrow U$, $S \leftrightarrow R$.

Problem-solving Tip

In using the ratio of lengths, we need to be careful in identifying corresponding sides of similar triangles, i.e. $PT : PU$ instead of $PT : TU$.

(ii)



Notice that $\triangle PUR$ and $\triangle PQU$ have a common height corresponding to the bases RU and UQ respectively. Let the common height be h cm.

$$\frac{\text{Area of } \triangle PQU}{\text{Area of } \triangle PUR} = \frac{\frac{1}{2} \times UQ \times h}{\frac{1}{2} \times RU \times h}$$

$$= \frac{UQ}{RU}$$

$$\frac{\text{Area of } \triangle PQU}{12} = \frac{2}{4}$$

$$\text{Area of } \triangle PQU = \frac{2}{4} \times 12$$

$$= 6 \text{ cm}^2$$

From Worked Example 3, we can conclude that the ratio of the areas of two triangles that have a *common height* h is equal to the ratio of the lengths of the bases of the two triangles, i.e.

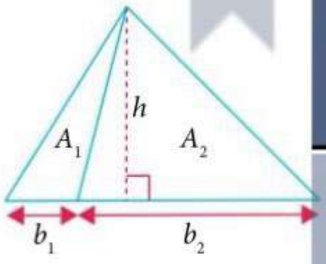


Fig. 11.2

$$\frac{A_2}{A_1} = \frac{b_2}{b_1},$$

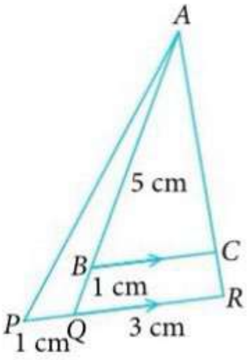
where A_1 and b_1 are the area and the length of the base of the first triangle respectively, and A_2 and b_2 are the area and the length of the base of the second triangle respectively.

Practise Now 3

Similar and Further Questions
Exercise 11A
Questions 13–17

In the figure, ACR, ABQ and PQR are straight lines. BC is parallel to QR, $AB = 5$ cm, $BQ = PQ = 1$ cm and $QR = 3$ cm. Given that the area of $\triangle AQR$ is 8 cm², find the area of

(i) $\triangle APQ$,
(ii) $\triangle ABC$.



Reflection

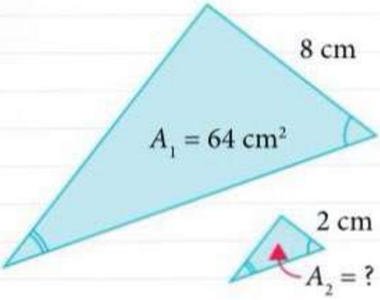
1. What is the relationship between the areas of similar figures and their corresponding lengths?
2. What is the relationship between the ratio of the areas of two triangles with common height and the ratio of the lengths of their bases?

Advanced Intermediate Basic

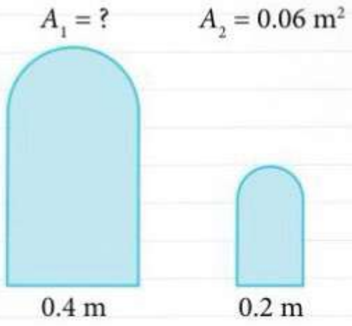
Exercise 11A

1. Find the unknown area of each of the following pairs of similar figures.

(a)

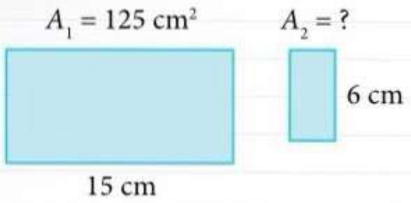


(b)

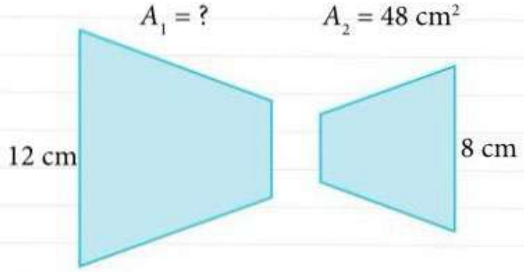


Exercise 11A

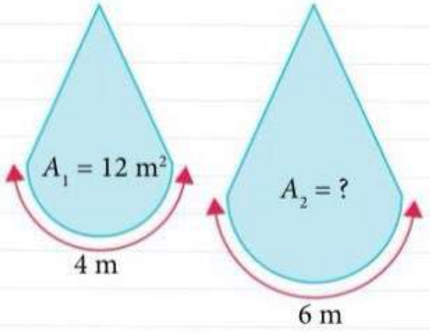
(c)



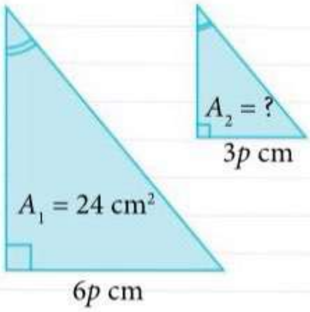
(d)



(e)



(f)



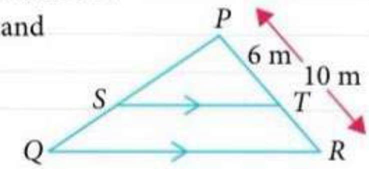
2. Find the ratio of the areas of two circles whose radii are 4 cm and 7 cm.

3. A triangular plot of land *PQR* has a water pipe *QR*. *ST* is another water pipe parallel to *QR*, where *S* lies on *PQ* and *T* lies on *PR*. *PT* = 6 m, *PR* = 10 m and the area of △*PST* is 24 m².

Find the area of the land occupied by

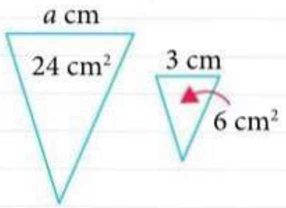
(i) △*PQR*,

(ii) *SQRT*.

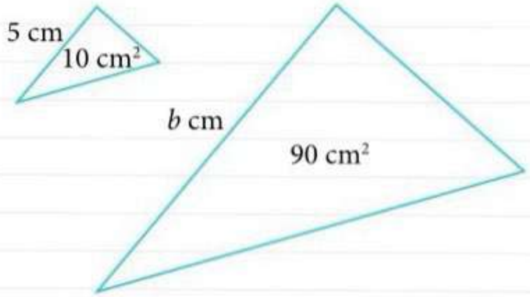


4. Find the unknown value in each of the following pairs of similar figures.

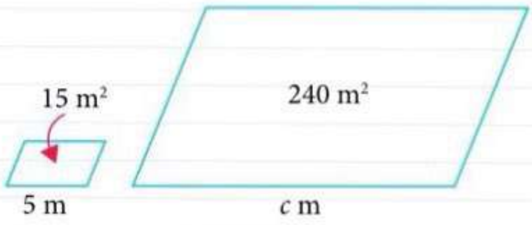
(a)



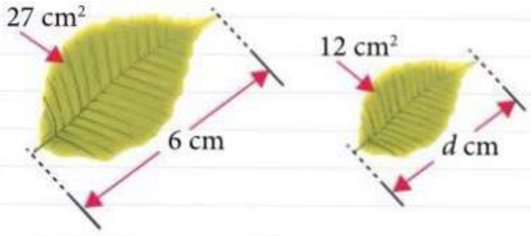
(b)



(c)

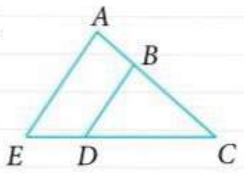


(d)



5. The perimeters of two similar regular hexagons are 10 m and 8 m. Given that the area of the larger hexagon is 200 m², find the area of the smaller hexagon.

6. In the figure below, △*CAE* is an enlargement of △*CBD* with a scale factor of $\frac{4}{3}$.

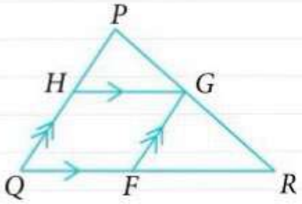


Given that the area of △*CBD* is 9 cm², find the area of *ABDE*.

Exercise 11A

7. In a scale drawing of a house, the width of a door that measures 150 cm is represented by a line 30 mm long. Find the actual land area, in square metres, occupied by the house if the corresponding area on the plan is 3250 cm^2.

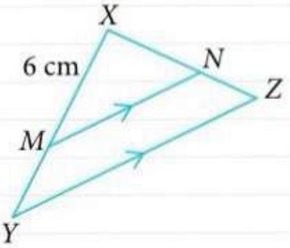
8. In the figure, *PHQ*, *PGR* and *QFR* are straight lines. *HG* is parallel to *QR*, *FG* is parallel to *QP* and $QF : FR = p : q$.



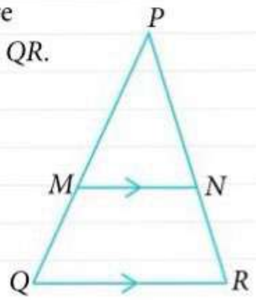
Find the ratio of the area of $\triangle PHG$ to that of $\triangle PQR$ in terms of p and q.

9. Two solid cones are geometrically similar and the height of one cone is 1.5 times that of the other. Given that the height of the smaller cone is 12 cm and its surface area is 124 cm^2, find
(i) the height, and **(ii)** the surface area
of the larger cone.

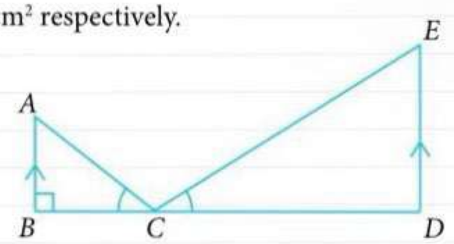
10. In the figure, $\triangle XYZ$ is an enlargement of $\triangle XMN$. Given that $XM = 6$ cm and that the areas of $\triangle XMN$ and *MYZN* are 14 cm^2 and 22 cm^2 respectively, find the length of *MY*.



11. In the figure, *PMQ* and *PNR* are straight lines. *MN* is parallel to *QR*. If the areas of $\triangle PMN$ and trapezium *MQRN* are in the ratio 9 : 16, find the ratio *MN* : *QR*.

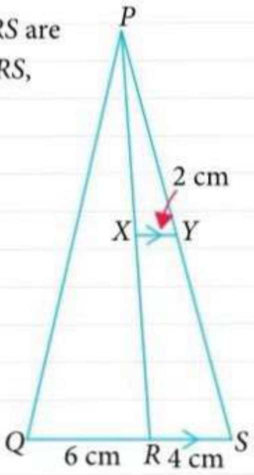


12. In the figure, *BCD* is a straight line and *BA* is parallel to *DE*. $\angle ABC = 90°$ and $\angle ACB = \angle ECD$. The areas of $\triangle ABC$ and $\triangle CDE$ are 25 cm^2 and 64 cm^2 respectively.



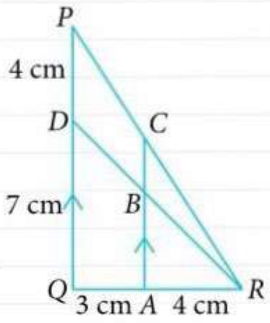
Given further that *CD* is 4.5 cm longer than *BC*, find the length of *BC*.

13. In the figure, *PYS*, *PXR* and *QRS* are straight lines. *XY* is parallel to *RS*, $XY = 2$ cm, $QR = 6$ cm and $RS = 4$ cm. Given that the area of $\triangle PXY$ is 10 cm^2, find the area of
(i) $\triangle PRS$,
(ii) $\triangle PQR$.



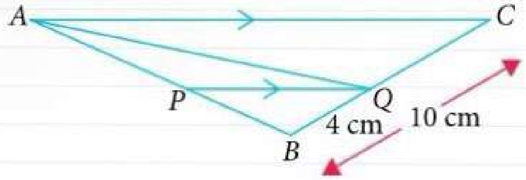
14. The perimeters of two similar triangles are in the ratio 3 : 4. The sum of their areas is 105 cm^2. Find the area of each triangle.

15. The diagram shows a $\triangle PQR$. *D* lies on *PQ*, *A* lies on *QR* and *C* lies on *PR*. *DR* meets *AC* at *B* and *AC* is parallel to *QP*. $AR = 4$ cm, $QA = 3$ cm, $DQ = 7$ cm and $PD = 4$ cm.
Find
(i) the length of *BC*,
(ii) the ratio of the area of $\triangle ARB$ to that of $\triangle BRC$,
(iii) the ratio of the area of $\triangle BRC$ to that of *ABDQ*.



Exercise 11A

16. In the figure, APB and BQC are straight lines, and PQ is parallel to AC.



Given that $BQ = 4$ cm, $BC = 10$ cm and the area of $\triangle BPQ$ is 8 cm², find the area of

(i) $\triangle ABC$, **(ii)** $\triangle PQC$,

(iii) $\triangle AQC$.

17. The areas of two similar triangles are 54 cm² and 96 cm². Given that the perimeter of the smaller triangle is $(x + 1)$ cm, find the sum of the perimeters of the two triangles in terms of x. Hence, give a possible value of x for which the sum is a whole number.

11.2 Volume of similar solids

In Section 11.1, we have learnt how to find the area of similar figures. In this section, we will learn how to find the volume of similar solids.

Investigation Volume and mass of similar solids

1. Table 11.3 shows three cubes. Are they similar? Explain your answer.

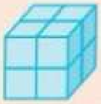
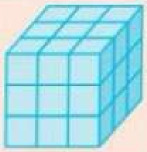
Cube			
Length of cube	1 unit	2 units	3 units
Volume of cube			

Table 11.3

2. Complete Table 11.3 to find the volume of each cube.

3. **(a)** The length of the second cube is double that of the first cube. What is the relationship between their volumes?

   (b) The length of the third cube is three times that of the first cube. What is the relationship between their volumes?

4. Let the length and the volume of a cube be l_1 and V_1 respectively.
Let the length and the volume of a second cube be l_2 and V_2 respectively.
Note that the two cubes are **similar**.
Express the following ratio of volumes in terms of l_1 and l_2.

$$\frac{V_2}{V_1} = \frac{\square}{\square}$$

5. Is the formula in Question 4 always true?
Let us investigate what happens if we have similar cylinders instead.
Fig. 11.3 shows two **similar** cylinders: they have exactly the same shape, i.e. the ratio of the corresponding lengths is a constant k, called the **scale factor** of the solids.
For example,

$$\frac{h_2}{h_1} = k$$

where h_1 and h_2 are the heights of the first and second cylinders respectively.

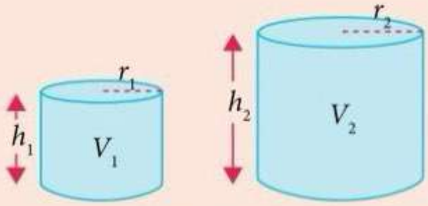


Fig. 11.3

The radii of the circular cross sections of the two cylinders are r_1 and r_2 respectively. What is the value of $\frac{r_2}{r_1}$?

6. The volume of the first cylinder is $V_1 = \pi r_1^{\,2} h_1$.
   - **(a)** Find the volume of the second cylinder, V_2, in terms of r_1 and h_1.
   - **(b)** Hence, find the volume of the second cylinder, V_2, in terms of V_1.

7. Express the following ratio of volumes in terms of k, then in terms of h_1 and h_2, and then in terms of r_1 and r_2.

$$\frac{V_2}{V_1} = \square = \left(\frac{h_2}{\square}\right)^{\square} = \left(\frac{\square}{\square}\right)^{\square}$$

8. If two similar solids are made of the same material, what is the relationship between their masses m_1 and m_2?
Let the density of the material be d. Then

$$d = \frac{m_1}{V_1} = \frac{m_2}{V_2}$$

where V_1 and V_2 are the volumes of the solids respectively.
Express the following in terms of V_1 and V_2.

$$\frac{m_2}{m_1} = \frac{\square}{\square}$$

Information

The density of a material is defined as the mass per unit volume of the material.

In general, the ratio of the volumes of two similar solids of the same density is the cube of the ratio of their corresponding lengths, and the ratio of their masses is equal to the ratio of their volumes, i.e.

$$\frac{V_2}{V_1} = \left(\frac{l_2}{l_1}\right)^3 \text{ and } \frac{m_2}{m_1} = \frac{V_2}{V_1},$$

where V_1, l_1 and m_1 are the volume, length and mass of the first solid respectively, and V_2, l_2 and m_2 are the volume, length and mass of the second similar solid respectively.

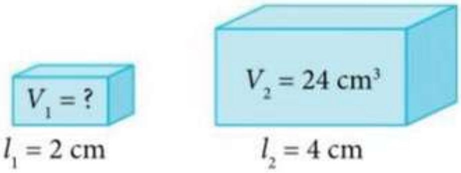
Big Idea

Proportionality
Since the ratio of the volumes of two similar solids is proportional to the cube of the ratio of their corresponding lengths, it can be determined using any pair of corresponding sides.

Worked Example 4

Finding the volume of similar solids

The figure shows two toy blocks which take the shape of a pair of similar cuboids. Find the volume, V_1, of the smaller block.



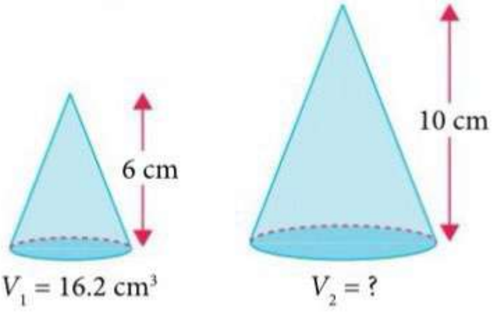
Solution

$$\begin{aligned}\frac{V_1}{V_2} &= \left(\frac{l_1}{l_2}\right)^3\\ \frac{V_1}{24} &= \left(\frac{2}{4}\right)^3\\ &= \left(\frac{1}{2}\right)^3\\ &= \frac{1}{8}\\ V_1 &= \frac{1}{8}\times 24\\ &= 3 \text{ cm}^3\end{aligned}$$

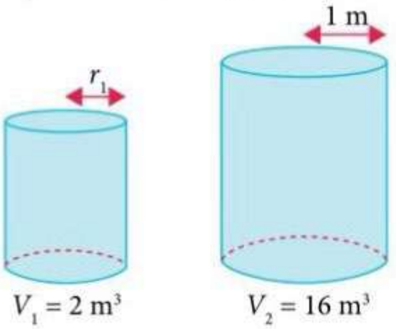
Practise Now 4

Similar and Further Questions
Exercise 11B
Questions 1(a)–(e), 2(a)–(c), 3, 4(a)–(d), 5, 9

1. The figure shows two chocolate hats which take the shape of a pair of similar cones. Find the volume, V_2, of the larger chocolate hat.



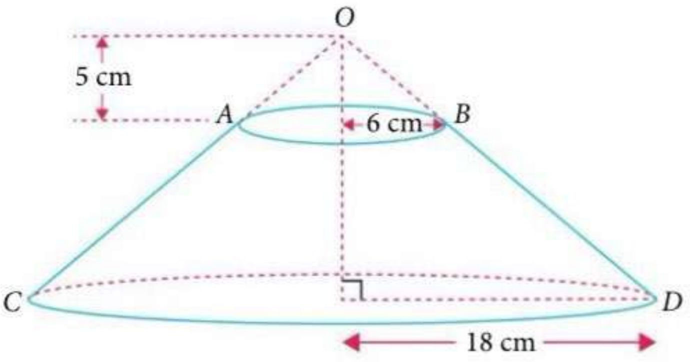
2. Find the unknown radius, r_1, for the following pair of similar cylinders.



Worked Example 5

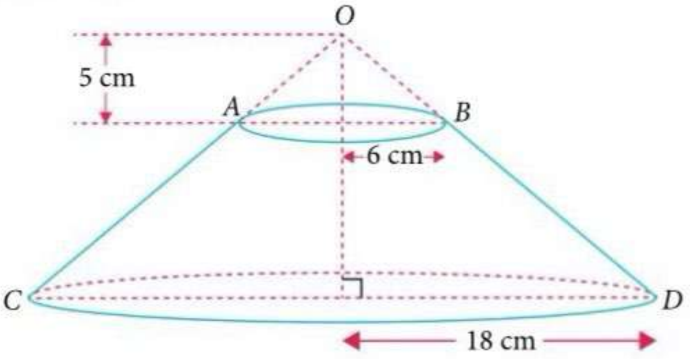
Finding volume of frustum

The figure shows a frustum which is obtained by removing the smaller cone *OAB* with a base radius of 6 cm from the bigger cone *OCD* that has a base radius of 18 cm. Given that the height of the smaller cone *OAB* is 5 cm, find the volume of the frustum.



Solution

Method 1:



$\angle AOB = \angle COD$ (common $\angle$)

$\angle ABO = \angle CDO$ (corr. $\angle$s, $AB \parallel CD$)

$\therefore \triangle OAB$ is similar to $\triangle OCD$ (AA Similarity Test) and hence cone *OAB* is similar to cone *OCD*.

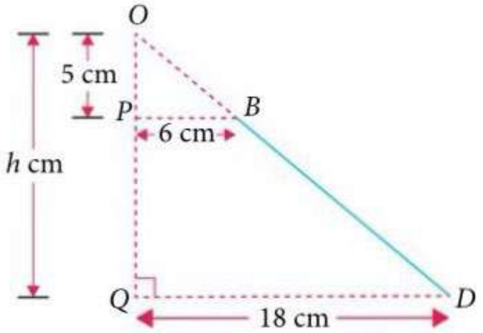
$$\frac{\text{Volume of cone } OCD}{\text{Volume of cone } OAB} = \left(\frac{18}{6}\right)^3$$
$$= 27$$

$$\text{Volume of cone } OCD = 27 \times \text{volume of cone } OAB$$
$$= 27 \times \frac{1}{3}\pi(6^2)(5)$$
$$= 1620\pi \text{ cm}^3$$

$$\therefore \text{volume of frustum} = \text{volume of cone } OCD - \text{volume of cone } OAB$$
$$= 1620\pi - \frac{1}{3}\pi(6^2)(5)$$
$$= 1620\pi - 60\pi$$
$$= 1560\pi$$
$$= 4900 \text{ cm}^3 \text{ (to 3 s.f.)}$$

Method 2:

Let the height of cone *OCD* be *h* cm.



$\angle POB = \angle QOD$ (common $\angle$)

$\angle OPB = \angle OQD$ (corr. $\angle$s, $PB \,//\, QD$)

$\therefore \triangle OPB$ is similar to $\triangle OQD$ (AA Similarity Test).

$$\frac{OQ}{OP} = \frac{QD}{PB}$$

$$\frac{h}{5} = \frac{18}{6}$$

$$= 3$$

$$h = 15$$

The height of the larger cone is 15 cm.

$\therefore$ volume of frustum

= volume of cone *OCD* – volume of cone *OAB*

$$= \frac{1}{3}\pi(18^2)(15) - \frac{1}{3}\pi(6^2)(5)$$

$$= \frac{1}{3}\pi(18^2 \times 15 - 6^2 \times 5)$$

$$= \frac{1}{3}\pi(4680)$$

$$= 1560\pi$$

$$= 4900 \text{ cm}^3 \text{ (to 3 s.f.)}$$

Big Idea

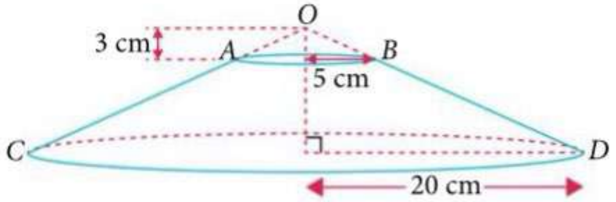
Proportionality

If two figures are similar, then all the ratios of the lengths of the corresponding dimensions (in this case, the radius and the height) are equal.
What can you say about the ratio of the perimeter of similar figures?

Practise Now 5

Similar and Further Questions
Exercise 11B
Question 15

The figure shows a frustum which is obtained by removing the smaller cone *OAB* with a base radius of 5 cm from the bigger cone *OCD* that has a base radius of 20 cm. Given that the height of the smaller cone *OAB* is 3 cm, find the volume of the frustum.



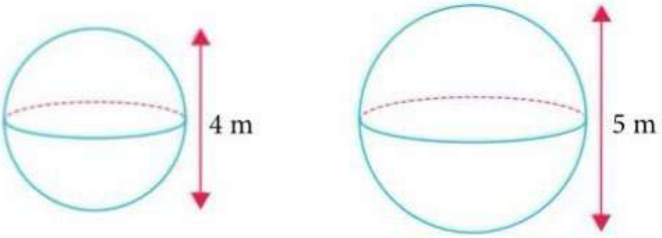
Introductory Problem Revisited

Now that you have gained some understanding of the relationship between the ratio of the volumes of similar solids and the ratio of their corresponding lengths, can you find the length of the larger cuboid?

Worked Example 6

Finding the mass of similar solids

Two solid spheres of diameters 4 m and 5 m are made of the same material. Given that the smaller sphere has a mass of 120 kg, find the mass of the larger sphere.



Mass = 120 kg　　Mass = ?

Solution

Let m_1, V_1 and l_1 be the mass, volume and diameter of the smaller sphere respectively.
Let m_2, V_2 and l_2 be the mass, volume and diameter of the larger sphere respectively.

$$\frac{m_2}{m_1} = \frac{V_2}{V_1}$$
$$= \left(\frac{l_2}{l_1}\right)^3$$
$$\frac{m_2}{120} = \left(\frac{5}{4}\right)^3$$
$$m_2 = \left(\frac{5}{4}\right)^3 \times 120$$
$$= 234\frac{3}{8} \text{ kg}$$

∴ the mass of the larger sphere is $234\frac{3}{8}$ kg.

Practise Now 6

Similar and Further Questions
Exercise 11B
Questions 6-8, 10-12, 16

1. Two similar solid triangular prisms made of the same material have heights 5 cm and 8 cm. Given that the smaller prism has a mass of 80 g, find the mass of the larger prism, giving your answer correct to the nearest integer.
2. The figure shows a figurine with a height of 20 cm and a mass of 3 kg. Bernard made a similar statue with a height of 2 m using the same material. Find the mass of the statue made by Bernard.

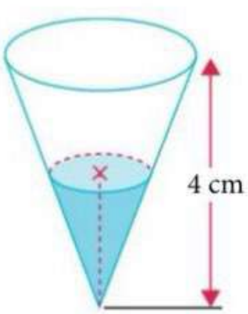
11.3 Solving problems involving similar solids

In this section, we shall learn how to solve problems involving similar solids.

Worked Example 7

Solving problem involving similar solids

The figure shows an inverted conical container of height 4 cm. It contains a volume of water which is equal to one-eighth of its full capacity.



Find

(i) the depth of the water,

(ii) the ratio of the area of the top surface of the water to the area of the top surface of the container.

Solution

(i) Let V_1 and h_1 be the volume and height of the smaller cone respectively, and V_2 and h_2 be the volume and height of the larger cone respectively.

$$\frac{V_1}{V_2} = \left(\frac{h_1}{h_2}\right)^3$$

$$\frac{1}{8} = \left(\frac{h_1}{4}\right)^3 \qquad \text{since } V_1 = \frac{1}{8}V_2\text{, then } \frac{V_1}{V_2} = \frac{1}{8}$$

$$\left(\frac{h_1}{4}\right)^3 = \frac{1}{8}$$

$$\frac{h_1}{4} = \sqrt[3]{\frac{1}{8}}$$

$$= \frac{1}{2}$$

$$h_1 = \frac{1}{2}\times 4$$

$$= 2 \text{ cm}$$

$\therefore$ the depth of the water is 2 cm.

(ii) The top surface of the water and that of the container are circles.

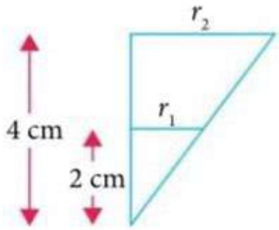
Let r_1 and r_2 be the radii of the smaller circle and the larger circle respectively.

Using similar triangles,

$$\frac{r_1}{r_2} = \frac{h_1}{h_2}$$

$$= \frac{2}{4}$$

$$= \frac{1}{2}$$



Let A_1 and A_2 be the areas of the smaller circle and larger circle respectively.

$$\frac{A_1}{A_2} = \left(\frac{r_1}{r_2}\right)^2$$
$$= \left(\frac{1}{2}\right)^2$$
$$= \frac{1}{4}$$

∴ the ratio of the area of the top surface of the water to the area of the top surface of the container is 1 : 4.

Practise Now 7

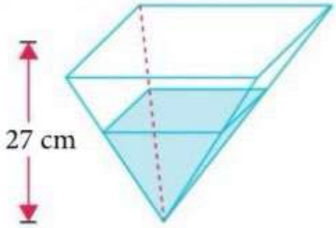
Similar and Further Questions
Exercise 11B
Questions 13, 14, 17

The figure shows a container in the shape of an inverted right pyramid of height 27 cm. It contains a volume of vegetable oil which is equal to one-sixth of its full capacity.

Find

(i) the depth of the vegetable oil,

(ii) the ratio of the area of the top surface of the vegetable oil to the area of the top surface of the container, giving your answer in the form 1 : *n*.



Thinking Time

Two similar cones with slant heights are given as shown.

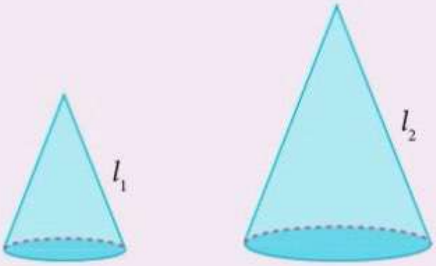


Fig. 11.4

Express the ratio of the total surface area of the smaller cone to that of the larger cone in terms of l_1 and l_2. Explain your answer.

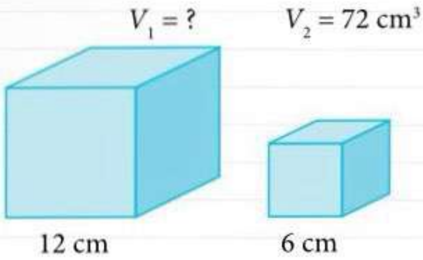
Reflection

1. What is the relationship between the volumes of similar solids and their corresponding lengths?
2. What is the relationship between the surface areas of similar solids and their corresponding lengths?
3. How is the ratio of the masses of similar solids related to the ratio of their volumes? When is this relationship not true?

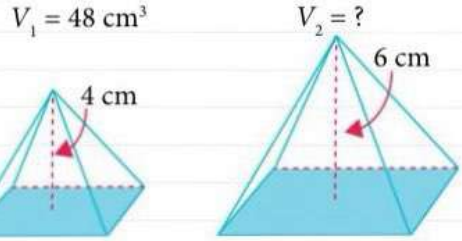
Exercise 11B

1. Find the unknown volume of each of the following pairs of similar solids.

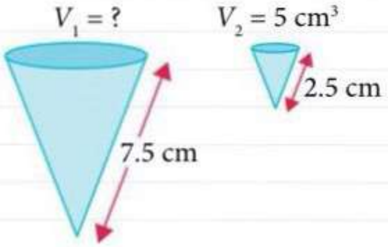
(a)



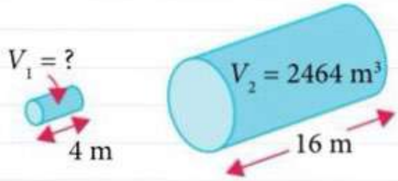
(b)



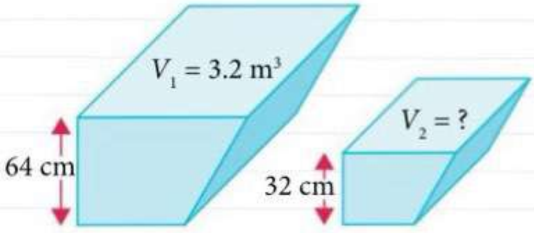
(c)



(d)



(e)

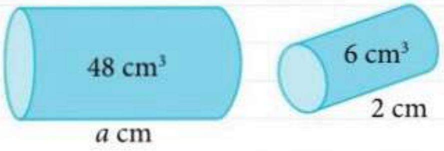


2. Find the ratio of the volumes of
   - **(a)** two similar solid cylinders of circumferences 10 cm and 8 cm,
   - **(b)** two similar solid cones of heights 9 cm and 12 cm,
   - **(c)** two solid spheres of radii 4 cm and 6 cm.

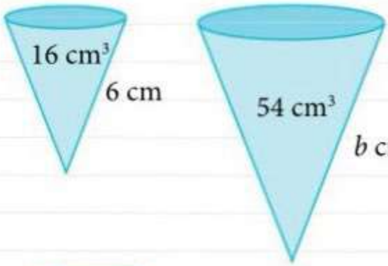
3. In a restaurant, a Junior glass has a height of 6 cm and a Senior glass has a height of 9 cm. Given that the capacity of a Senior glass is 540 cm^3, find the capacity of the Junior glass.

4. Find the unknown value in each of the following pairs of similar solids.

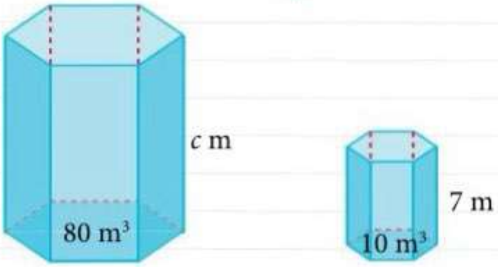
(a)



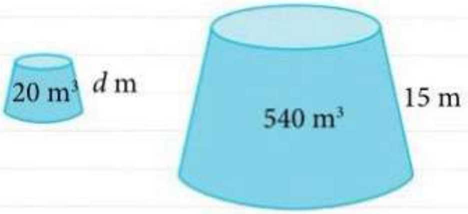
(b)



(c)



(d)



5. The areas of the bases of two similar cones are in the ratio 9 : 16.
   - **(i)** Find the ratio of the heights of the cones.
   - **(ii)** Given that the volume of the larger cone is 448 cm^3, find the volume of the smaller cone.

6. The masses of two spheres of the same material are 640 kg and 270 kg. Find the ratio of their diameters.

Exercise 11B

7. A certain brand of chilli flakes comes in similar bottles of two sizes – 'mini' and 'regular'. The 'mini' bottle has a mass of 280 g and a height of 15 cm. Given that the 'regular' bottle has a mass of 750 g, find its height.

8. Two similar solid candy canes have heights 4 cm and 7 cm.

- **(i)** Find the ratio of the total surface areas of the candy canes.
- **(ii)** Given that the smaller candy cane has a mass of 10 g, find the mass of the larger candy cane.

9. The volume of one sphere is 4 times that of another sphere. Given that the radius of the smaller sphere is 3 cm, find the radius of the larger sphere.

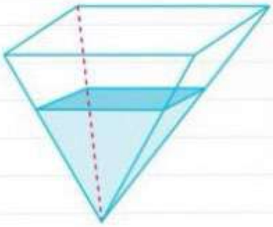
10. The mass of a glass figurine of height 6 cm is 500 g. Find the mass of a similar glass figurine with a height of 4 cm.

11. A train is 10 m long and its mass is 72 tonnes. A similar model made of the same material is 40 cm long. (1 tonne = 1000 kg)

- **(i)** Find the mass of the model.
- **(ii)** Given that the tank of the model train contains 0.85 litres of water when it is full, find the capacity of the tank of the train, giving your answer correct to the nearest integer.

12. The masses of two similar glass tanks are 8.58 kg and 4.29 kg. Given that the larger tank has a base area of 12.94 m^2, find the base area of the smaller tank.

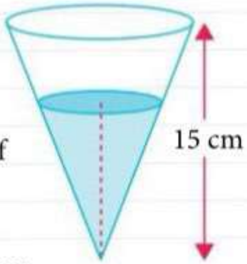
13. The figure shows a container in the shape of an inverted right pyramid which contains some water. The area of the top surface of the container is 63 cm^2 and the area of the top surface of the water is 28 cm^2.



Find

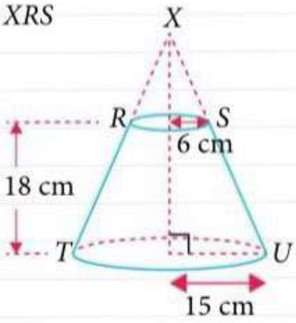
- **(i)** the depth of the water if its volume is 336 cm^3,
- **(ii)** the ratio of the depth of the water to the height of the container,
- **(iii)** the capacity of the container.

14. The figure below shows an inverted conical container of height 15 cm found in a laboratory. It contains a volume of mercury which is equal to $\frac{8}{27}$ of its full capacity. Find



- **(i)** the depth of the mercury,
- **(ii)** the area of the mercury that is exposed to the air if the area of the top surface of the container is 45 cm^2,
- **(iii)** the capacity of the container.

15. The figure shows a frustum which is obtained by removing the smaller cone XRS with a base radius of 6 cm from the bigger cone XTU that has a base radius of 15 cm. Given that the height of the frustum is 18 cm, find its volume.



16. A clay model has a mass of x^2 kg and a height of 30 cm. A similar clay model has a mass of $(x + 0.3)$ kg and a height of 20 cm. Find the values of x.

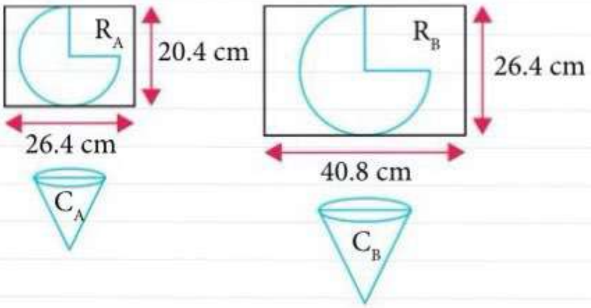
Exercise 11B

17. Two pieces of rectangular paper R$_A$ and R$_B$ are as shown below. The largest possible $\frac{3}{4}$-circle is cut out from each piece of paper to make paper cones C$_A$ and C$_B$.

(a) Find the ratio of the area of C$_A$ to that of C$_B$.

(b) Are the two cones similar? Explain.

(c) Vasi claimed that the ratio of the volume of C$_A$ to that of C$_B$ is the same as that found in part **(a)**. Do you agree? Explain your reasoning.



Looking Back

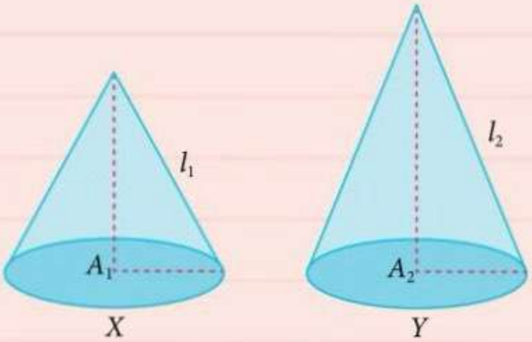
In this chapter, we used the ideas of **proportionality** to investigate the relationship between the ratio of the area and volume of similar figures and solids and the ratio of their corresponding lengths. These relationships provide a means for us to determine the dimensions of the similar objects without actually measuring them.

The ideas we have learnt are not only useful in the drawing of scale models to figure out how to build actual structures. They can also be used in forensic science to estimate the height and build of criminals. We also use these ideas in aeronautical engineering to test and refine the designs of parts of an aircraft, such as the engine, before the actual construction. Can you think of other real-world instances where we can apply the ideas we have learnt?

Summary

If X and Y are two similar solids, then

- ratio of their corresponding lengths is $\frac{l_2}{l_1}$,
- ratio of their areas, $\frac{A_2}{A_1} = \left(\frac{l_2}{l_1}\right)^2$, and
- ratio of their volumes, $\frac{V_2}{V_1} = \left(\frac{l_2}{l_1}\right)^3$.



X
Volume of X is V_1

Y
Volume of Y is V_2

Answer Keys

Chapter 1 Further Sets

Practise Now 1

1. **(i)** $C \cap D = \{e, g\}$
2. **(i)** $E = \{6, 12, 18\}$,
   $F = \{3, 6, 9, 12, 15, 18\}$
   (ii) $E \cap F = \{6, 12, 18\}$
   (iv) Yes
3. **(i)** $G = \{1, 2, 3, 4, 6, 12\}$,
   $H = \{5, 7, 11, 13\}$
   (ii) $G \cap H = \varnothing$

Practise Now 2

1. **(ii)** $C \cup D = \{p, q, r, s, t, u, v\}$
2. **(i)** $E = \{1, 2, 4, 8\}$,
   $F = \{1, 2, 4, 8, 16\}$
   (iii) $E \cup F = \{1, 2, 4, 8, 16\}$
   (iv) Yes
3. **(i)** $G = \{7, 14, 21, 28, 35, 42, 49, 56\}$,
   $H = \{9, 18, 27, 36, 45, 54\}$
   (iii) $G \cup H = \{7, 9, 14, 18, 21, 27, 28, 35, 36, 42, 45, 49, 54, 56\}$

Practise Now 3

1. **(i)** $\xi = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$,
   $A = \{1, 3, 5, 7, 9\}$,
   $B = \{3, 6, 9\}$
   (iii) (a) $(A \cup B)' = \{2, 4, 8\}$
   (b) $A \cap B' = \{1, 5, 7\}$

Practise Now 4

(i) $\xi = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20\}$,
$F = \{1, 2, 4, 5, 10, 20\}$,
$G = \{2, 4, 6, 8, 10, 12, 14, 16, 18, 20\}$,
$H = \{1, 2, 3, 4, 6, 8, 9, 12, 16, 18\}$
(iii) (a) $\{5\}$
(b) $\{1, 2, 4, 5, 7, 10, 11, 13, 15, 17, 19, 20\}$

Practise Now 6

(a) $X' \cap Y$
(b) $(X' \cap Y) \cup (X \cap Y')$
(c) Y'
(d) $X \cup Y'$

Practise Now 7

1. 15
2. 29

Introductory Problem Revisited

11

Practise Now 8

1. **(i)** 8 **(ii)** 0
2. **(i)** 19 **(ii)** 17
   (iii) 8 **(iv)** 3

Exercise 1A

1. **(i)** $A \cap B = \{2, 4\}$
2. **(i)** $C \cap D = \{$yellow, pink, blue$\}$
3. **(i)** $E \cap F = \varnothing$
4. **(ii)** $G \cup H$
   $= \{$apple, orange, banana, grape, durian, pear, strawberry$\}$
5. **(ii)** $I \cup J = \{v, w, x, y, z\}$
6. **(ii)** $K \cup L = \{11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21\}$
7. **(i)** $M = \{1, 4, 9, 16, 25, 36, 49, 64\}$,
   $N = \{1, 8, 27, 64\}$
   (ii) $M \cap N = \{1, 64\}$
8. **(i)** $P = \{8, 16, 24, 32\}$,
   $Q = \{4, 8, 12, 16, 20, 24, 28, 32\}$
   (ii) $P \cap Q = \{8, 16, 24, 32\}$
   (iv) Yes
9. **(i)** $R = \{1, 2, 3, 6, 9, 18\}$,
   $S = \{10, 12, 14, 15, 16\}$
   (ii) $R \cap S = \varnothing$
10. **(i)** $T = \{3, 6, 9, 12, 15, 18\}$,
    $V = \{1, 2, 3, 6, 9, 18\}$
    (iii) $T \cup V = \{1, 2, 3, 6, 9, 12, 15, 18\}$
11. **(i)** $W = \{4, 8, 12\}$,
    $X = \{1, 2, 3, 4, 6, 8, 12, 24\}$
    (iii) $W \cup X = \{1, 2, 3, 4, 6, 8, 12, 24\}$
    (iv) Yes
12. **(i)** $Y = \{1, 5, 25\}$,
    $Z = \{6, 12, 18, 24\}$
    (iii) $Y \cup Z = \{1, 5, 6, 12, 18, 24, 25\}$
13. $A \cap B$
    $= \{(x, y) : (x, y)$ are the coordinates of the points of intersection between the curve $y = x^2 - 3x + 2$ and the line $y = 0\}$
14. $C \cap D = \{(3, 14), (-1, 2)\}$
15. $E \cup F = \{y : y$ is a real number such that $y \geqslant -4\}$

Exercise 1B

1. **(i)** $\xi = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}$,
   $I = \{4, 8, 12\}$,
   $J = \{1, 2, 4, 8\}$
   (iii) (a) $(I \cup J)'$
   $= \{3, 5, 6, 7, 9, 10, 11, 13, 14, 15\}$
   (b) $I \cap J' = \{12\}$
2. **(i)** $\xi = \{4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18\}$,
   $Y = \{6, 9, 12, 15, 18\}$,
   $Z = \{9, 18\}$
   (iii) (a) $(Y \cup Z)'$
   $= \{4, 5, 7, 8, 10, 11, 13, 14, 16, 17\}$
   (b) $Y \cap Z' = \{6, 12, 15\}$
3. **(i)** $\xi = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$,
   $P = \{2, 3, 5, 7, 11\}$,
   $Q = \{4, 6, 8, 9, 10\}$
   (iii) (a) $P \cup Q = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$
   (b) $(P \cup Q)' = \{0, 1\}$
   (c) $P' \cap Q = \{4, 6, 8, 9, 10\}$
5. **(a)** A **(b)** B
   (c) $A \cap B$ **(d)** $A \cup B$
   (e) A' **(f)** B'
6. 31
7. 15
9. **(ii) (a)** $\{e, g, s\}$
   (b) $\{e, g, i, s, u\}$
10. **(a) (i)** $\{1, 2, 3, 4, 5, 6, 7, 8\}$
    (ii) $\{2, 4\}$
    (b) $\{a, b, c\}$
15. **(a)** $X \cap Y'$ **(b)** $X' \cap Y$
    (c) $(X \cup Y)'$ **(d)** $X' \cup Y'$
    (e) $X \cup Y'$ **(f)** $X' \cup Y$
    (g) $X \cap Y'$ **(h)** $X' \cup Y$
    (i) $Y \cup Z$
    (j) $X' \cap (Y \cup Z)$
16. 14
17. 22
18. **(i)** 5 **(ii)** 0
19. **(i)** 16 **(ii)** 24
20. **(i)** 10 **(ii)** 1
21. **(i)** 7 **(ii)** 11
24. **(i)** A **(ii)** ξ
    (iii) $\varnothing$ **(iv)** A
25. **(i)** A **(ii)** B
    (iii) Not possible to simplify further
    (iv) Not possible to simplify further
    (v) A **(vi)** $\varnothing$
    (vii) Not possible to simplify further
    (viii) B
26. **(i)** $(P \cap Q') \cup (P' \cap Q)$
    (ii) $(P \cap Q') \cup (P' \cap Q) \cup (P \cup Q)'$
27. 4
28. 27
29. **(iii)** 12
30. **(ii)** 3 **(iii)** 16
31. 11
32. **(i)** 24 **(ii)** 14
    (iii) 9 **(iv)** 0
33. **(i)** 6 **(ii)** 0
    (iii) 28 **(iv)** 15

Chapter 2 Probability of Combined Events

Practise Now 1

(i) {C, L, E_1, V, E_2, R}
(ii) **(a)** $\frac{1}{3}$ **(b)** $\frac{1}{6}$
(c) $\frac{1}{3}$ **(d)** 0
(iii) **(a)** $\frac{1}{3}$ **(b)** $\frac{2}{3}$

Practise Now 2

(i) $\frac{1}{4}$ **(ii)** $\frac{3}{4}$
(iii) $\frac{1}{6}$ **(iv)** $\frac{5}{12}$

Practise Now 3

1. **(ii)** **(a)** $\frac{1}{6}$ **(b)** $\frac{5}{12}$
(c) $\frac{1}{4}$
2. **(i)** $\frac{2}{5}$ **(ii)** $\frac{6}{25}$
(iii) $\frac{8}{25}$

Practise Now 4

1. **(i)** 2, 6, 7;
3, 4, 7, 8;
4, 5, 9;
5, 6, 9, 10;
6, 7, 10, 11;
8, 11, 12

1, 2, 5, 6;
2, 10, 12;
3, 6, 15;
4, 8, 20, 24;
5, 10, 25, 30;

6, 12, 36
(ii) **(a)** $\frac{1}{2}$ **(b)** $\frac{1}{3}$
(c) $\frac{1}{6}$ **(d)** 0
(iii) **(a)** $\frac{1}{4}$ **(b)** $\frac{1}{3}$
(c) $\frac{5}{24}$ **(d)** 1
2. **(i)** **(a)** $\frac{1}{9}$ **(b)** $\frac{5}{9}$
(ii) **(a)** 2, 3; 2, 2, 3; 3, 3
(b) $\frac{1}{3}$ **(c)** $\frac{8}{9}$

Practise Now 5

1. **(i)** $\frac{3}{8}$ **(ii)** $\frac{7}{8}$
2. **(ii)** **(a)** $\frac{5}{8}$ **(b)** $\frac{1}{4}$
(c) $\frac{1}{2}$ **(d)** $\frac{1}{8}$

Practise Now 6

(a) $\frac{1}{9}$ **(b)** $\frac{2}{9}$
(c) $\frac{1}{3}$ **(d)** $\frac{2}{9}$

Practise Now 7

(i) $\frac{1}{13}$ **(ii)** $\frac{3}{13}$
(iii) $\frac{4}{13}$ **(iv)** $\frac{9}{13}$
(v) $\frac{1}{4}$ **(vi)** $\frac{4}{13}$

Practise Now 8

(ii) **(a)** $\frac{11}{30}$ **(b)** $\frac{73}{168}$
(c) $\frac{5}{6}$ **(d)** $\frac{95}{168}$

Practise Now 9

(i) $\frac{7}{12}$ **(ii)** $\frac{5}{12}$
(iii) $\frac{35}{144}$ **(iv)** $\frac{5}{12}$
(v) $\frac{35}{72}$

Practise Now 10

(i) $\frac{2}{9}$ **(ii)** $\frac{1}{9}$

Practise Now 11

1. **(iii)** **(a)** $\frac{16}{63}$ **(b)** $\frac{32}{63}$
(c) $\frac{43}{63}$
2. **(iii)** **(a)** $\frac{7}{30}$ **(b)** $\frac{7}{15}$
(c) $\frac{49}{120}$

Exercise 2A

1. **(a)** {1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}
(b) {head, tail}
(c) {B_1, B_2, B_3, B_4, W_1, W_2}
(d) {S_1, T_1, U, D, E, N, T_2, S_2}
2. **(i)** {G_1, G_2, Y_1, Y_2, Y_3}
(ii) **(a)** $\frac{3}{5}$ **(b)** $\frac{2}{5}$
(iii) **(a)** 0 **(b)** 1
3. **(i)** {P, R, O, B_1, A, B_2, I_1, L, I_2, T, Y}
(ii) **(a)** $\frac{2}{11}$ **(b)** $\frac{1}{11}$
(c) $\frac{3}{11}$ **(d)** 0
(iii) **(a)** $\frac{4}{11}$ **(b)** $\frac{7}{11}$
4. 110
5. **(i)** $\frac{3}{8}$ **(ii)** $\frac{1}{4}$
(iii) $\frac{3}{8}$
6. **(i)** $\frac{1}{8}$ **(ii)** $\frac{1}{4}$
(iii) $\frac{5}{8}$
7. **(ii)** **(a)** $\frac{1}{6}$ **(b)** $\frac{5}{6}$
(c) $\frac{1}{2}$ **(d)** $\frac{1}{2}$
8. **(ii)** **(a)** $\frac{1}{3}$ **(b)** $\frac{1}{3}$
(c) $\frac{2}{3}$
9. **(i)** 0, 1, 2, 3, 4, 5;
2, 3, 5, 6;
2, 3, 4, 5, 6, 7;
3, 4, 5, 6, 7, 8;
4, 6, 7, 8, 9;
5, 6, 7, 8, 9, 10
(ii) 36
(iii) **(a)** $\frac{1}{9}$ **(b)** $\frac{17}{36}$
(c) $\frac{19}{36}$ **(d)** $\frac{1}{2}$
(e) $\frac{1}{2}$
(iv) 7
10. **(i)** 12, 13;
12, 13, 14;
13, 15
28, 35;
32, 48;
36, 45, 54
(ii) **(a)** $\frac{4}{9}$ **(b)** $\frac{2}{3}$
(c) $\frac{8}{9}$
(iii) **(a)** $\frac{2}{9}$ **(b)** $\frac{7}{9}$
(c) $\frac{5}{9}$
11. **(ii)** **(a)** $\frac{1}{8}$ **(b)** $\frac{3}{8}$
(c) $\frac{1}{2}$ **(d)** $\frac{1}{8}$
12. **(i)** $\frac{1}{8}$ **(ii)** $\frac{3}{8}$
(iii) $\frac{3}{8}$
13. **(ii)** **(a)** $\frac{1}{6}$ **(b)** $\frac{1}{3}$
14. **(i)** $\frac{70}{81}$ **(ii)** $\frac{8}{27}$
(iii) $\frac{5}{9}$ **(iv)** $\frac{11}{81}$
15. **(i)** $\frac{1}{52}$ **(ii)** $\frac{1}{2}$
(iii) $\frac{1}{4}$ **(iv)** $\frac{3}{4}$
16. **(i)** {11, 14, 16, 41, 44, 46, 61, 64, 66}
(ii) **(a)** $\frac{1}{9}$ **(b)** $\frac{2}{9}$
(c) $\frac{1}{3}$ **(d)** $\frac{2}{3}$
17. $\frac{1}{6}$
18. **(i)** **(a)** $\frac{1}{5}$ **(b)** $\frac{1}{5}$
(c) $\frac{4}{5}$ **(d)** $\frac{6}{25}$
(ii) $\frac{2}{5}$ **(iii)** $\frac{4}{25}$
19. **(i)** 2, 3, 4, 5, 6; 2, 4, 8, 10, 12
(ii) **(a)** $\frac{1}{4}$ **(b)** $\frac{3}{4}$
(c) $\frac{1}{3}$ **(d)** $\frac{5}{6}$
(e) $\frac{1}{3}$
20. **(i)** 1, 2, 3, 4, 5;
1, 0, 2, 3;
2, 1, 0, 1, 2, 3;
3, 2, 1, 0, 1, 2;
4, 3, 2, 1, 0, 1;
5, 3, 2, 1
(ii) **(a)** $\frac{5}{18}$ **(b)** $\frac{5}{6}$
(c) $\frac{1}{2}$ **(d)** $\frac{4}{9}$
(e) $\frac{1}{3}$
21. **(ii)** **(a)** $\frac{2}{3}$ **(b)** $\frac{4}{9}$
(c) $\frac{2}{3}$ **(d)** $\frac{2}{3}$
(e) $\frac{4}{9}$ **(f)** $\frac{2}{9}$
(g) $\frac{1}{3}$
22. **(i)** $\frac{1}{2}$ **(ii)** $\frac{3}{10}$
(iii) $\frac{2}{5}$ **(iv)** $\frac{1}{5}$
23. {(head, 1), (head, 2), (head, 3), (head, 4), (head, 5), (head, 6), (tail, 1), (tail, 2), (tail, 3), (tail, 4), (tail, 5), (tail, 6)}
24. **(i)** **(a)** $\frac{4}{15}$ **(b)** $\frac{3}{5}$
(c) $\frac{11}{15}$
(ii) $\frac{4}{15}$
25. $\frac{5}{24}$
26. **(i)** $x = 3, y = 15$
(ii) **(a)** $\frac{5}{8}$
(b) $\frac{11}{16}$

Exercise 2B

1. **(i)** $\frac{5}{11}$ **(ii)** $\frac{4}{11}$
(iii) $\frac{9}{11}$ **(iv)** $\frac{2}{11}$
2. **(i)** $\frac{7}{15}$ **(ii)** $\frac{1}{3}$
(iii) $\frac{4}{5}$ **(iv)** $\frac{1}{5}$

3. **(i)** $\frac{5}{6}$ **(ii)** $\frac{1}{6}$

4. **(i)** $\frac{4}{7}$ **(ii)** $\frac{5}{14}$
(iii) $\frac{3}{14}$

5. **(i)** $\frac{61}{120}$ **(ii)** $\frac{59}{120}$
(iii) $\frac{13}{24}$

6. **(i)** $\frac{3}{17}$ **(ii)** $\frac{2}{17}$
(iii) $\frac{5}{17}$ **(iv)** $\frac{10}{17}$
(v) $\frac{13}{17}$ **(vi)** $\frac{7}{17}$

7. **(i)** $\frac{2}{13}$ **(ii)** $\frac{11}{13}$
(iii) $\frac{5}{13}$ **(iv)** $\frac{5}{13}$
(v) $\frac{7}{13}$

8. **(ii)** **(a)** $\frac{7}{15}$ **(b)** $\frac{17}{30}$
(c) $\frac{8}{15}$ **(d)** $\frac{13}{30}$

9. **(ii)** **(a)** $\frac{8}{15}$ **(b)** $\frac{167}{385}$
(c) $\frac{4}{5}$ **(d)** $\frac{218}{385}$

10. **(ii)** **(a)** Yes **(b)** No
(c) No **(d)** Yes
(e) No **(f)** No

11. **(i)** 4 **(ii)** No

Exercise 2C

1. **(i)** Arranged vertically:
$\frac{5}{9}, \frac{4}{9};$
$\frac{5}{9}, \frac{4}{9}, \frac{5}{9}, \frac{4}{9}$
(ii) **(a)** $\frac{5}{9}$ **(b)** $\frac{4}{9}$
(c) $\frac{20}{81}$ **(d)** $\frac{4}{9}$

2. **(i)** Arranged vertically:
$\frac{3}{5}, \frac{2}{5};$
$\frac{3}{5}, \frac{2}{5}, \frac{3}{5}, \frac{2}{5}$
(ii) **(a)** $\frac{9}{25}$ **(b)** $\frac{12}{25}$
(c) $\frac{2}{5}$

3. **(i)** $\frac{7}{12}$ **(ii)** $\frac{7}{24}$
(iii) $\frac{3}{8}$

4. **(i)** 0.035 **(ii)** 0.065
(iii) 0.38

5. **(i)** Arranged vertically:
$\frac{5}{8};$
$\frac{2}{7}, \frac{5}{7}, \frac{3}{7}, \frac{4}{7}$
(ii) **(a)** $\frac{15}{56}$ **(b)** $\frac{15}{28}$
(c) $\frac{3}{28}$

6. **(iii)** **(a)** $\frac{1}{3}$ **(b)** $\frac{8}{15}$
(c) $\frac{2}{3}$

7. **(iii)** **(a)** $\frac{1}{15}$ **(b)** $\frac{2}{15}$
(c) $\frac{1}{9}$

8. **(iii)** **(a)** $\frac{2}{3}$ **(b)** $\frac{15}{22}$
(c) $\frac{5}{22}$ **(d)** $\frac{5}{11}$
(e) $\frac{2}{3}$

9. **(i)** $\frac{1}{5}$
(ii) **(a)** $\frac{1}{10}$ **(b)** $\frac{2}{5}$
(c) $\frac{3}{10}$ **(d)** $\frac{3}{5}$

10. **(i)** Arranged vertically:
$\frac{1}{6}, \frac{1}{3}, \frac{1}{2};$
$\frac{5}{6}, \frac{1}{6}, \frac{5}{6}, \frac{1}{6}, \frac{5}{6}, \frac{1}{6};$
$\frac{5}{6}, \frac{1}{6}, \frac{5}{6}, \frac{1}{6}, \frac{5}{6}, \frac{1}{6}, \frac{5}{6},$
$\frac{1}{6}, \frac{5}{6}, \frac{1}{6}, \frac{5}{6}, \frac{1}{6}$
(ii) **(a)** $\frac{5}{108}$ **(b)** $\frac{1}{72}$
(c) $\frac{1}{36}$ **(d)** $\frac{5}{18}$
(e) $\frac{7}{24}$

11. **(i)** Arranged vertically:
$\frac{1}{2}, \frac{1}{4};$
$\frac{1}{4}, \frac{3}{4}, \frac{1}{4}, \frac{3}{4}, \frac{1}{4}, \frac{3}{4};$
40, 20, 30, 60
(ii) **(a)** $\frac{3}{4}$ **(b)** $\frac{1}{4}$
(iii) **(a)** $\frac{3}{8}$ **(b)** $\frac{9}{16}$
(c) $\frac{15}{16}$ **(d)** $\frac{1}{16}$

12. **(i)** $\frac{9}{80}$ **(ii)** $\frac{11}{80}$
(iii) $\frac{299}{800}$

13. **(i)** 0.001 08
(ii) 0.597 52
(iii) 0.402 48
(iv) 0.045 36

14. **(i)** **(a)** $\frac{7}{30}$ **(b)** $\frac{2}{5}$
(c) $\frac{11}{30}$
(ii) $\frac{1}{10}$

15. **(ii)** **(a)** 0 **(b)** $\frac{17}{33}$
(c) $\frac{5}{44}$

16. **(i)** $\frac{1}{37}$ **(ii)** $\frac{508}{1295}$

17. **(i)** $\frac{3}{65}$ **(ii)** $\frac{21}{520}$
(iii) $\frac{63}{260}$

18. **(i)** Arranged vertically:
$\frac{n-8}{n};$
$\frac{n-8}{n-1}, \frac{8}{n-1}, \frac{n-9}{n-1}$
(iii) −13 or 14
(v) $\frac{48}{91}$

19. **(i)** Arranged vertically:
$\frac{1}{5}, \frac{4}{5};$
$\frac{1}{4}, \frac{3}{4}, \frac{1}{4}, \frac{3}{4}$
(ii) Yes

20. $\frac{1}{7}$

21. **(i)** **(a)** $\frac{25}{216}$
(b) $\frac{125}{1296}$
(c) $\frac{671}{1296}$
(ii) **(a)** $\frac{5}{108}$
(b) $\frac{1}{36}$

Chapter 3 Statistical Data Analysis

Practise Now 1A

(ii) **(a)** 32 **(b)** 18
(c) 28
(iv) $\frac{4}{5}$

Practise Now 1B

(i) 35; 55; 65; 70
(iii) **(a)** 50 **(b)** $\frac{41}{70}$
(c) 6.4
(iv) $\frac{35}{69}$

Practise Now 2

1. **(i)** 14; 30.5; 44
(ii) 30 **(iii)** 49

2. **(i)** 26; 15.5; 41
(ii) 25.5; 45

Practise Now 3

100 minutes

Practise Now 4

(i) 75; 27; 121.5
(ii) 94.5; 135
(iii) 12; 126
(iv) 51

Practise Now 5

(i) **(a)** 62 minutes
(b) 18 minutes
(ii) **(a)** 70 minutes
(b) 23 minutes
(iii) School B
(iv) School A

Practise Now 6

(i) 13; 23

Practise Now 7

(i) 17.9 cm; 18 cm; 17.1 cm; 1.8 cm

Practise Now 8

(ii) Strong positive correlation
(iv) 12.9 litres **(v)** No

Practise Now 9

(ii) Strong negative correlation
(iv) 12.7 seconds **(v)** No

Exercise 3A

1. **(i)** 15; 24; 35; 52; 71; 91; 105; 115; 120
(ii) **(a)** 24 **(b)** 15
(c) 80
(iv) $\frac{1}{8}$

2. **(ii)** **(a)** 26 **(b)** 4
(c) 5
(iv) $\frac{4}{31}$

3. **(ii)** **(a)** 3 **(b)** 9
(c) 32
(iv) $\frac{2}{5}$

4. **(i)** **(a)** 26 **(b)** 24
(c) 81%
(ii) $\frac{6}{25}$

5. **(i)** **(a)** 180 **(b)** $\frac{4}{5}$
(c) 29.8
(ii) $\frac{13}{20}$

6. **(i)** **(a)** 30 **(b)** 70%
(c) 4
(ii) $\frac{1}{20}$

8. **(i)** 16; 40; 61; 73; 80
(iii) **(a)** 12 **(b)** $\frac{1}{5}$
(c) 36
(iv) $\frac{87}{632}$
9. **(i)** **(a)** 40
(b) $26\frac{2}{3}$% or 26.7%
(c) 47
(iii) $\frac{29}{30}$
(iv) 0; 1; 4; $x \leqslant 30$, 11; $x \leqslant 40$, 23; $x \leqslant 50$, 46; $x \leqslant 60$, 56
(v) **(a)** 1; 3; 7; 12; 23; 10; 4; 0
(b) 41.5 minutes
10. **(i)** **(a)** 55% **(b)** 52
(c) 24 **(e)** $\frac{1139}{1580}$
(ii) **(a)** 2; 6; 18; 44; 10
(b) 55.875 g
11. **(i)** 230; 161; 98; 70; 46; 27
(iii) **(a)** 46 **(b)** 52
(c) 63
(iv) $\frac{1}{5}$
12. **(iii)** $10 \leqslant x < 15$
(iv) $\frac{11}{245}$
13. **(i)** **(a)** 320 **(b)** $\frac{42}{73}$
(c) 31.6
(ii) $\frac{51}{73}$
14. **(i)** **(a)** 13 **(b)** $\frac{23}{50}$
(c) 23
(iii) $\frac{23}{110}$
(iv) 3; 8; $u \leqslant 20$, 20; $u \leqslant 25$, 54; $u \leqslant 30$, 76; $u \leqslant 35$, 90; 100
15. **(i)** Group B
(ii) Group C

Exercise 3B

1. **(a)** 8; 4; 6; 8; 4
(b) 29; 58.5; 65; 71; 12.5
(c) 31; 12; 18; 29.5; 17.5
(d) 19.7; 6.7; 11.1; 15.1; 8.4
2. **(i)** 1; 7.5; 24
(ii) 23; 29
3. 25 hours
4. **(i)** \$97; \$88; \$105
(ii) \$17; \$60
(iii) \$85; \$110
5. **(i)** **(a)** 35 cm
(b) 30 cm; 39 cm
(c) 31 cm; 36 cm
(d) 4
(ii) $\frac{1}{2}$
6. **(i)** **(a)** 42 cents
(b) 58 cents
(ii) **(a)** 26 cents
(b) 24 cents
(iii) School B
7. **(i)** **(a)** 350 kg
(b) 500 kg
(ii) **(a)** 170 kg
(b) 190 kg
(iii) Factory Q
8. **(i)** Soil A **(ii)** Soil A
9. **(i)** 6 years **(ii)** 5.5 years
(iii) 4 years; 7 years
(iv) 3 years
10. **(i)** **(a)** 10 minutes; 13 minutes; 15.25 minutes
(b) 5.25 minutes
(ii) $26\frac{2}{3}$% or 26.7%
(iii) Median waiting time
11. **(i)** 21; 28; 34
(ii) 38 **(iii)** 8.5
(iv) 2.63% **(v)** No
12. **(a)** 25 minutes
(b) 200
(c) 52 minutes
(d) **(i)** Weekday
(ii) Yes
13. **(i)** **(a)** 84 **(b)** 51.5
(c) 57
(ii) **(a)** 229 **(b)** 146
(c) 97
(iii) City Y
14. {3, 8, 9, 10, 19, 19, 38}
15. **(i)** **(a)** 5.5 hours; 7 hours; 6 hours; 7.2 hours
(b) 1.5 hours
(ii) $\frac{2}{5}$ **(iii)** $\frac{2}{13}$

Exercise 3C

1. **(i)** Group A: 8 days; 8 days
Group B: 7.875 days; 25 days
2. **(i)** 13.4 minutes; 22 minutes
(ii) 14.4 minutes; 22 minutes
3. **(i)** Train A: 5.275 minutes; 7 minutes
Train B: 4.975 minutes; 8 minutes
4. **(i)** **(a)** 25 minutes
(b) 8 minutes
5. **(i)** **(a)** City A: 51.9 °C;
City B: 48.6 °C
(b) City A: 20 °C;
City B: 25 °C
(c) City A: $50 \leqslant x < 55$
City B: $45 \leqslant x < 50$
(d) City A: $50 \leqslant x < 55$
City B: $45 \leqslant x < 50$
(ii) City A **(iii)** City B
6. **(i)** Class P: 2.17 hours; 1.5 hours; 0 hour; 5 hours
Class Q: 3.92 hours; 4 hours; 4 hours; 4 hours
(ii) Class Q **(iii)** Class Q
7. **(i)** Yes **(ii)** No
(iii) 55.3 kg; 40 kg

Exercise 3D

1. **(a)** Strong negative correlation
(b) Moderate positive correlation
(c) Strong positive correlation
(d) No correlation
2. \$6900
3. **(ii)** No correlation
4. **(ii)** Strong positive correlation
(iv) 137 mm Hg **(v)** No
5. **(ii)** Strong negative correlation
(iv) 67 years **(v)** No

Chapter 4 Vectors

Practise Now 1A

$\overrightarrow{AB} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$, 5 units;
$\mathbf{c} = \begin{pmatrix} -2 \\ -2 \end{pmatrix}$, 2.83 units;
$\overrightarrow{DE} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$, 5 units;
$\mathbf{f} = \begin{pmatrix} 0 \\ -3 \end{pmatrix}$, 3 units

Practise Now 2

(i) **(a)** $x = 4, y = 4\frac{1}{2}$
(b) $-\mathbf{a} = \begin{pmatrix} -6 \\ \frac{1}{2} \end{pmatrix}$
(ii) **(a)** $y = \frac{105 - 24x}{2}$

Practise Now 3

(i) $\overrightarrow{PB} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$
(ii) $\overrightarrow{BQ} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$
(iii) $\overrightarrow{PR} = \begin{pmatrix} -4 \\ -1 \end{pmatrix}$
(iv) Yes; No

Practise Now 4

1. **(ii)** $\mathbf{a} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$, $\mathbf{a} + \mathbf{b} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$
(iii) $\mathbf{a} + \mathbf{b} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$
(iv) $|\mathbf{a}| = \sqrt{34}$ units, $|\mathbf{b}| = \sqrt{20}$ units, $|\mathbf{a} + \mathbf{b}| = \sqrt{26}$ units
(v) No
2. **(a)** $\begin{pmatrix} 8 \\ 12 \end{pmatrix}$
(b) $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$

Practise Now 5

(i) $\overrightarrow{PR}$ **(ii)** $\overrightarrow{PR}$
(iii) $\overrightarrow{PQ}$

Practise Now 6

(a) $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ **(b)** 6; −7

Practise Now 7

(ii) $\mathbf{r} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$, $\mathbf{s} = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$, $\mathbf{r} - \mathbf{s} = \begin{pmatrix} -3 \\ 3 \end{pmatrix}$
(iii) $\mathbf{r} - \mathbf{s} = \begin{pmatrix} -3 \\ 3 \end{pmatrix}$

(iv) $|\mathbf{r}| = \sqrt{34}$ units,
$|\mathbf{s}| = \sqrt{40}$ units,
$|\mathbf{r} - \mathbf{s}| = \sqrt{18}$ units

(v) No

Practise Now 8

(i) $\mathbf{b} - \mathbf{a}$ (ii) $\mathbf{a} - \mathbf{b}$
(iii) $\mathbf{m} - \mathbf{n}$ (iv) $\mathbf{v} + \mathbf{w}$
(v) $-\mathbf{w} - \mathbf{v}$

Practise Now 9

(i) $\overrightarrow{PR} = \mathbf{q}$
(ii) $\overrightarrow{RQ} = -\mathbf{p}$
(iii) $\overrightarrow{OR} = \mathbf{p} + \mathbf{q}$
(iv) $\overrightarrow{RO} = -\mathbf{p} - \mathbf{q}$
(v) $\overrightarrow{PQ} = \mathbf{q} - \mathbf{p}$
(vi) $\overrightarrow{QP} = \mathbf{p} - \mathbf{q}$

Practise Now 10

(a) $\overrightarrow{AC}$ (b) $\overrightarrow{CB}$
(c) $\overrightarrow{AB} - \overrightarrow{BC}$ (d) $\overrightarrow{RQ}$
(e) $\overrightarrow{PR}$ (f) $\overrightarrow{SQ}$

Practise Now 11

(a) (i) $\begin{pmatrix} -7 \\ 9 \end{pmatrix}$
(ii) $\begin{pmatrix} -3 \\ 0 \end{pmatrix}$
(b) (i) $x = 13, y = -10$
(ii) $x = -1, y = -10$

Practise Now 12

1. (a) (i) Parallel
(ii) Not parallel
(iii) Parallel
(b) $\begin{pmatrix} 8 \\ -6 \end{pmatrix}; \begin{pmatrix} -4 \\ 3 \end{pmatrix}$

2. −9

Practise Now 13

1. (i) $\begin{pmatrix} -5 \\ 5 \end{pmatrix}$
(ii) $\begin{pmatrix} 4 \\ 8 \end{pmatrix}$

2. $x = 1.5, y = 3$

Practise Now 14

(a) $\overrightarrow{OP} = \begin{pmatrix} 8 \\ -2 \end{pmatrix}$,
$\overrightarrow{OQ} = \begin{pmatrix} -1 \\ 7 \end{pmatrix}$,
$\overrightarrow{PQ} = \begin{pmatrix} -9 \\ 9 \end{pmatrix}$
(b) $B(2, -2)$

Practise Now 15

$C(1, -9)$

Practise Now 16

(i) (a) $\overrightarrow{AC} = 8\mathbf{a} + 4\mathbf{b}$
(b) $\overrightarrow{DF} = 2\mathbf{a}$
(c) $\overrightarrow{CF} = -6\mathbf{a}$
(d) $\overrightarrow{EF} = \frac{1}{3}(6\mathbf{a} - 4\mathbf{b})$
(ii) (a) $\frac{1}{9}$ (b) $\frac{1}{3}$

Practise Now 17

(i) $\overrightarrow{PQ} = 3\mathbf{q} - 9\mathbf{p}$;
$\overrightarrow{RS} = \frac{3}{2}(\mathbf{q} - 2\mathbf{p})$
(ii) $\overrightarrow{TS} = \frac{3}{4}(\mathbf{q} - 3\mathbf{p})$

Exercise 4A

1. (a) 5 units (b) 13 units
(c) 7.28 units
(d) 6.5 units
(e) 8 units

2. (a) $\begin{pmatrix} -12 \\ 7 \end{pmatrix}$
(b) $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$
(c) $\begin{pmatrix} -4 \\ -8 \end{pmatrix}$
(d) $\begin{pmatrix} 3 \\ 1.2 \end{pmatrix}$
(e) $\begin{pmatrix} 0 \\ -3\frac{1}{4} \end{pmatrix}$

3. $a = -2, b = 2\frac{1}{2}$

4. (i) 7 units
(ii) (a) $\overrightarrow{DC} = \begin{pmatrix} 7 \\ 0 \end{pmatrix}$
(b) $\overrightarrow{DA} = \begin{pmatrix} 3 \\ -4 \end{pmatrix}$

5. $\overrightarrow{AB} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$, 5 units;
$\overrightarrow{CD} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$, 2.24 units;
$\overrightarrow{RS} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}$, 2 units;
$\overrightarrow{TU} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$, 4 units;
$\mathbf{p} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$, 4.24 units;
$\mathbf{q} = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$, 2.24 units

7. (i) (a) $x = 4, y = 5\frac{1}{2}$
(b) $-\mathbf{a} = \begin{pmatrix} -1 \\ 3\frac{1}{2} \end{pmatrix}$
(ii) (a) $y = \frac{93 - 4x}{14}$

9. (i) $\overrightarrow{AY} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$
(ii) $\overrightarrow{YB} = \begin{pmatrix} 2 \\ 6 \end{pmatrix}$
(iii) $\overrightarrow{AC} = \begin{pmatrix} -3 \\ -2 \end{pmatrix}$
(iv) Yes; No

10. $n = \pm\sqrt{40}$

12. $s = 2, t = 1$

13. (i) (b) $\overrightarrow{DC}$ and $\overrightarrow{HG}$
(ii) (a) $\overrightarrow{JA}$, $\overrightarrow{GD}$ and $\overrightarrow{FE}$
(b) $\overrightarrow{GF}$, $\overrightarrow{KJ}$ and $\overrightarrow{LA}$
(c) $\overrightarrow{AD}$, $\overrightarrow{JG}$ and $\overrightarrow{IH}$
(d) $\overrightarrow{EG}$
(v) (a) $\overrightarrow{CB}$
(b) $\overrightarrow{GD}$
(c) $\overrightarrow{FG}$

Exercise 4B

1. (a) $\begin{pmatrix} 9 \\ 9 \end{pmatrix}$
(b) $\begin{pmatrix} -1 \\ -4 \end{pmatrix}$
(c) $\begin{pmatrix} -12 \\ -3 \end{pmatrix}$

2. (i) Yes (ii) Yes

3. (i) $\overrightarrow{LN}$ (ii) $\overrightarrow{LN}$
(iii) $\overrightarrow{LP}$

4. (a) $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ (b) $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$
(c) $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

5. (a) −9; 1 (5;
(c) $q; -p$

6. (a) 0 (b) 0
(c) 0

7. (i) $\mathbf{p} - \mathbf{q}$ (ii) $\mathbf{q} - \mathbf{p}$
(iii) $\mathbf{b} - \mathbf{a}$ (iv) $\mathbf{a} + \mathbf{b}$
(v) $\mathbf{s} - \mathbf{r}$ (vi) $\mathbf{r} + \mathbf{s}$
(vii) $-\mathbf{m} - \mathbf{n}$ (viii) $\mathbf{n} - \mathbf{m}$

8. (a) $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ (b) $\begin{pmatrix} 2 \\ 7 \end{pmatrix}$
(c) $\begin{pmatrix} 0 \\ 4 \end{pmatrix}$ (d) $\begin{pmatrix} 3 \\ 8 \end{pmatrix}$

10. (i) $\overrightarrow{PR}$ (ii) $\overrightarrow{SR}$
(iii) $\overrightarrow{SR}$ (iv) $\overrightarrow{ST}$
(v) $\overrightarrow{PR}$ (vi) $\overrightarrow{RS}$

12. (i) $\overrightarrow{RT} = \mathbf{s}$ (ii) $\overrightarrow{TS} = -\mathbf{r}$
(iii) $\overrightarrow{OT} = \mathbf{r} + \mathbf{s}$
(iv) $\overrightarrow{RS} = \mathbf{s} - \mathbf{r}$
(v) $\overrightarrow{SR} = \mathbf{r} - \mathbf{s}$

13. (a) $\overrightarrow{RT}$ (b) $\overrightarrow{TS}$
(c) $\overrightarrow{ST}$ (d) $\overrightarrow{RS} - \overrightarrow{ST}$
(e) $\overrightarrow{RT}$ (f) $\overrightarrow{US}$

14. (a) $x = 10, y = -7$
(b) $x = 9, y = 1$
(c) $x = -3, y = 10$
(d) $x = 3, y = 5$

16. (ii) Yes (iii) Yes

17. (i) (a) $\overrightarrow{PR}$
(b) $\overrightarrow{RQ}$
(c) $\overrightarrow{PQ}$
(ii) (a) $\overrightarrow{SR} = \mathbf{a}$
(b) $\overrightarrow{PR} = \mathbf{a} + \mathbf{b}$
(c) $\overrightarrow{SQ} = \mathbf{a} - \mathbf{b}$

18. (a) $\overrightarrow{KS}$ (b) $\overrightarrow{QS}$
(c) $\overrightarrow{PR}$ (d) $\overrightarrow{PS}$
(e) $\overrightarrow{PR}$ (f) 0

Exercise 4C

1. (a) Parallel
(b) Not parallel
(c) Parallel

2. (a) $\begin{pmatrix} 16 \\ -14 \end{pmatrix}; \begin{pmatrix} -24 \\ 21 \end{pmatrix}$
(b) $\begin{pmatrix} 1 \\ 3 \end{pmatrix}; \begin{pmatrix} -2 \\ -6 \end{pmatrix}$
(c) $\begin{pmatrix} -3 \\ -1 \end{pmatrix}; \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

3. **(a)** $\begin{pmatrix} 17 \\ -4 \end{pmatrix}$

(b) $\begin{pmatrix} 12 \\ 7\frac{1}{2} \end{pmatrix}$

(c) $\begin{pmatrix} -1 \\ 13 \end{pmatrix}$

4. **(a)** $\overrightarrow{OA} = \begin{pmatrix} 4 \\ 7 \end{pmatrix}$

(b) $\overrightarrow{OB} = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$

(c) $\overrightarrow{OC} = \begin{pmatrix} 6 \\ -1 \end{pmatrix}$

(d) $\overrightarrow{OD} = \begin{pmatrix} -4 \\ -9 \end{pmatrix}$

5. **(i)** $\overrightarrow{PQ} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$

(ii) $\overrightarrow{QR} = \begin{pmatrix} 0 \\ 7 \end{pmatrix}$

(iii) $\overrightarrow{RP} = \begin{pmatrix} 1 \\ -5 \end{pmatrix}$

(iv) $\overrightarrow{PR} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$

6. **(a)** Parallel
(b) Parallel
(c) Not parallel

7. **(a)** 8 **(b)** –4

8. **(a)** $x = 2, y = 1$
(b) $x = -7, y = 4$
(c) $x = 3, y = 1$

10. $\overrightarrow{LM} = 2\mathbf{a} + \mathbf{b}$;
$\overrightarrow{PR} = \frac{3}{2}\mathbf{b} - \mathbf{a}$;
$\overrightarrow{ST} = 2\mathbf{a} + 2\mathbf{b}$;
$\overrightarrow{XY} = \frac{5}{2}\mathbf{a} - 2\mathbf{b}$

11. $B(-5, 4)$

12. **(i)** $\overrightarrow{CD} = \begin{pmatrix} 6 \\ -10 \end{pmatrix}$
(ii) $B(7, -8)$
(iii) $C(2, 5)$

14. $k = \frac{1}{3}$; $\mathbf{v} = \begin{pmatrix} -45 \\ 24 \end{pmatrix}$

15. **(i)** $2\overrightarrow{AB} + 5\overrightarrow{CD} = \begin{pmatrix} -1 \\ 30 \end{pmatrix}$
(ii) –4.5

16. **(i)** $\overrightarrow{LM} = \begin{pmatrix} t+3 \\ 4 \end{pmatrix}$
(ii) 29 **(iii)** 4 or –10

17. **(i)** $Q(10, -5)$
(ii) $-\frac{1}{4}$ **(iii)** $\frac{y}{x}$
(iv) $\overrightarrow{PQ} = k\begin{pmatrix} x \\ y \end{pmatrix}$, where k is a constant

Exercise 4D

1. $C(9, 11)$

2. **(i)** $\overrightarrow{CM} = -\frac{1}{2}\mathbf{q}$
(ii) $\overrightarrow{DB} = \mathbf{p} - \mathbf{q}$
(iii) $\overrightarrow{AM} = \mathbf{p} + \frac{1}{2}\mathbf{q}$
(iv) $\overrightarrow{MD} = \frac{1}{2}\mathbf{q} - \mathbf{p}$

3. **(i)** $\overrightarrow{BC} = \frac{4}{3}\mathbf{q}$
(ii) $\overrightarrow{AD} = \mathbf{q} - \mathbf{p}$
(iii) $\overrightarrow{CA} = \mathbf{p} - \frac{4}{3}\mathbf{q}$

4. **(i)** $\overrightarrow{MR} = \mathbf{a} + 2\mathbf{b}$
(ii) $\overrightarrow{RN} = -\frac{8}{3}\mathbf{b}$
(iii) $\overrightarrow{NM} = \frac{2}{3}\mathbf{b} - \mathbf{a}$

5. $\overrightarrow{BM} = \frac{1}{2}\mathbf{a} - \mathbf{b}$

6. **(i)** $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$
(ii) $\overrightarrow{AC} = \frac{2}{5}(\mathbf{b} - \mathbf{a})$
(iii) $\overrightarrow{OC} = \frac{1}{5}(2\mathbf{b} + 3\mathbf{a})$

7. **(i)** $\overrightarrow{BC} = \mathbf{v} - \mathbf{u}$
(ii) $\overrightarrow{AM} = \frac{1}{2}\mathbf{u}$
(iii) $\overrightarrow{AN} = \frac{1}{2}\mathbf{v}$
(iv) $\overrightarrow{MN} = \frac{1}{2}(\mathbf{v} - \mathbf{u})$
Parallel

8. **(a)** $S(2, 2)$ **(b)** $S(0, -2)$

9. **(i)** $\overrightarrow{BC} = \mathbf{v} - \mathbf{u}$
(ii) $\overrightarrow{BE} = \frac{2}{5}(\mathbf{v} - \mathbf{u})$
(iii) $\overrightarrow{AD} = \frac{3}{2}\mathbf{u} + \mathbf{v}$
(iv) $\overrightarrow{AE} = \frac{1}{5}(3\mathbf{u} + 2\mathbf{v})$
(v) $\overrightarrow{BD} = \mathbf{v} + \frac{1}{2}\mathbf{u}$

10. **(i)** $\overrightarrow{PR} = 15\mathbf{b} - 15\mathbf{a}$
(ii) $\overrightarrow{PA} = \frac{15}{4}(\mathbf{b} - \mathbf{a})$
(iii) $\overrightarrow{OA} = \frac{15}{4}(\mathbf{b} + 3\mathbf{a})$
(iv) $\overrightarrow{OB} = 15\mathbf{a} + 5\mathbf{b}$

11. **(i)** $\overrightarrow{PC} = 20\mathbf{q} - 8\mathbf{p}$
(ii) $\overrightarrow{PB} = 5\mathbf{q} - 2\mathbf{p}$
(iii) $\overrightarrow{OB} = 6\mathbf{p} + 5\mathbf{q}$
(iv) $\overrightarrow{QB} = 6\mathbf{p} - 3\mathbf{q}$

12. **(i)** **(a)** $\overrightarrow{PQ} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$

(b) $\overrightarrow{SR} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

(c) $\overrightarrow{RQ} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

(d) $\overrightarrow{TQ} = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$

(ii) $\frac{1}{2}$

13. **(i)** **(a)** $\overrightarrow{QP} = \mathbf{p} - \mathbf{q}$
(b) $\overrightarrow{QS} = \frac{2}{5}(\mathbf{p} - \mathbf{q})$
(c) $\overrightarrow{OS} = \frac{1}{5}(2\mathbf{p} + 3\mathbf{q})$
(d) $\overrightarrow{ST} = \frac{1}{10}(9\mathbf{q} - 4\mathbf{p})$
(ii) **(b)** collinear;
$RS : ST = 2 : 3$

14. **(i)** $\overrightarrow{BC} = \begin{pmatrix} 2 \\ 8 \end{pmatrix}$
(ii) $\overrightarrow{AM} = \begin{pmatrix} 5 \\ -1 \end{pmatrix}$
(iii) $D(3, 10)$

15. **(i)** **(a)** $\overrightarrow{SA} = \frac{2}{3}\mathbf{b}$
(b) $\overrightarrow{QB} = \frac{2}{3}\mathbf{a}$
(c) $\overrightarrow{PB} = \frac{2}{3}\mathbf{a} + \mathbf{b}$
(d) $\overrightarrow{QS} = \mathbf{a} - \mathbf{b}$
(e) $\overrightarrow{BA} = \frac{1}{3}(\mathbf{a} - \mathbf{b})$
(ii) **(a)** $\frac{1}{3}$ **(b)** $\frac{1}{9}$
(c) $\frac{1}{18}$

16. **(i)** **(a)** $\overrightarrow{RS} = -3\mathbf{a} - 7\mathbf{b}$
(b) $\overrightarrow{RT} = -2\mathbf{a} - 8\mathbf{b}$
(c) $\overrightarrow{RQ} = \mathbf{a} - 11\mathbf{b}$
(iii) $\overrightarrow{QS} = 4(\mathbf{b} - \mathbf{a})$
(iv) **(a)** $\frac{3}{4}$ **(b)** $\frac{3}{4}$
(c) $\frac{1}{2}$

17. **(i)** **(a)** $\overrightarrow{OT} = 4\mathbf{q} - 3\mathbf{p}$
(b) $\overrightarrow{AT} = 4\mathbf{q} - 4\mathbf{p}$
(c) $\overrightarrow{OB} = \mathbf{p} + \mathbf{q}$
(d) $\overrightarrow{BT} = 3\mathbf{q} - 4\mathbf{p}$
(e) $\overrightarrow{TR} = \frac{3}{4}(4\mathbf{p} - 3\mathbf{q})$
(iii) **(a)** $\frac{3}{4}$ **(b)** $\frac{9}{16}$

Chapter 5 Relations an Functions

Practise Now 1

(a) Yes **(b)** No

Practise Now 2

(a) 44 **(b)** –66
(c) $-2\frac{2}{3}$ **(d)** 2
(e) 36 **(f)** $-2\frac{1}{2}$
(g) 7 **(h)** 3
(i) $-1\frac{2}{3}$

Practise Now 3

(a) $2b - 5$ **(b)** $7b + 5$
(c) $18b - 28$

Practise Now 4

(a) $36x^2 - 15x + 2$
(b) $16x^2 + 38x + 23$
(c) $4x^4 - 29x^2 + 53$

Practise Now 5

$a = 1\frac{4}{5}, b = -\frac{2}{5}; 1\frac{2}{5}, 47$

Practise Now 6

$\mathrm{f}^{-1}(x) = \frac{x-3}{8}$

Practise Now 7

$\mathrm{f}^{-1}(x) = \frac{x+4}{7}; 2, 0, \frac{29}{49}$

Practise Now 8

$\mathrm{f}^{-1}(x) = \frac{5x}{x-2}; x = 2; 7\frac{1}{2}, 3, -\frac{5}{7}$

Practise Now 9

$p = -4, q = 27$

Practise Now 10

$\mathrm{gf}(x) = \frac{16x^2}{(x-1)^2}$,
$\mathrm{fg}(x) = \frac{2}{(2x+3)(2x+5)}; 7\frac{1}{9}, \frac{2}{99}$

Practise Now 11

1. $a = -\frac{b}{2} - 1$
2. **(a)** 0 **(b)** 1

Exercise 5A

1. **(a)** Yes **(b)** No
(c) Yes **(d)** Yes
(e) No **(f)** Yes

3. 8, −28, −1, −7

4. **(a)** 3 **(b)** 9
(c) 5 **(d)** 10

5. **(a)** 18 **(b)** −17
(c) 8 **(d)** 1
(e) 2

6. **(a)** $a^2 + 5$ **(b)** $a^2 + 2a + 6$
(c) $4a$

7. **(a)** $\frac{1}{2}x(x-1)$
(b) $\frac{1}{2}(x+1)(x+2)$
(c) x **(d)** $\frac{1}{2}x^2(x^2+1)$

8. **(a)** **(i)** $3\frac{1}{2}$ **(ii)** $5\frac{1}{4}$
(iii) $2\frac{1}{2}$ **(iv)** 45
(b) 20, 28

9. **(a)** 5 **(b)** −2
(c) $\frac{2}{7}$ **(d)** 3
(e) 1 **(f)** $\frac{6}{7}$

10. **(a)** $3a^2 - 5a$ **(b)** 1 or 4
(c) $a^4 - 4a^2 - 5a + 8$

11. $m = -2\frac{1}{4}, c = 7\frac{1}{4}; \frac{1}{2}, 16\frac{1}{4}$

12. 13, 17, 21
(a) No **(b)** No
(c) No **(d)** No

13. $2, \frac{1}{8}, -\frac{3}{4}, 5\frac{1}{4}$
(a) No **(b)** No
(c) $\frac{9}{17}$
(d) $\frac{3}{4}a + \frac{1}{2}, \frac{3}{2}a + \frac{1}{2}, 1\frac{1}{4} - 2a$
(e) 30 **(f)** $\frac{3}{22}$

14. $p = 5, q = 6$; 106, 10

Exercise 5B

1. $f^{-1}(x) = 4x + 12$

2. $f^{-1}(x) = x + 7; 10, 14, 2, 7\frac{1}{3}$

3. $g^{-1}(x) = \frac{x-4}{3}; -\frac{1}{3}, -2\frac{2}{3}, -1\frac{1}{6}, -1\frac{7}{12}$

4. $h^{-1}(x) = \frac{x-6}{5}; 0, \frac{4}{5}, -1\frac{7}{25}, 1\frac{3}{10}$

5. $f^{-1}(x) = \frac{8-x}{3}; -\frac{1}{3}, 6\frac{2}{3}, 1\frac{5}{9}, 2\frac{35}{48}$

6. $3, 8, \frac{2}{3}, \frac{1}{3}$

7. **(a)** $6\frac{2}{3}$ **(b)** 40
(c) 44 **(d)** 0
(e) $36\frac{1}{3}$ **(f)** −35

8. $a = 2, b = -24$

9. $f^{-1}(x) = \frac{2x}{4x+5}; x = -\frac{5}{4}; \frac{8}{21}, \frac{12}{19}$

10. $f^{-1}(x) = \frac{2x-1}{x-3}; x = 3; 4\frac{1}{2}, 3\frac{1}{4}$

11. $p = 5, q = -3$; 8, 14

12. $a = 16, b = 12;$
$f^{-1}(x) = \frac{x+3}{4}; 2\frac{1}{2}, -\frac{5}{8}$

13. $a = 1, b = 2; f^{-1}(x) = x - 2$

14. $p = 1\frac{2}{3}, q = -6\frac{2}{3};$
$f^{-1}(x) = \frac{3}{5}x + 4$

15. $7, 3\frac{1}{2}$

Exercise 5C

1. **(a)** $fg(x) = -3x + 5,$
$gf(x) = -3x + 31$
(b) $fg(x) = 2x^2 + 7,$
$gf(x) = 4x^2 - 12x + 14$
(c) $fg(x) = \frac{4}{x} - 1,$
$gf(x) = \frac{4}{x-1}$
(d) $fg(x) = \frac{8}{x} - 7,$
$gf(x) = \frac{4}{2x-1} - 3$
(e) $fg(x) = \frac{3}{x+1} + 2,$
$gf(x) = \frac{3}{x+3}$
(f) $fg(x) = 12x + 15,$
$gf(x) = 12x + 5$
(g) $fg(x) = \frac{2}{x} + 1,$
$gf(x) = \frac{2}{x-2} + 3$
(h) $fg(x) = \left(\frac{1+2x}{x-1}\right)^2,$
$gf(x) = \frac{1+2x^2}{x^2-1}$
(i) $fg(x) = \frac{3(x-3)}{x-1},$
$gf(x) = \frac{7-2x}{2(2-x)}$

2. **(i)** $fg(x) = 3x + 3,$
$gf(x) = 3x + 5$
(ii) 12, 14, 0, 2

3. **(i)** $fg(x) = 8x^2 + 8x + 5,$
$gf(x) = 4x^2 + 7$
(ii) 5, 11, 101, 43

4. **(i)** −10 **(ii)** 8

5. **(i)** $fg(x) = 2kx + 5k - 3$
(ii) $\frac{2}{5}$

6. **(i)** $fg(x) = \frac{2}{3x-4},$
$gf(x) = \frac{6}{x} - 4$

7. **(i)** $fg(x) = 2ax + 2b + 3,$
$gf(x) = 2ax + 3a + b$
(ii) $a = \frac{1}{2}, b = -1\frac{1}{2}$
(iii) $-\frac{1}{2}$

8. **(i)** $fg(x) = 2x^2 + 9,$
$gf(x) = 4x^2 - 4x + 6$
(ii) 2.58 or −0.581

Chapter 6 Further Trigonometry

Practise Now 1

1. **(a)** 0.995 **(b)** 0.629

2. 0.905

Practise Now 2

1. **(a)** $\frac{3}{5}$ **(b)** $-\frac{4}{5}$
(c) $\frac{4}{3}$

2. **(i)** 13 units
(ii) **(a)** $\frac{12}{13}$ **(b)** $-\frac{12}{13}$
(c) $\frac{12}{5}$
(iii) $D(8, 9)$

Practise Now 3

(a) 24.5° or 155.5°
(b) 103.5°
(c) 84.0°

Practise Now 4

1. 298 m^2

2. 17.2 cm^2

Practise Now 5

1. 3.59

2. 53.1°

Practise Now 6

(i) 82.3°
(ii) 8.01 cm
(iii) 10.7 cm

Practise Now 7

1. **(i)** 34.7° **(ii)** 103.3°
(iii) 17.5 cm

2. **(i)** 52.1° **(ii)** 31.1°
(iii) 8.11 cm

Practise Now 8

1. 134.0°; 3.89 cm

2. 96.8°; 5.96 cm

Practise Now 9

(i) 16.1 cm
(ii) 69.5°
(iii) 39.5°

Practise Now 10

96.8°

Exercise 6A

1. **(a)** sin 70° **(b)** sin 4°
(c) sin 82° **(d)** −cos 81°
(e) −cos 73° **(f)** −cos 5°

2. **(a)** 0.530 **(b)** 0.819

3. **(a)** 3.535 **(b)** 0.707
(c) −2.121

4. **(a)** $\frac{4}{5}$ **(b)** $-\frac{3}{5}$
(c) $\frac{4}{3}$

5. **(i)** 9
(ii) **(a)** $\frac{9}{41}$ **(b)** $-\frac{40}{41}$
(c) $\frac{9}{40}$

6. **(a)** 31.3° **(b)** 48.6°
(c) 61.0° **(d)** 20.2°

7. **(a)** 148.7° **(b)** 131.4°
(c) 119.0° **(d)** 159.8°

8. **(a)** 47.9° **(b)** 40.9°
(c) 60° **(d)** 9.9°

9. **(a)** 48.9° or 131.1°
(b) 72.2° or 107.8°
(c) 28.1° or 151.9°
(d) 103.8° **(e)** 141.5°
(f) 58.4°

10. **(a)** $\frac{8}{17}$ **(b)** $-\frac{15}{17}$
(c) $\frac{8}{15}$

11. **(i)** **(a)** $\frac{3}{5}$ **(b)** $-\frac{4}{5}$
(c) $\frac{3}{8}$
(ii) $D(0, 1)$

12. **(a)** $\frac{5}{13}$ **(b)** $-\frac{12}{13}$
(c) $\frac{5}{27}$

13. **(a)** 18.0° or 142.0°
(b) 134.1°

Exercise 6B

1. **(a)** 34.2 cm^2 **(b)** 29.4 cm^2
(c) 41.5 m^2 **(d)** 31.7 m^2
(e) 27.4 cm^2 **(f)** 70.7 m^2
2. 117 cm^2
3. 9040 cm^2
4. 2.76 m^2
5. **(i)** 633 cm^2 **(ii)** 29.5 cm
6. 23 000 m^2
7. **(i)** 27.5° **(ii)** 10.5 cm
(iii) 6.22 cm^2
8. 3.23
9. 22.7 cm
10. 34.4° and 145.6°
11. 10.2° and 169.8°
12. $x = 16$, $y = 25$
13. **(i)** 10.8 cm^2 **(ii)** 104.5°

Exercise 6C

1. **(a)** $\angle C = 62°$, $b = 10.7$ cm, $c = 9.76$ cm
(b) $\angle F = 79.3°$, $d = 4.43$ m, $f = 6.96$ m
(c) $\angle H = 38°$, $g = 11.5$ mm, $i = 5.29$ mm
2. 11.8 cm
3. 15.6 cm
4. **(a)** $\angle B = 26.9°$, $\angle C = 61.1°$, $c = 13.4$ cm
(b) $\angle A = 55.6°$, $\angle C = 26.4°$, $c = 7.81$ m
(c) $\angle A = 113.3°$, $\angle B = 31.7°$, $a = 15.2$ cm
5. **(i)** 39.2° **(ii)** 39.8°
(iii) 13.6 cm
6. **(i)** 39.6° **(ii)** 49.4°
(iii) 8.80 cm
7. **(i)** 12.5 m **(ii)** 41.7°
(iii) 13.4 m
8. **(i)** 6.92 m **(ii)** 40.1 m^2
9. **(i)** 9.40 cm **(ii)** 5 cm
(iii) 4.92 cm
10. **(i)** No **(ii)** 9.47 cm
(iii) 4.3 cm
11. 14.7 km^2
12. **(i)** 2.64 cm **(ii)** 55.8°
(iii) 49.4°
13. $\angle KJL = 127.1°$, $JK = 5.71$ mm
14. 19.6 m
15. **(i)** 9.18 cm **(ii)** 0.734 km
(iii) 0.321 km^2
16. **(i)** 130.1°
(ii) $6\frac{2}{3}$ cm or 6.67 cm

Exercise 6D

1. 6.24 cm
2. 4.57 cm
3. 9.45 cm
4. $\angle X = 48.2°$, $\angle Y = 58.4°$, $\angle Z = 73.4°$
5. 34.5°
6. 88.5°
7. **(i)** 9 m **(ii)** 15.1 m
8. **(i)** 6.12 m **(ii)** 7 m
9. **(i)** 3.46 cm **(ii)** 5.29 cm
(iii) 90°
10. **(i)** 22.6° **(ii)** 4.84 m
(iii) 6.86 m
11. **(i)** 7.94 **(ii)** 81.0°
12. **(i)** 73.4° **(ii)** 1.92 cm
(iii) 2.18 cm
13. **(i)** 20 km **(ii)** 89.6°
(iii) 225 km^2
14. 93.8°; 9.29 cm
15. **(i)** No **(ii)** $\frac{131}{144}$
(iii) 6.78
16. **(i)** $-\frac{1}{20}$ **(ii)** 6.57 cm
17. 7.09 cm

Chapter 7 Applications of Trigonometry

Practise Now 1

1. 63.8 m
2. 19.0 m

Practise Now 2

1. **(i)** 354 m **(ii)** 14.3°
2. 45.2 m

Practise Now 3

1. **(a)** 050° **(b)** 330°
(c) 230° **(d)** 150°
2. **(a)** 123° **(b)** 231°
(c) 303° **(d)** 051°

Practise Now 4

1. 208 m
2. **(i)** 245°
(ii) $AC = 310$ m; $BC = 317$ m

Practise Now 5

(i) 49.4 km
(ii) 199.3°

Practise Now 6

(a) **(i)** 200° **(ii)** 2.92 km
(iii) 4.92 km
(b) 1.25 km

Introductory Problem Revisited

180 m

Practise Now 7

1. **(i)** 58.0° **(ii)** 28.1°
(iii) 74.2°
2. **(i)** 14.6 cm **(ii)** 28.3°

Practise Now 8

1. **(i)** 33.7° **(ii)** 53.1°
(iii) 30.8°
2. **(i)** 32.0° **(ii)** 35.3°
(iii) 23.8°

Practise Now 9

(i) 170 m
(ii) 8.4°

Practise Now 10

68.8 m

Exercise 7A

1. 119 m
2. 52.1 m
3. 27.6°
4. 63.1 m
5. 36.3 m
6. 68.7°
7. 9.74 m
8. 40.3 m
9. 35.0
10. 210
11. 63.5 m
12. 10.3 m
13. 28.0
14. **(i)** 81.3 m **(ii)** 93.5 m

Exercise 7B

1. **(a)** 033° **(b)** 118°
(c) 226° **(d)** 321°
2. **(a)** 055° **(b)** 165°
(c) 317° **(d)** 235°
(e) 345° **(f)** 137°
3. **(a)** 036° **(b)** 216°
(c) 073° **(d)** 253°
(e) 296° **(f)** 116°
4. **(i)** 34.6 km **(ii)** 35.5 km
5. 40.2 km
6. **(i)** 310° **(ii)** 270°
(iii) 220°
7. **(i)** 315° **(ii)** 267°
(iii) 032°
8. 028° or 216°
9. **(a)** 218 m **(b)** 436 m
10. 7.97 km
11. **(i)** 696 m **(ii)** 038.9°
12. **(i)** 126.6° **(ii)** 16.9 m
(iii) 305.8°
13. **(i)** 553 m **(ii)** 40.5°
(iii) 184.5° **(iv)** 331 m
14. 31.2 km; 080.2°
15. **(a)** **(i)** 65.9 km
(ii) 139.0°
(b) 1009 hours

Exercise 7C

1. **(a)** $\angle RPV$ **(b)** $\angle RQV$
   (c) $\angle QSU$ **(d)** $\angle PVS$
   (e) $\angle TVW$ **(f)** $\angle TQW$
2. **(a)** 7.07 cm **(b)** 45°
   (c) 8.66 cm **(d)** 35.3°
3. **(i)** 9.22 cm **(ii)** 6 cm
   (iii) 33.1° **(iv)** 45°
4. **(i)** 13 cm **(ii)** 53.1°
   (iii) 67.4°
5. **(i)** 33.7° **(ii)** 10 cm
   (iii) 21.8°
6. **(i)** 56.3° **(ii)** 5.55 m
   (iii) 10.6 m **(iv)** 52.0°
7. 38.6 m
8. **(i)** 14.1 cm **(ii)** 28.7 cm
   (iii) 63.8°
9. **(i)** 113 cm **(ii)** 15.7°
   (iii) 50.2°
10. **(i)** 25.8° **(ii)** 44.8°
11. **(i)** 26.6° **(ii)** 57.5°
12. **(i)** 053.1° **(ii)** 61.2 m
    (iii) 4.7°
13. **(a)** **(i)** 227.0°
    (ii) 120°
    (iii) 330°
    (b) 18.0°
14. **(a)** **(i)** 053.1°
    (ii) 326.3°
    (b) **(i)** 12.1 m
    (ii) 7.7°
15. **(a)** **(i)** 380 m
    (ii) 33 400 m^2
    (iii) 176 m
    (b) 7.8°
16. **(i)** 53.1° **(ii)** 30°
17. **(i)** 72.0° **(ii)** 55.5 cm^2
18. 39.3 m; 46.8 m
19. **(i)** 023.5° **(ii)** 38.6°

Chapter 8 Arc Length and Sector Area

Practise Now 1

1. **(i)** 99.5 cm
   (ii) 108 cm
2. 6.65

Introductory Problem Revisited

3.77 m

Practise Now 2

1. 13.2 m
2. **(i)** 29.1 cm **(ii)** 80.8 cm

Practise Now 3

1. **(i)** 247 cm^2 **(ii)** 459 cm^2
2. **(ii)** 1450 cm^2

Practise Now 4

(ii) 122 m^2

Practise Now 5

1. **(ii)** 6.72 cm^2
2. 16.8%

Exercise 8A

1. **(a)** 11.4 cm **(b)** 32.7 cm
   (c) 63.5 cm **(d)** 53.7 cm
2. **(a)** **(i)** 11.9 cm
   (ii) 62.6 cm
   (b) **(i)** 31.3 cm
   (ii) 101 cm
   (c) **(i)** 44.5 cm
   (ii) 101 cm
3. **(a)** 16.0 cm **(b)** 28.0 cm
4. **(a)** 49° **(b)** 80°
   (c) 263° **(d)** 346°
5. 1.18 m
6. 191.0°
7. **(a)** 17.0 cm **(b)** 37.0 cm
8. $\left(18+\frac{25}{3}\pi\right)$ cm
9. **(i)** 105° **(ii)** 12.8 cm
10. **(i)** 61.8° **(ii)** 30.5 cm
11. **(i)** 67.7 cm **(ii)** 198 cm
12. **(i)** 56.6 cm **(ii)** 155 cm
13. **(ii)** 48.2 cm
14. **(i)** 27.3° **(ii)** 54.5°
    (iii) 32.4 cm
15. 44.4 cm
16. 43.0 cm

Exercise 8B

1. **(a)** 8.80 cm, 30.8 cm^2, 22.8 cm
   (b) 108.0°, 66 mm, 1155 mm^2
   (c) 28.0 mm, 132 mm, 188 mm
   (d) 84.0 cm, 9240 cm^2, 388 cm
   (e) 225.1°, 385 m^2, 83 m
   (f) 15.3 cm, 20.1 cm, 50.8 cm
2. **(a)** **(i)** 17.7 cm
   (ii) 12.8 cm^2
   (b) **(i)** 8.22 cm
   (ii) 2.14 cm^2
   (c) **(i)** 26.7 cm
   (ii) 44.0 cm^2
3. **(a)** $14\frac{2}{3}$ cm or 14.7 cm; 103 cm^2
   (b) 24.2 cm; 169 cm^2
   (c) 30.8 cm; 216 cm^2
   (d) 52.8 cm; 370 cm^2
4. **(a)** 385 cm^2; 22.0 cm
   (b) $898\frac{1}{3}$ cm^2 or 898 cm^2; 51.3 cm
   (c) 1155 cm^2; 66.0 cm
   (d) 2117.5 cm^2; 121 cm
5. **(a)** 9.33 cm **(b)** 12.0 cm
6. **(a)** 60.3° **(b)** 165.8°
   (c) 303.5° **(d)** 26.7°
7. **(a)** 43.6 cm; 118 cm^2
   (b) 33.2 cm; 40.8 cm^2
   (c) 263 cm; 1640 cm^2
8. **(i)** 100° **(ii)** 42.0 cm
9. 84 cm^2
10. **(ii)** 32.5 cm^2
11. $1.47p^2$ cm^2
12. $\frac{4}{25}$
13. 0.716 m^2
14. **(i)** 132° **(ii)** 13.0 cm
    (iii) 92.7 cm **(iv)** 200 cm^2
15. **(i)** 8.49 cm **(ii)** 21.4 cm
    (iii) 20.5 cm^2
16. **(b)** 19.4 cm^2
    (c) **(i)** 619 cm^3
    (ii) 35

Chapter 9 Geometrical Properties of Circles

Practise Now 1

1. 13 cm
2. 12.6 cm

Practise Now 2

1. 6.14 cm or 32.6 cm
2. 22.6 cm

Practise Now 3

1. 16 cm
2. 14 cm

Practise Now 4

1. **(i)** 23.2° **(ii)** 11.4 cm
   (iii) 23.625 cm^2
2. **(i)** 3.9 **(ii)** 64.0°
   (iii) 7.10 cm^2

Practise Now 5

1. **(i)** 104° **(ii)** 14.1 cm
   (iii) 155 cm^2
2. 32°

Practise Now 6

1. **(i)** 56° **(ii)** 118°
2. $x = 110°, y = 55°$
3. 146°

Practise Now 7

1. $x = 55°, y = 70°, z = 20°$
2. **(i)** 25° **(ii)** 50°
   (iii) 105°

Practise Now 8

1. **(i)** 44° **(ii)** 25°
   (iii) 69°
2. 20°

Practise Now 9

1. **(i)** 113° **(ii)** 21°
2. 111°

Practise Now 10

1. **(ii)** 7.80 cm
2. **(ii)** 7.14 cm

Practise Now 11

(i) 33° **(ii)** 24°

Exercise 9A

1. 5 cm
2. 28.8 cm
3. 15 cm
4. 13 m
5. 13.7 cm
6. **(a)** $a = 12, b = 67.4$
(b) $c = 11.0, d = 61.9$
(c) $e = 6, f = 50.2$
7. 8 cm
8. 11 cm
9. **(a)** $a = 12$
(b) $x = 11, y = 90$
10. 24°
11. **(i)** 26° **(ii)** 122°
12. $\frac{90^\circ + x}{2}$
13. **(a)** $a = 49, b = 14$
(b) $c = 58, d = 15$
(c) $e = 34, f = 14.8$
(d) $g = 35, h = 55$
(e) $i = 8, j = 67.4$
(f) $k = 12.6, l = 50.0$
14. **(i)** 28° **(ii)** 59°
(iii) 26.3 cm **(iv)** 369 cm^2
15. 18.0 cm^2
16. 17 cm
17. **(i)** 14.2 cm **(ii)** 377 cm^2
18. 24.7 cm
19. 1 cm or 7 cm
20. 8.39 cm
21. **(i)** 44° **(ii)** 25°
22. **(i)** 7.5 cm **(ii)** 67.4°
(iii) 34.4 cm^2
23. 138°
24. 9 cm
25. 7 m
26. 64°
27. 51°
28. 63.3 cm
29. 5.05 cm
30. 45 cm

Exercise 9B

1. **(a)** 80 **(b)** 30
(c) 40 **(d)** 115
(e) 125 **(f)** 50
(g) 35 **(h)** 28
2. **(a)** 50 **(b)** 45
(c) 30 **(d)** 60
3. **(a)** 50 **(b)** 12
4. $x = 50^\circ, y = 25^\circ$
5. 60°
6. 65°
7. **(a)** 40 **(b)** 36
(c) 47 **(d)** 130
8. **(i)** 70° **(ii)** 70°
9. 270°
10. 37°
11. **(i)** 62° **(ii)** 47°
12. **(i)** 64° **(ii)** 64°
13. 78.5 cm^2
14. 70
15. 31°
16. 32°
17. 45°
18. 125°
19. **(i)** 90° **(ii)** 55°
20. 40°
21. **(i)** 35° **(ii)** 131°
22. **(i)** $(131 - x)^\circ$
(ii) $(149 - x)^\circ$
(iii) 50 **(iv)** 99°
23. **(b)** **(i)** 8 cm
(ii) 9 cm
24. **(ii)** $\triangle PCB$
25. **(ii)** $16\frac{1}{3}$ cm or 16.3 cm
26. 36°
27. 18°
28. $\angle P = 65^\circ, \angle Q = 55^\circ, \angle R = 60^\circ$
29. **(ii)** $10\frac{1}{6}$ cm or 10.2 cm

Chapter 10 Geometrical Transformations

Practise Now 1A

$y = x$

Practise Now 1B

(i) $A'(-3, -1), B'(-1, -5)$
(ii) $A''(7, -1), B''(5, -5)$

Practise Now 2

(b) (4, 3), 90° clockwise

Practise Now 3

(a) $A'(4, -1), B'(5, 2), C'(7, 2), D'(8, -1)$

Practise Now 4

(i) (6, 9) **(ii)** $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$
(iii) (8, 2)

Practise Now 8

(a) (−1, 0) **(b)** 2

Practise Now 10

Reflection in the line $y = 4$, enlargement with centre (1, 4) and scale factor $1\frac{1}{2}$

Practise Now 11

(c) Reflection in the line $y = -x$

Introductory Problem Revisited

$X(10, -5)$

Exercise 10A

1. **(a)** **(i)** (3, − 4)
(ii) (−1, −3)
(iii) (3, −3)
(iv) (−3, 4)
(v) (3, 2)
(vi) $(p, -q)$
(b) **(i)** (−3, 4)
(ii) (1, 3)
(iii) (−3, 3)
(iv) (3, −4)
(v) (−3, −2)
(vi) $(-p, q)$
2. **(a)** **(i)** (−1, 11)
(ii) (−1, −5)
(b) No
3. (1, 2)
4. **(ii)** (2, −2)
5. **(i)** (−2, 1) **(ii)** (6, 1)
(iii) (−6, 1) **(iv)** (10, 1)
6. **(i)** $y = x$ **(ii)** $x = 0$
7. **(a)** $x = 2$ **(b)** $y = 4$
(c) $y = x + 1$
(d) $y = -x + 2$
(e) $y = -x + 1$
(f) $y = 2x - 2$
8. **(i)** (1, −2) **(ii)** (2, 1
9. (3, 10)
10. **(ii)** (−2, 2)
11. **(i)** **(a)** (5, 2)
(b) (5, 2)
(ii) Yes
(iii) (3, 3)
12. **(i)** **(a)** (4, 5)
(b) (4, −1)
(ii) No
(iii) (4, 2)

Exercise 10B

2. **(a)** (3, −3) **(b)** (7, 6)
(c) (9, −2)
3. **(a)** (6, 5) **(b)** (7, 0)
4. 120° anticlockwise rotation about origin,
240° anticlockwise rotation about origin,
5. **(ii)** (2, 0), 180°
6. **(a)** **(i)** (6, 3)
(ii) 180°
(b) **(i)** (4.9, 1.9)
(ii) 90°
(c) (2, 2), (0, 0), (2, −1.2)
7. **(i)** 6 **(ii)** (4.5, 1)
(iii) (1, 3.5)
8. **(i)** 78° **(ii)** 31°

Exercise 10C

1. **(a)** $A''(5, -2), B''(6, 0), C''(8, 0), D''(8, -1)$
2. $P'(4, 1), Q'(10, 3), R'(5, -2)$
3. $\begin{pmatrix} -4 \\ 5 \end{pmatrix}$, (−5, 0)
4. **(a)** (4, 10) **(b)** (13, −4)
(c) (10, 0) **(d)** (10, 0)
(e) (1, 14)
5. **(i)** (4, 8)
(ii) $p = 0, q = 0$
(iii) $h = 6, k = 12$
(iv) (−2, −4)

Exercise 10D

4. **(a)** (4, 2) **(b)** (6, 6)
(c) (−4, −2) **(d)** (3.5, 3)
7. $P(-1.5, -1.5), Q(-0.5, -2.5), R(-2, -2)$

8. $L(-0.5, -1)$, $M(0, 1)$, $N(3, -1.5)$

9. **(i)** (2, 1)
(ii) $P(10, 0)$, $Q(4, 6)$

10. **(a)** $\triangle ABC$
(b) $PBQR$

11. **(a)** (5, 5), 2
(b) (0, 1), 2
(c) (4, 3), $-\frac{1}{3}$
(d) (5.5, 3), $\frac{1}{3}$

12. **(a)** (7, 6), 2
(b) (2, 1), 3
(c) (4, 6), $1\frac{1}{2}$
(d) (4, 6), $-\frac{1}{3}$
(e) (3, 2), 2
(f) (4, 5), −2

13. **(i)** $P(-6, 5)$, $Q(0, -3)$
(ii) 10 units

14. $A(1, 1)$, $B(5, 2)$, $C(2, 3)$

15. $A(1, 1)$, $C(0, 3)$

16. 20 cm

Exercise 10E

1. **(a)** (−3, 7) **(b)** (1, 7)

2. (−2, −5)

3. **(a)** (3, −2) **(b)** (1, 4)

4. (−3, −1)

5. **(a)** (8, 4) **(b)** (6, 3)

6. **(i)** (5, 5) **(ii)** (5, 11)
(iii) 6

7. Rotation of 180° about origin

8. Reflection in the line $y = -x$

10. (3, −2), (6, −1)

11. $B(1, 8)$, $C(2, 3)$, $D(-5, -1)$

12. (3, 3)

13. **(ii)** Reflection in the line AOC

14. **(i)** $A'(1, 1)$, $B'(3, 2)$, $C'(4, -1)$
(ii) $P'(5, 0)$, $Q'(5, 3)$, $R'(8, 3)$, $S'(8, 0)$

15. **(i)** (12, 8) **(ii)** (−28, −8)

Chapter 11 Area and Volume of Similar Figures and Solids

Practise Now 1

(a) 98 cm^2
(b) 10.7 m^2

Practise Now 2

3.6 m

Practise Now 3

(i) $2\frac{2}{3}$ cm^2 or 2.67 cm^2
(ii) $5\frac{5}{9}$ cm^2 or 5.56 cm^2

Practise Now 4

1. 75 cm^3
2. 0.5 m

Practise Now 5

4950 cm^3

Introductory Problem Revisited

58.7 cm

Practise Now 6

1. 328 g
2. 3000 kg

Practise Now 7

(i) 14.9 cm
(ii) 1 : 3.30

Exercise 11A

1. **(a)** 4 cm^2 **(b)** 0.24 m^2
(c) 20 cm^2 **(d)** 108 cm^2
(e) 27 m^2 **(f)** 6 cm^2

2. $\frac{16}{49}$

3. **(i)** $66\frac{2}{3}$ m^2 or 66.7 m^2
(ii) $42\frac{2}{3}$ m^2 or 42.7 m^2

4. **(a)** 6 **(b)** 15
(c) 20 **(d)** 4

5. 128 m^2

6. 7 cm^2

7. 812.5 m^2

8. $\frac{p^2}{p^2+2pq+q^2}$

9. **(i)** 18 cm **(ii)** 279 cm^2

10. 3.62 cm

11. 3 : 5

12. 7.5 cm

13. **(i)** 40 cm^2 **(ii)** 60 cm^2

14. 37.8 cm^2; 67.2 cm^2

15. **(i)** $2\frac{2}{7}$ cm or 2.29 cm
(ii) $\frac{7}{4}$ **(iii)** $\frac{64}{231}$

16. **(i)** 50 cm^2 **(ii)** 12 cm^2
(iii) 30 cm^2

17. $\frac{7}{3}(x+1)$ cm; 35

Exercise 11B

1. **(a)** 576 cm^3 **(b)** 162 cm^3
(c) 135 cm^3 **(d)** 38.5 m^3
(e) 0.4 m^3

2. **(a)** $\frac{64}{125}$ **(b)** $\frac{27}{64}$
(c) $\frac{8}{27}$

3. 160 cm^3

4. **(a)** 4 **(b)** 9
(c) 14 **(d)** 5

5. **(i)** 3 : 4 **(ii)** 189 cm^3

6. $\frac{4}{3}$

7. 20.8 cm

8. **(i)** $\frac{16}{49}$ **(ii)** 53.6 g

9. 4.76 cm

10. 148 g

11. **(i)** 4.608 kg **(ii)** 13 281 l

12. 8.15 m^2

13. **(i)** 36 cm **(ii)** $\frac{2}{3}$
(iii) 1134 cm^3

14. **(i)** 10 cm **(ii)** 20 cm^2
(iii) 225 cm^3

15. 6620 cm^3

16. 3.65 or −0.277

17. **(a)** $\frac{289}{484}$ **(b)** Yes
(c) No